中　國　物　理　學　報
第七卷第五期, 309—315 頁, 1950 年 9 月

# 關於 $^2$H, $^3$H, $^3$He, $^4$He 諸原子核結合能之問題

## 第一部

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我們應用變分法, 以摩勒—羅森費德之原子核位能爲基礎, 計算 $^2$H, $^3$H, $^3$He, $^4$He 諸原子核之結合能。因所選之波函數簡單, 計算很容易。計算結果與實驗所得結果符合, 惟 $^4$He 之結合能過大, 指明所採用之原子核位能尙須加以修改。

要從量的方面了解原子核之結合能有兩種困難。第一, 我們尙缺乏關於原子核作用力之準確知識。第二, 研究多體問題時, 數學非常複雜, 尤其個體近似方法已不能應用。現在暫以尙被認爲有力之介子學說作爲此計算之假定, 依照與實驗結果之比較, 我們可以改進數學上之技巧, 或修改所採用之作用力定律。

現在我們採取摩勒—羅森費德之公式, 作爲 $j$ 與 $k$ 兩核子間之位能

$$\left.\begin{aligned} V_{jk} &= \{g_0^2 + f_0^2(\sigma_j, \sigma_k)\}(\tau_j, \tau_k)\varphi_{jk}, \\ \varphi_{jk} &= r_{jk}^{-1}\ \exp\ (-\lambda r_{jk}), \qquad r_{jk} = |\boldsymbol{r}_j - \boldsymbol{r}_k| \end{aligned}\right\} \tag{1}$$

其作用力程 $\lambda$ 與介子之靜止質量 $\mu$ 之關係爲 $\hbar\lambda = \mu c$。對於第 $j$ 核子, 同位自旋算子 $\tau_j = (\tau_{jx}, \tau_{jy}, \tau_{jz})$ 作用於電荷波函數 $P(j)$ 或 $N(j)$ 之方式與自旋算子 $\sigma_j = (\sigma_{jx}, \sigma_{jy}, \sigma_{jz})$ 作用於自旋波函數 $\alpha(j)$ 或 $\beta(j)$ 之方式相同, 即

$$\left.\begin{aligned} &\sigma_{jx}\alpha(j) = \beta(j), && \sigma_{jy}\alpha(j) = i\beta(j), && \sigma_{jz}\alpha(j) = \alpha(j) \\ &\sigma_{jx}\beta(j) = \alpha(j), && \sigma_{jy}\beta(j) = -i\alpha(j), && \sigma_{jz}\beta(j) = -\beta(j) \\ &\tau_{jx}P(j) = N(j), && \tau_{jy}P(j) = iN(j), && \tau_{jz}P(j) = P(j) \\ &\tau_{jx}N(j) = P(j), && \tau_{jy}N(j) = -iP(j), && \tau_{jz}N(j) = -N(j) \end{aligned}\right\} \tag{2}$$

用 $r_j$ 表第 $j$ 核子之空間坐標向量。常數 $g_0^2$ 及 $f_0^2$ 已曾由 $^2$H 之最低能階 $^3S$ 與 $^1S$ 之能量數值定出，經過數值積分得下結果

$$g_0^2/\hbar c = 0{\cdot}027, \quad f_0^2/\hbar c = 0{\cdot}009 + 0{\cdot}56\,\mu/M \tag{3}$$

後者內含介子質量 $\mu$ 與核子質量 $M$ 之比值。如假定 $\mu = M/10, f_0^2/\hbar c = 0{\cdot}065$. 此外兩核子間尚有庫倫位能

$$\Delta V_{jk} = \left(\frac{1+\tau_{jz}}{2}\right)\left(\frac{1+\tau_{kz}}{2}\right)\frac{e^2}{r_{jk}}, \tag{4}$$

$$\frac{e^2}{\hbar c} = \frac{1}{137} = 0{\cdot}0073$$

經原子核庫倫位能很小，可以看作一小的改正值。暫時不計庫倫位能，質量數爲 $A$ 之原子核之哈密爾敦函數爲

$$H = \sum_{j=1}^{A} T_j + \frac{1}{2}\sum_{j,k=1}^{A} V_{jk}, \tag{5}$$

其中第 $j$ 核子之動能即常用之 $T_j = \boldsymbol{p}_j^2/2M$, $\boldsymbol{p}_j$ 爲其動量。其總動量，總自旋及總同位自旋均爲運動常數

$$\boldsymbol{p} = \sum_{j=1}^{A}\boldsymbol{p}_j, \quad \sigma = \sum_{j=1}^{A}\sigma_j, \quad \tau = \sum_{j=1}^{A}\tau_j \tag{6}$$

故駐態可用 $I, M_I, J, M_J$ 四量子數之不同值加以區別，四量子數之定義爲

$$\left.\begin{aligned} \left(\frac{\hbar}{2}\sigma\right)^2\psi &= I(I+1)\hbar^2\psi, \quad \frac{\hbar}{2}\sigma_z\psi = M_I\hbar\psi, \\ \left(\frac{\hbar}{2}\tau\right)^2\psi &= J(J+1)\hbar^2\psi, \quad \frac{\hbar}{2}\tau_z\psi = M_J\hbar\psi. \end{aligned}\right\} \tag{7}$$

