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准坐标下广义力学系统的 Lie 对称定理及其逆定理*

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研究准坐标下广义力学系统的 Lie 对称性与守恒量. 首先,对准坐标下广义力学系统定义无限小生成元,并应用微分方程在无限小变换下不变性的 Lie 方法,建立系统的确定方程. 其次,给出结构方程和守恒量的形式. 最后,研究 Lie 对称性逆问题(由已知积分求 Lie 对称)并举例说明结果的应用.

关键词:广义力学,准坐标,Lie 对称,确定方程,结构方程,守恒量

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1　引　　言

力学系统的守恒量,或第一积分,不仅具有数学重要性,而且表现为深刻的物理规律,它已成为近代分析力学的一个重要研究方向. 1979 年 Lutzky[1]等人把 Lie 研究微分方程不变性的扩展群方法引入力学领域,提出了使运动微分方程不变的 Lie 对称性,从此力学领域 Lie 对称性的研究得到迅速的发展. 文献[2]讨论了力学系统的对称性和不变量,文献[3]讨论了非 Chetaev 型非完整系统的 Lie 对称性和守恒量,文献[4]讨论了准坐标下完整力学系统的 Lie 对称与守恒量. 但这些研究内容,都局限于普通分析力学的范畴.

本文根据广义力学中 Lagrange 方程在无限小变换下的不变性,给出准坐标下广义力学中 Lagrange 系统的 Lie 对称性与守恒量,并举例说明结果的应用.

2　广义力学系统的运动微分方程

假设系统的位形由 n 个广义坐标 $q_1, q_2, \cdots, q_n$ 来确定. 选准速度为广义速度的线性式,即

$$\omega_s = \sum_{r=1}^{n} a_{sr} \dot{q}_r, \tag{1}$$

其中系数 a_{sr} 仅依赖于广义坐标 q. 设由(1)式可解出广义速度

$$\dot{q}_s = \sum_{r=1}^{n} b_{sr} \omega_r, \tag{2}$$

此处

$$b_{sr} a_{rk} = \delta_{sk}, \quad \delta_{sk} = \begin{cases} 1, s = k, \\ 0, s \neq k. \end{cases}$$

由(2)式可有

$$\delta q_s = \sum_{r=1}^{n} b_{sr} \delta \pi_r . \tag{3}$$

定义对准坐标的偏导数为

$$\frac{\partial}{\partial \pi_s} = \sum_{r=1}^{n} b_{rs} \frac{\partial}{\partial q_r}, \tag{4}$$

在广义力学中 Lagrange 函数为[5]

$$L = L(q_s, \dot{q}_s, \ddot{q}_s, \cdots, \overset{(\delta)}{q}_s, t), \quad (s = 1, 2, \cdots, n). \tag{5}$$

令 $\widetilde{L}$ 为 Lagrange 函数 L 中借助关系(2)式消去 $\dot{q}_s, \ddot{q}_s, \cdots, \overset{(\delta)}{q}_s$ 而由准速度 $\omega_r, \dot{\omega}_r, \ddot{\omega}_r, \cdots, \overset{(\delta-1)}{\omega}_r$ 表示的表达式,即

$$\widetilde{L}(q_s, \omega_s, \dot{\omega}_s, \ddot{\omega}_s, \cdots, \overset{(\delta-1)}{\omega}_s, t) = L\Big[q_s, \Big(\sum_{r=1}^{n} b_{sr}\omega_r\Big), \Big(\sum_{r=1}^{n} b_{sr}\omega_r\Big)', \cdots, \Big(\sum_{r=1}^{n} b_{sr}\omega_r\Big)^{(\delta-1)}, t\Big], \tag{6}$$

于是由文献[6],准坐标下非保守系统的 Hamilton 原理可写为

$$\int_{t_0}^{t_1} \Big(\delta \widetilde{L} + \sum_{s=1}^{n} \widetilde{Q}_s \delta \pi_s\Big) \mathrm{d}t = 0, \tag{7}$$

其中

$$\widetilde{Q}_s = \sum_{k=1}^{n} Q_k b_{ks},$$

注意到

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta \overset{(\delta-1)}{\pi_s}\right) = \delta \overset{(\delta-1)}{\omega_s} . \tag{8}$$

假设想象的邻近路径不是任意选取的，而是在瞬时 t_0 和 t_1 与真实路径相重合，因此有

$$\delta\pi_s\big|_{t_1} = \delta\pi_s\big|_{t_0} = 0,$$
$$\delta \overset{(\alpha)}{\omega_s}\big|_{t_1} = \delta \overset{(\alpha)}{\omega_s}\big|_{t_0} = 0, \tag{9}$$
$$(\alpha = 0,1,2,\cdots,\delta-1).$$

下面由 Hamilton 原理(7)式来求准坐标下广义力学系统的运动微分方程.