靜止之原子核，其波函數 $\psi$ 中僅含核子間之相對坐標，故 $\boldsymbol{p}\psi = 0$。$\psi$ 之形狀受包里原則之限制。$A \leqq 4$ 諸原子核之最低態，其 $\psi$ 之形狀見第一表，其中空間波函數 $\varphi$ 對於 $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots \boldsymbol{r}_A$ 完全對稱，而電荷自旋波函數均屬行列式之形狀。例如

$$\det[P\alpha, N\beta] = \begin{vmatrix} P(1)\alpha(1), & P(2)\alpha(2) \\ N(1)\beta(1), & N(2)\beta(2) \end{vmatrix}.$$

## 第一表 $\psi$ 之形狀

| $A$ | $I$ | $M_I$ | $J$ | $M_J$ | 原子核 | $\psi$ |
|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 0 | 0 | $^4$He | $\varphi(1,2,3,4)(4!)^{-\frac{1}{2}}\det[P\alpha,P\beta,N\alpha,N\beta]$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $^3$He | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha,P\beta,N\alpha]$ |
| 3 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $^3$He | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha,P\beta,N\beta]$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $^3$H | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha,N\alpha,N\beta]$ |
| 3 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $^3$H | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\beta,N\alpha,N\beta]$ |
| 2 | 1 | 1 | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\alpha,N\alpha]$ |
| 2 | 1 | 0 | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\left\{\frac{1}{\sqrt{2}}\det[P\alpha,N\beta]+\frac{1}{\sqrt{2}}\det[P\beta,N\alpha]\right\}$ |
| 2 | 1 | $-1$ | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\beta,N\beta]$ |
| 2 | 0 | 0 | 1 | 1 | $^2$He | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\alpha,P\beta]$ |
| 2 | 0 | 0 | 1 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\left\{\frac{1}{\sqrt{2}}\det[P\alpha,N\beta]+\frac{1}{\sqrt{2}}\det[N\alpha,P\beta]\right\}$ |
| 2 | 0 | 0 | 1 | $-1$ | $^2n$ | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[N\alpha,N\beta]$ |

最低態之能量由 $\int\psi^* H\psi\,d\tau/\int\psi^*\psi\,d\tau$ 之極小值決定, $\int\cdots d\tau$ 包含對所有核子自旋及電荷坐標之總和及空間坐標之積分。按第一表所示 $\psi$ 之形狀代入, 利用 (2) 式及 $\alpha(j),\beta(j)$ 間及 $P(j),N(j)$ 間之正一關係來計算自旋及電荷之總和, 我們得到

$$\frac{\int\psi^* H\psi\,d\tau}{\int\psi^*\psi\,d\tau}=\frac{\int\varphi^* K\varphi\,d(r)}{\int\varphi^*\varphi\,d(r)} \tag{8}$$

其中之 $K$ 可寫成下式[1]

$$\left.\begin{aligned} &K=\sum_{j=1}^{A} T_j - G\frac{1}{2}\sum_{j,k=1}^{A}\varphi_{jk} \\ &A=2, I=1\text{ 時,}\qquad G=3g_0^2+3f_0^2 \\ &A=2, I=0\text{ 時,}\qquad G=-g_0^2+3f_0^2 \\ &A=3\text{ 或 }A=4\text{ 時,}\qquad G=g_0^2+3f_0^2 \end{aligned}\right\} \tag{9}$$

$\int\cdots d(r)$ 只包含對所有核子空間坐標之積分。對 He 原子核可將庫倫能之改正項加於 (8) 式, 即

$$\frac{\int\varphi^*(e^2/r_{12})\,\varphi\, d(r)}{\int\varphi^*\,\varphi\, d(r)} \tag{10}$$

今取視爲合理之 $\varphi$ ($a$ 爲參變數)

$$\varphi(1,2,\cdots,A)=\exp\left(-\frac{1}{\sqrt{2}}a\lambda\varrho\right),\qquad (a>0) \tag{11}$$

其中

$$\varrho^2=\frac{1}{2A}\sum_{j,k=1}^{A} r_{jk}^2=\frac{1}{2A}\sum_{j,k=1}^{A}|\boldsymbol{r}_j-\boldsymbol{r}_k|^2 \tag{12}$$

$\varrho^2$ 爲 $\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_A$ 之完全對稱函數, 僅含諸相對坐標。爲方便起見, 作下列坐標及動量之直角變換, 可得

$$\left.\begin{aligned} &\text{(i)}\quad A=2:\quad \boldsymbol{R}=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{\sqrt{2}},\quad \varrho_1=\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{\sqrt{2}}; \\ &\text{(ii)}\quad A=3:\quad \boldsymbol{R}=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3}{\sqrt{3}},\quad \varrho_1=\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{\sqrt{2}},\quad \varrho_2=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2-2\boldsymbol{r}_3}{\sqrt{6}} \\ &\text{(iii)}\quad A=4:\quad \boldsymbol{R}=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3+\boldsymbol{r}_4}{\sqrt{4}},\quad \varrho_1=\frac{\boldsymbol{r}_1-\boldsymbol{r}_2}{\sqrt{2}}, \\ &\qquad\qquad \varrho_2=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2-2\boldsymbol{r}_3}{\sqrt{6}},\quad \varrho_3=\frac{\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3-3\boldsymbol{r}_4}{\sqrt{12}}; \end{aligned}\right\} \tag{13}$$

及 $$\sum_{j=1}^{A} r_j^2 = \boldsymbol{R}^2 + \sum_{i=1}^{A-1} \varrho_i^2, \quad d(r) \equiv \prod_{j=1}^{A} d(r_j) = d(R) \prod_{i=1}^{A-1} d(\varrho_i) \equiv d(R)\, d(\varrho); \tag{14}$$

$$\varrho^2 = \frac{1}{2A} \sum_{j=1}^{A} \sum_{k=1}^{A} \{r_j^2 + r_k^2 - 2(\boldsymbol{r}_j, \boldsymbol{r}_k)\} = \sum_{j=1}^{A} r_j^2 - R^2 = \sum_{i=1}^{A-1} \varrho_i^2 \tag{15}$$

後者亦可寫成

$$\varrho^2 = \sum_{j=1}^{A} \left(\boldsymbol{r}_j - \frac{1}{\sqrt{A}} \boldsymbol{R}\right)^2, \qquad \left(\frac{1}{\sqrt{A}} \boldsymbol{R} = \frac{1}{A} \sum_{k=1}^{A} \boldsymbol{r}_k\right) \tag{16}$$

這是各個核子與全原子核質量中心間距離平方之和。引用 $3(A-1)$ 度 $\varrho$ 空間，則總動能爲

$$\frac{\int \varphi^* \sum_{j=1}^{A} T_j\, \varphi\, d(r)}{\int \varphi^* \varphi\, d(r)}$$

$$= \frac{-\frac{\hbar^2}{2M} \int_0^\infty e^{-\frac{1}{\sqrt{2}} a \lambda \varrho} \left\{ \left( \frac{d^2}{d\varrho^2} + \frac{3A-4}{\varrho} \frac{d}{d\varrho} \right) e^{-\frac{1}{\sqrt{2}} a \lambda \varrho} \right\} \varrho^{3A-4}\, d\varrho}{\int_0^\infty e^{-\sqrt{2} a \lambda \varrho} \varrho^{3A-4}\, d\varrho}$$

$$= \frac{\hbar^2 \lambda^2}{M} \cdot \frac{1}{4} a^2 \tag{17}$$

參用 (9) 式可得總位能

$$-\frac{A(A-1)}{2} G \frac{\int \varphi^* \varphi_{12}\, \varphi\, d(r)}{\int \varphi^* \varphi\, d(r)}$$

1： 此處所討論之諸原子核，如用更爲一般性之位能公式，如

$$V_{jk} = \{A + B(\tau_j, \tau_k) + C(\sigma_j, \sigma_k) + D(\tau_j, \tau_k)(\sigma_j, \sigma_k)\}\, \varphi_{jk},$$

則所得結果亦與 (9) 式相同，但其係數間之關係爲 $g_0^2 = B - C$, $f_0^2 = D - \frac{1}{3} A + \frac{2}{3} C$.

$$= -\frac{A(A-1)}{2} G \frac{\int e^{-\sqrt{2}a\lambda\varrho}\left(\frac{e^{-\sqrt{2}\lambda\varrho_1}}{\sqrt{2}\varrho_1}\right) d(\varrho)}{\int e^{-\sqrt{2}a\lambda\varrho} d(\varrho)} \tag{18}$$

如 $A=2$, 即僅含二核子時

$$\varrho = \varrho_1, \quad d(\varrho) = 4\pi\varrho^2 d\varrho,$$

令
$$x = \sqrt{2}(1+a)\lambda\varrho, \quad y = \sqrt{2}\,a\lambda\varrho, \text{ 則}$$

$$\frac{\int \varphi^* \varphi_{12} \varphi\, d(r)}{\int \varphi^* \varphi\, d(r)} = \frac{1}{\sqrt{2}} \frac{\int_0^\infty e^{-x} x\, dx}{\{\lambda\sqrt{2}(1+a)\}^2} \frac{\{\lambda\sqrt{2}\,a\}^3}{\int_0^\infty e^{-y} y^2\, dy} = \frac{\lambda a^3}{2(1+a)^2} \tag{19}$$

因此

$$\frac{\int \varphi^* K\varphi\, d(r)}{\int \varphi^* \varphi\, d(r)} \equiv \frac{\hbar^2\lambda^2}{M}\varepsilon, \quad \frac{MG}{\hbar^2\lambda} \equiv k, \tag{20}$$

其中

$$\varepsilon = \varepsilon(a) = \frac{1}{4}a^2 - \frac{1}{2}k\frac{a^3}{(1+a)^2}, \qquad (A=2) \tag{21}$$