由于

$$\delta\int_{t_0}^{t_1}\widetilde{L}\,\mathrm{d}t = \int_{t_0}^{t_1}\sum_{s=1}^{n}\left(\frac{\partial\widetilde{L}}{\partial q_s}\delta q_s + \frac{\partial\widetilde{L}}{\partial\omega_s}\delta\omega_s + \frac{\partial\widetilde{L}}{\partial\dot{\omega}_s}\delta\dot{\omega}_s + \cdots + \frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\delta\overset{(\delta-1)}{\omega_s}\right)\mathrm{d}t, \tag{10}$$

利用分部积分法及时端条件(9)，得到

$$\int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial q_s}\delta q_s = \int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial q_s}\sum_{r=1}^{n}b_{sr}\delta\pi_r\mathrm{d}t = \int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\pi_s}\delta\pi_s\mathrm{d}t, \tag{11}$$

$$\begin{aligned}\int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\omega_s}\delta\omega_s\mathrm{d}t &= \int_{t_0}^{t_1}\Bigg[\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\omega_s}(\delta\pi_s)' - \sum_{s=1}^{n}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\frac{\partial\widetilde{L}}{\partial\omega_l}\omega_k\delta\pi_s\right)\Bigg]\mathrm{d}t \\ &= \left(\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\omega_s}\delta\pi_s\right)\Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1}\sum_{s=1}^{n}\Bigg[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{L}}{\partial\omega_s}\right)\delta\pi_s + \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\frac{\partial\widetilde{L}}{\partial\omega_l}\omega_k\delta\pi_s\Bigg]\mathrm{d}t \\ &= -\int_{t_0}^{t_1}\sum_{s=1}^{n}\Bigg[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{L}}{\partial\omega_s}\right) + \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\frac{\partial\widetilde{L}}{\partial\omega_l}\omega_k\Bigg]\delta\pi_s\mathrm{d}t,\end{aligned} \tag{12}$$

同理

$$\int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_s}\delta\dot{\omega}_s\mathrm{d}t = \int_{t_0}^{t_1}\sum_{s=1}^{n}\left\{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\widetilde{L}}{\partial\dot{\omega}_s}\right) + \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\widetilde{L}}{\partial\dot{\omega}_l}\right)\omega_k\right\}\delta\pi_s\mathrm{d}t, \tag{13}$$

$$\int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\ddot{\omega}_s}\delta\ddot{\omega}_s\mathrm{d}t = -\int_{t_0}^{t_1}\sum_{s=1}^{n}\left\{\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left(\frac{\partial\widetilde{L}}{\partial\ddot{\omega}_s}\right) + \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\widetilde{L}}{\partial\ddot{\omega}_l}\right)\omega_k\right\}\delta\pi_s\mathrm{d}t, \tag{14}$$

……

$$\begin{aligned}\int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\delta\overset{(\delta-1)}{\omega_s}\mathrm{d}t &= \int_{t_0}^{t_1}\sum_{s=1}^{n}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\delta\overset{(\delta-2)}{\omega_s}\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right]\delta\overset{(\delta-2)}{\omega_s}\right\}\mathrm{d}t = \left[\sum_{s=1}^{n}\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\delta\overset{(\delta-2)}{\omega_s}\right]\Bigg|_{t_0}^{t_1} \\ &\quad - \int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right]\delta\overset{(\delta-2)}{\omega_s}\mathrm{d}t \\ &= -\int_{t_0}^{t_1}\sum_{s=1}^{n}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right]\delta\overset{(\delta-3)}{\omega_s}\right] - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right]\delta\overset{(\delta-3)}{\omega_s}\right\}\mathrm{d}t \\ &= \int_{t_0}^{t_1}\sum_{s=1}^{n}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right]\delta\overset{(\delta-3)}{\omega_s}\mathrm{d}t = \cdots \\ &= (-1)^{\delta}\int_{t_0}^{t_1}\sum_{s=1}^{n}\left[\frac{\mathrm{d}^{\delta}}{\mathrm{d}t^{\delta}}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}}\right] + \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\frac{\mathrm{d}^{\delta-1}}{\mathrm{d}t^{\delta-1}}\left[\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_l}}\right]\omega_k\right]\delta\pi_s\mathrm{d}t,\end{aligned} \tag{15}$$