如 $\mu = M/10$, 則 $\hbar^2\lambda^2/M = 9{\cdot}38$ Mev, $I=1$ 時 $k = 2{\cdot}76$, $I=0$ 時 $k=1{\cdot}68$, 然 $\varepsilon(a)$, 於 $a>0$, 僅在 $k \geqq 1{\cdot}893$ 時有一極小值, 故 $A=2$, $I=0$ 之原子核是不穩定的[2]。 此結論與實驗結果相符合。 $I=1$ 時, $\varepsilon$ 之極小值爲 $\varepsilon = -0{\cdot}227$, 在 $a = 2{\cdot}07$ 處。 相當之結合能爲 $2{\cdot}13$ Mev, 與實驗結果密合。 見第二表。

$$A=3 \text{ 時 } d(\varrho) = 4\pi\varrho_1^2 d\varrho_1 \cdot 4\pi\varrho_2^2 d\varrho_2,$$

$$A=4 \text{ 時 } d(\varrho) = 4\pi\varrho_1^2 d\varrho_1 \cdot 4\pi\varrho_2^2 d\varrho_2 \cdot 4\pi\varrho_3^2 d\varrho_3$$

變換坐標爲極坐標

$$\varrho_1 = \varrho\cos\theta, \quad \varrho_2 = \varrho\sin\theta \qquad (A=3)$$

或球體坐標

$$\varrho_1 = \varrho\cos\theta, \quad \varrho_2 = \varrho\sin\theta\cos\varphi, \quad \varrho_3 = \varrho\sin\theta\sin\varphi \qquad (A=4)$$

2: $\mu/M$ 爲任何值時, 此結論均可應用, 因 $A=2$, $I=0$ 時 $k$ 與 $\mu/M$ 之值無關。

計算位能之積分，可得

$$\varepsilon(a)=\frac{1}{4}a^2-\frac{2}{5\pi}k\frac{a^3}{(a^2-1)^3}\left(8a^4+9a^2-2-30\frac{a^4}{\sqrt{a^2-1}}\tan^{-1}\sqrt{\frac{a-1}{a+1}}\right) \qquad (A=3) \qquad (22)$$

$$\varepsilon(a)=\frac{1}{4}a^2-\frac{105}{64}k\frac{a^3}{(1+a)^5}\left(a^3+\frac{47}{35}a^2+\frac{5}{7}a+\frac{1}{7}\right), \qquad (A=4) \qquad (23)$$

仍假定 $\mu=M/10$, $A=3$, $A=4$ 時 $k$ 均爲 2·22。 對 He 原子核應加以庫倫改正

$$\left.\begin{aligned}\Delta\varepsilon&=\frac{M}{\mu}\cdot\frac{e^2}{\hbar c}\cdot\frac{16}{15\pi}a \quad (^3\mathrm{He})\\ \Delta\varepsilon&=\frac{M}{\mu}\cdot\frac{e^2}{\hbar c}\cdot\frac{35}{128}a \quad (^4\mathrm{He})\end{aligned}\right\} \qquad (24)$$

經數值計算 $\varepsilon$ 或 $\varepsilon+\Delta\varepsilon$ 之極小值列於第二表。 $\mu=M/6$ 時之結果亦列入該表內，以備與實驗結果比較。 此處應注意，庫倫能量確爲很小之改正，約爲 3%。例如 $^4$He 內，$\mu=M/10$ 時改正項約計 1·18 Mev, $\mu=M/6$ 時，改正項約計 1·77 Mev。

第 二 表　結合能（單位爲 Mev.）

| 原子核 | 自旋態 | $\mu=M/10$ | | | $\mu=M/6$ | | | 實驗結合能 |
|---|---|---|---|---|---|---|---|---|
| | | $a$ | $\varepsilon$ 或 $\varepsilon+\Delta\varepsilon$ | 結合能 | $a$ | $\varepsilon$ 或 $\varepsilon+\Delta\varepsilon$ | 結合能 | |
| $^2$H | $I=1$ | 2·07 | −·227 | 2·13 | 1·52 | −·066 | 1·73 | 2·19 |
| $^3$H | $I=\frac{1}{2}$ | 3·59 | −·774 | 7·25 | 2·85 | −·401 | 10·44 | 8·39 |
| $^3$He | $I=\frac{1}{2}$ | 3·49 | −·683 | 6·40 | 2·80 | −·358 | 9·34 | 7·65 |
| $^4$He | $I=0$ | 6·33 | −3·48 | 32·6 | 5·64 | −2·29 | 59·7 | 28·20 |
| $^3$H − $^3$He | | | | ·85 | | | 1·10 | ·74 |

CHINESE JOURNAL OF PHYSICS
Vol. 7, No. 5, 1950, pp. 316 — 323

# ON THE BINDING ENERGIES OF THE ATOMIC NUCLEI $^2$H, $^3$H, $^3$He and $^4$He

## PART I.

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*(Received September 3, 1949)*

### *ABSTRACT*

The binding energies of the atomic nuclei $^2$H, $^3$H, $^3$He and $^4$He are obtained by means of the variational method, using as a basis the Møller-Rosenfeld nuclear potential. A simple choice of the wave function renders the calculation elementary. Agreement with experimental results is good, except that the binding energy obtained for $^4$He is excessive, which indicates the necessity of further improving upon the nuclear potential.

In attempting to understand quantitatively the binding energies of the atomic nuclei, we face two difficulties. The first concerns the physical law of nuclear force, of which we still lack precise knowledge. The second involves the mathematical complexity of many-body problem, and particularly the failure of individual particle approximation. At the moment it seems reasonable to adopt, as a working hypothesis, the law of nuclear force as derived from the yet provisional meson theory. Our endeavour here is rather to develop the mathematical technique of dealing with many-body problems appropriate to the closed packing of the nuclear particles, the nucleons, in the nuclei. Guided by comparison with experimental results, we may then hope to improve upon either our mathematical technique or the adopted law of nuclear force.

In this paper we adopt, for the potential energy between a pair of nucleons, say $j$ and $k$, the Møller-Rosenfeld expression

$$\left.\begin{aligned} V_{jk} &= \{ g_0^2 + f_0^2(\sigma_j, \sigma_k) \} (\tau_j, \tau_k)\, \varphi_{jk}, \\ \varphi_{jk} &= r_{jk}^{-1} \exp(-\lambda r_{jk}), \quad r_{jk} = |\boldsymbol{r}_j - \boldsymbol{r}_k|. \end{aligned}\right\} \tag{1}$$

The range $\lambda$ of nuclear force is related to the rest mass $\mu$ of the meson by $\hbar\lambda=\mu c$. The isotopic spin operator $\tau_j=(\tau_{jx},\tau_{jy},\tau_{jz})$ acts on the charge wave function $P(j)$ or $N(j)$ in the same way as the spin operator $\sigma_j=(\sigma_{jx},\sigma_{jy},\sigma_{jz})$ acts on the spin wave function $\alpha(j)$ or $\beta(j)$ of the $j$th nucleon, namely

$$\left.\begin{aligned}
\sigma_{jx}\alpha(j)&=\beta(j), & \sigma_{jy}\alpha(j)&=\ \ i\beta(j), & \sigma_{jz}\alpha(j)&=\ \ \alpha(j),\\
\sigma_{jx}\beta(j)&=\alpha(j), & \sigma_{jy}\beta(j)&=-i\alpha(j), & \sigma_{jz}\beta(j)&=-\beta(j),\\
\tau_{jx}P(j)&=N(j), & \tau_{jy}P(j)&=\ \ iN(j), & \tau_{jz}P(j)&=\ \ P(j),\\
\tau_{jx}N(j)&=P(j), & \tau_{jy}N(j)&=-iP(j), & \tau_{jz}N(j)&=-N(j).
\end{aligned}\right\} \tag{2}$$

The spatial co-ordinate vector of the $j$th nucleon is given by $\boldsymbol{r}_j$. The universal constants $g_0^2$ and $f_0^2$ have been determined from the energy values of the lowest $^3S$ and $^1S$ levels of the $^2$H nucleus, by numerically integrating the wave equation, with the results

$$g_0^2/\hbar c=\cdot 027,\quad f_0^2/\hbar c=\cdot 009+\cdot 56\,\mu/M, \tag{3}$$

the latter involving the ratio of the masses $\mu$ of the meson and $M$ of the nucleon. We assume $\mu=M/10$, so $f_0^2/\hbar c=\cdot 065$. In addition we have the Coulomb potential between a pair of nucleons

$$\Delta V_{jk}=\left(\frac{1+\tau_{jz}}{2}\right)\left(\frac{1+\tau_{kz}}{2}\right)\frac{e^2}{r_{jk}},$$

with

$$e^2/\hbar c=1/137=\cdot 0073. \tag{4}$$

For light atomic nuclei we shall treat the Coulomb energy as a small correction. Neglecting for the moment the Coulomb energy, we have, for the Hamiltonian of the atomic nucleus of mass number $A$,

$$H=\sum_{j=1}^{A}T_j+\tfrac{1}{2}\sum_{j,k=1}^{A}V_{jk}, \tag{5}$$

where the kinetic energy of the $j$th nucleon is given by the usual expression $T_j=\boldsymbol{p}_j^2/2M$, $\boldsymbol{p}_j$ being the momentum. The total momentum, the total spin, and the total isotopic spin are all constants of motion

$$p=\sum_{j=1}^{A} p_j, \quad \sigma=\sum_{j=1}^{A} \sigma_j, \quad \tau=\sum_{j=1}^{A} \tau_j. \tag{6}$$

The stationary states can therefore be classified according to various values assigned to the quantum numbers $I$, $M_I$, $J$ and $M_J$, defined by

$$\begin{aligned}(\tfrac{1}{2}\hbar\sigma)^2\psi=I(I+1)\hbar^2\psi, \quad \tfrac{1}{2}\hbar\sigma_z\psi=M_I\hbar\psi,\\ (\tfrac{1}{2}\hbar\tau)^2\psi=J(J+1)\hbar^2\psi, \quad \tfrac{1}{2}\hbar\tau_z\psi=M_J\hbar\psi.\end{aligned} \tag{7}$$

TABLE 1. *Form of* $\psi$

| $A$ | $I$ | $M_I$ | $J$ | $M_J$ | Nucleus | $\psi$ |
|---|---|---|---|---|---|---|
| 4 | 0 | 0 | 0 | 0 | $^4$He | $\varphi(1,2,3,4)(4!)^{-\frac{1}{2}}\det[P\alpha, P\beta, N\alpha, N\beta]$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $^3$He | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha, P\beta, N\alpha]$ |
| 3 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $^3$He | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha, P\beta, N\beta]$ |
| 3 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $^3$H | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\alpha, N\alpha, N\beta]$ |
| 3 | $\frac{1}{2}$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $^3$H | $\varphi(1,2,3)(3!)^{-\frac{1}{2}}\det[P\beta, N\alpha, N\beta]$ |
| 2 | 1 | 1 | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\alpha, N\alpha]$ |
| 2 | 1 | 0 | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\left\{\frac{1}{\sqrt{2}}\det[P\alpha, N\beta] + \frac{1}{\sqrt{2}}\det[P\beta, N\alpha]\right\}$ |
| 2 | 1 | $-1$ | 0 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\beta, N\beta]$ |
| 2 | 0 | 0 | 1 | 1 | $^2$He | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[P\alpha, P\beta]$ |
| 2 | 0 | 0 | 1 | 0 | $^2$H | $\varphi(1,2)(2!)^{-\frac{1}{2}}\left\{\frac{1}{\sqrt{2}}\det[P\alpha, N\beta] + \frac{1}{\sqrt{2}}\det[N\alpha, P\beta]\right\}$ |
| 2 | 0 | 0 | 1 | $-1$ | $^2n$ | $\varphi(1,2)(2!)^{-\frac{1}{2}}\det[N\alpha, N\beta]$ |

For the nucleus at rest, the wave function $\psi$ involves only the relative co-ordinates between the nucleons, so $\boldsymbol{p}\,\psi=0$. The form of $\psi$ is further restricted by the Pauli principle. For $A \leqq 4$, the $\psi$ for the lowest states are of the form as given in Table 1, where the spatial wave function $\varphi$ is a totally symmetric function of the arguments $\boldsymbol{r}_1,\cdots,\boldsymbol{r}_A$. The charge-spin wave functions are all of the form of determinants, e.g.

$$\det[P\alpha, N\beta] = \begin{vmatrix} P(1)\,\alpha(1), & P(2)\,\alpha(2) \\ N(1)\,\beta(1), & N(2)\,\beta(2) \end{vmatrix}.$$

The energy value of the lowest state is to be determined from the minimum of $\int\psi^* H\psi\,d\tau/\int\psi^*\psi\,d\tau$, where $\int\cdots d\tau$ includes summation over the spin and charge co-ordinates as well as integration over the spatial co-ordinates of all the nucleons. After substituting for $\psi$ the expression given in Table 1, we can evaluate the summation over the spin and charge co-ordinates by means of (2) and the orthonormal relations between $\alpha(j)$ and $\beta(j)$, or between $P(j)$ and $N(j)$. Thus we obtain, after some calculation,

$$\frac{\int\psi^* H\psi\,d\tau}{\int\psi^*\psi\,d\tau} = \frac{\int\varphi^* K\varphi\,d(r)}{\int\varphi^*\varphi\,d(r)}, \tag{8}$$

where $K$ can be written in the form[1]

$$K = \sum_{j=1}^{A} T_j - G\,\tfrac{1}{2}\sum_{j,k=1}^{A}\varphi_{jk},$$

with

$$\left.\begin{aligned} G &= 3\,g_0^2 + 3f_0^2 && \text{for } A=2, \quad I=1, \\ G &= -\,g_0^2 + 3f_0^2 && \text{for } A=2, \quad I=0, \\ G &= \;\;\, g_0^2 + 3f_0^2 && \text{for } A=3 \text{ or } A=4. \end{aligned}\right\} \tag{9}$$

1. It is interesting to note that, for the nuclei considered here, the result (9) also follows from the more general potential

$$V_{jk} = \{A + B(\tau_j,\tau_k) + C(\sigma_j,\sigma_k) + D(\tau_j,\tau_k)(\sigma_j,\sigma_k)\}\,\varphi_{jk},$$

with then the following combinations for $g_0^2$ and $f_0^2$, viz.

$$g_0^2 = B - C, \quad f_0^2 = D - \frac{1}{3}A + \frac{2}{3}C.$$

The integration $\int\cdots d(r)$ includes only integration over the spatial co-ordinates of all the nucleons. For He nucleus we can now take account of the Coulomb energy by adding to (8) the correction term

$$\frac{\int \varphi^* (e^2/r_{12})\, \varphi\, d(r)}{\int \varphi^* \varphi\, d(r)}. \tag{10}$$

It appears reasonable to take ($a$ being a parameter)

$$\varphi(1, \cdots, A) = \exp\left(-\frac{1}{\sqrt{2}}\, a\, \lambda\, \varrho\right), \quad (a > 0) \tag{11}$$

where

$$\varrho^2 = \frac{1}{2A} \sum_{j,k=1}^{A} r_{jk}{}^2 = \frac{1}{2A} \sum_{j,k=1}^{A} |\boldsymbol{r}_j - \boldsymbol{r}_k|^2 \tag{12}$$

is a totally symmetric function of $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_A$, involving only the relative co-ordinates. For the evaluation of the integrals (8) and (10), it is convenient to make the following orthogonal change of variables for the co-ordinates :

(i) for $A = 2$: $\boldsymbol{R} = \dfrac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{\sqrt{2}}$, $\varrho_1 = \dfrac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\sqrt{2}}$;

(ii) for $A = 3$: $\boldsymbol{R} = \dfrac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3}{\sqrt{3}}$, $\varrho_1 = \dfrac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\sqrt{2}}$, $\varrho_2 = \dfrac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3}{\sqrt{6}}$

(iii) for $A = 4$: $\boldsymbol{R} = \dfrac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \boldsymbol{r}_4}{\sqrt{4}}$, $\varrho_1 = \dfrac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\sqrt{2}}$,

$$\varrho_2 = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - 2\boldsymbol{r}_3}{\sqrt{6}}, \quad \varrho_3 = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 - 3\boldsymbol{r}_4}{\sqrt{12}}; \tag{13}$$

and also similarly for the momenta. We have

$$\sum_{j=1}^{A} \boldsymbol{r}_j^2 = \boldsymbol{R}^2 + \sum_{i=1}^{A-1} \varrho_i^2, \quad d(r) \equiv \prod_{j=1}^{A} d(r_j) = d(R) \prod_{i=1}^{A-1} d(\varrho_i) \equiv d(R)\, d(\varrho), \tag{14}$$

$$\varrho^2 = \frac{1}{2A} \sum_{j=1}^{A} \sum_{k=1}^{A} \left\{ \boldsymbol{r}_j^2 + \boldsymbol{r}_k^2 - 2(\boldsymbol{r}_j, \boldsymbol{r}_k) \right\} = \sum_{j=1}^{A} \boldsymbol{r}_j^2 - \boldsymbol{R}^2 = \sum_{i=1}^{A-1} \varrho_i^2. \tag{15}$$

The latter can also be expressed as

$$\varrho^2 = \sum_{j=1}^{A} \left(r_j - \frac{1}{\sqrt{A}} \boldsymbol{R}\right)^2, \quad \left(\frac{1}{\sqrt{A}} \boldsymbol{R} = \frac{1}{A} \sum_{k=1}^{A} r_k\right), \tag{16}$$

that is the sum of the squares of the distances of all nucleons, as measured from the center of mass of the whole nucleus. Thus the total kinetic energy is, by introducing spherical co-ordinates in the $3(A-1)$-dimensional $\varrho$-space, given by

$$\frac{\int \varphi^* \sum_{j=1}^{A} T_j \varphi \, d(r)}{\int \varphi^* \varphi \, d(r)}$$

$$= \frac{-\dfrac{\hbar^2}{2M} \displaystyle\int_0^\infty e^{-\frac{1}{\sqrt{2}} a\lambda\varrho} \left\{ \left( \frac{d^2}{d\varrho^2} + \frac{3A-4}{\varrho} \frac{d}{d\varrho} \right) e^{-\frac{1}{\sqrt{2}} a\lambda\varrho} \right\} \varrho^{3A-4} \, d\varrho}{\displaystyle\int_0^\infty e^{-\sqrt{2} a\lambda\varrho} \varrho^{3A-4} \, d\varrho}$$

$$= \frac{\hbar^2 \lambda^2}{M} \cdot \frac{1}{4} a^2, \tag{17}$$

while the total potential energy is, on comparing with (9), given by

$$-\frac{A(A-1)}{2} G \frac{\int \varphi^* \varphi_{12} \varphi \, d(r)}{\int \varphi^* \varphi \, d(r)}$$

$$= -\frac{A(A-1)}{2} G \frac{\int e^{-\sqrt{2} a\lambda\varrho} (e^{-\sqrt{2} a\lambda\varrho_1 / \sqrt{2}\varrho_1}) \, d(\varrho)}{\int e^{-\sqrt{2} a\lambda\varrho} \, d(\varrho)} \tag{18}$$

Consider now the case of two nucleons only, $A=2$. Then we have $\varrho = \varrho_1$ and $d(\varrho) = 4\pi\varrho^2 d\varrho$, so, with $x = \sqrt{2}(1+a)\lambda\varrho$ and $y = \sqrt{2}\, a\lambda\varrho$, we get

$$\frac{\int \varphi^* \varphi_{12} \varphi \, d(r)}{\int \varphi^* \varphi \, d(r)} = \frac{1}{\sqrt{2}} \frac{\int_0^\infty e^{-x} x \, dx}{\{\sqrt{2}(1+a)\lambda\}^2} \cdot \frac{\{\sqrt{2}\, a\lambda\}^3}{\int_0^\infty e^{-y} y^2 \, dy} = \frac{\lambda a^3}{2(1+a)^2}, \tag{19}$$

and hence

$$\frac{\int \varphi^* K \varphi \, d(r)}{\int \varphi^* \varphi \, d(r)} = \frac{\hbar^2 \lambda^2}{M} \varepsilon, \quad \frac{MG}{\hbar^2 \lambda} = k, \tag{20}$$

with

$$\varepsilon = \frac{1}{4} a^2 - \frac{1}{2} k \frac{a^3}{(1+a)^2}, \quad (A=2). \tag{21}$$

For $\mu = M/10$, we have $\hbar^2 \lambda^2 / M = 9{\cdot}38$ Mev, while $k = 2{\cdot}76$ for $I=1$ or $k = 1{\cdot}68$ for $I=0$. Now $\varepsilon$ possesses a minimum for some $a > 0$ only when $k \geqq 1{\cdot}893$, hence the nuclei with $A=2$ and $I=0$ are unstable.[2] This is in agreement with experimental finding. For the triplet state, $I=1$, the minimum value $\varepsilon = -0{\cdot}227$ occurs at $a = 2{\cdot}07$, which gives a binding energy of $2{\cdot}13$ Mev in close agreement with the experimental data. See Table 2.

For $A=3$ or $A=4$, we have $d(\varrho) = 4\pi \varrho_1^2 d\varrho_1 \cdot 4\pi \varrho_2^2 d\varrho_2$ or $d(\varrho) = 4\pi \varrho_1^2 d\varrho_1 \cdot 4\pi \varrho_2^2 d\varrho_2 \cdot 4\pi \varrho_3^2 d\varrho_3$. It suffices to mention that further change to polar co-ordinates ($\varrho_1 = \varrho \cos\theta$, $\varrho_2 = \varrho \sin\theta$) or spherical co-ordinates ($\varrho_1 = \varrho\cos\theta$, $\varrho_2 = \varrho \sin\theta \cos\varphi$, $\varrho_3 = \varrho \sin\theta \sin\varphi$) is to be made for the evaluation of the potential-energy integrals. The results are of the form (20), with

$$\varepsilon = \frac{1}{4} a^2 - \frac{2}{5\pi} k \frac{a^3}{(a^2-1)^3} \left( 8a^4 + 9a^2 - 2 - 30 \sqrt{\frac{a^4}{a^2-1}} \tan^{-1} \sqrt{\frac{a-1}{a+1}} \right), \quad (A=3) \tag{22}$$

$$\varepsilon = \frac{1}{4} a^2 - \frac{105}{64} k \frac{a^3}{(1+a)^5} \left( a^3 + \frac{47}{35} a^2 + \frac{5}{7} a + \frac{1}{7} \right), \quad (A=4), \tag{23}$$

The numerical value for $k$ is $2{\cdot}22$ both for $A=3$ and for $A=4$, assuming still $\mu = M/10$. For He nucleus we have to add the Coulomb correction

$$\Delta\varepsilon = \frac{M}{\mu} \cdot \frac{e^2}{\hbar c} \cdot \frac{16}{15\pi} a \ (^3\text{He}), \quad \text{or} \quad \Delta\varepsilon = \frac{M}{\mu} \cdot \frac{e^2}{\hbar c} \cdot \frac{35}{128} a \ (^4\text{He}). \tag{24}$$

The minimum of $\varepsilon$ or $\varepsilon + \Delta\varepsilon$ is obtained by numerical calculation, with results shown in Table 2. Similar results calculated for $\mu = M/6$ are

2. This conclusion holds for all values of $\mu/M$, because for $A=2$, $I=0$, $k$ is actually independent of $\mu/M$.

also contained in this table, as well as comparison with experimental results. It may be noted that the Coulomb energy turns out to be indeed a small correction of about 3%, for example for $^4$He it is 1·18 Mev for $\mu=M/10$ or 1·77 Mev for $\mu=M/6$.

TABLE 2 *Binding energies*

(in Mev units)

| Nucleus | State of the spin | $\mu=M/10$ | | | $\mu=M/6$ | | | Exper. binding energy |
|---|---|---|---|---|---|---|---|---|
| | | $a$ | $\varepsilon$ or $\varepsilon+\varDelta\varepsilon$ | binding energy | $a$ | $\varepsilon$ or $\varepsilon+\varDelta\varepsilon$ | binding energy | |
| $^2$H | triplet | 2·07 | − ·227 | 2·13 | 1·52 | − ·066 | 1·73 | 2·19 |
| $^3$H | doublet | 3·59 | − ·774 | 7·25 | 2·85 | − ·401 | 10·44 | 8·39 |
| $^3$He | doublet | 3·49 | − ·683 | 6·40 | 2·80 | − ·358 | 9·34 | 7·65 |
| $^4$He | singlet | 6·33 | −3·48 | 32·6 | 5·64 | −2·29 | 59·7 | 28·20 |
| $^3$H − $^3$He | | | | ·85 | | | 1·10 | ·74 |