将(11)—(15)式代入(10)式中，得

$$\begin{aligned}\delta\int_{t_0}^{t_1}\widetilde{L}\,\mathrm{d}t = \int_{t_0}^{t_1}\sum_{s=1}^{n}\Bigg\{&\frac{\partial\widetilde{L}}{\partial\pi_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\omega_s} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_s} - \frac{\mathrm{d}^3}{\mathrm{d}t^3}\frac{\partial\widetilde{L}}{\partial\ddot{\omega}_s} + \cdots + (-1)^{\delta}\frac{\mathrm{d}^{\delta}}{\mathrm{d}t^{\delta}}\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}} \\ &- \sum_{l=1}^{n}\sum_{k=1}^{n}\gamma_{ks}^{l}\left[\frac{\partial\widetilde{L}}{\partial\omega_l} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_l} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\widetilde{L}}{\partial\ddot{\omega}_l} + \cdots + (-1)^{(\delta-1)}\frac{\mathrm{d}^{\delta-1}}{\mathrm{d}t^{\delta-1}}\frac{\partial\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_l}}\right]\omega_k\Bigg\}\delta\pi_s\mathrm{d}t,\end{aligned} \tag{16}$$

再将(16)式代入 Hamilton 原理(7)式，有

$$\int_{t_0}^{t_1}\sum_{s=1}^{n}\left\{\frac{\partial\widetilde{L}}{\partial\pi_s} + \sum_{\alpha=1}^{\delta}(-1)^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\partial\widetilde{L}}{\partial\overset{(\alpha-1)}{\omega_s}}\right.$$

$$-\sum_{l=1}^{n}\sum_{k=1}^{n}\left[(-1)^{\alpha}\sum_{\alpha=0}^{\delta-1}\gamma_{ks}^{l}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\partial\widetilde{L}}{\partial\overset{(\alpha)}{\omega_l}}\omega_k\right]$$

$$+\widetilde{Q}_s\Big\}\delta\pi_s\mathrm{d}t=0, \tag{17}$$

由于时间 t_0,t_1 是任意的,故由上式可得

$$\frac{\partial\widetilde{L}}{\partial\pi_s}+\sum_{\alpha=1}^{\delta}(-1)^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\partial\widetilde{L}}{\partial\overset{(\alpha-1)}{\omega_s}}=\widetilde{R}_s, \tag{18}$$

此处

$$\widetilde{R}_s=-\widetilde{Q}_s+\sum_{l=1}^{n}\sum_{k=1}^{n}(-1)^{\alpha}\sum_{\alpha=0}^{\delta-1}\gamma_{ks}^{l}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\partial\widetilde{L}}{\partial\overset{(\alpha)}{\omega_l}}\omega_k, \tag{19}$$

Boltzmann 三标记符号

$$\gamma_{ks}^{l}=\sum_{m=1}^{n}\sum_{r=1}^{n}\left(\frac{\partial a_{lm}}{\partial q_r}-\frac{\partial a_{lr}}{\partial q_m}\right)b_{rk}b_{ms}, \tag{20}$$

设方程(18)是非奇异的,即

$$\det\left(\frac{\partial^2\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_k}\partial\overset{(\delta-1)}{\omega_s}}\right)\neq 0,$$

则由方程(18)得到准坐标下广义力学系统运动方程的显式

$$\overset{(2\delta-1)}{\omega_s}=g_s(q,\omega,\dot{\omega},\ddot{\omega},\cdots,\overset{(2\delta-2)}{\omega},t). \tag{21}$$

定义准坐标下广义力学系统的广义动量为

$$\widetilde{p}_{s\alpha}=\sum_{j=0}^{\delta-\alpha}(-1)^{j}\frac{\mathrm{d}^{j}}{\mathrm{d}t^{j}}\left(\frac{\partial\widetilde{L}}{\partial\overset{(j+\alpha-1)}{\omega_s}}\right)$$
$$(\alpha=1,2,\cdots,\delta). \tag{22}$$

3 无限小变换与生成元

引入时间和准坐标的无限小群变换

$$\widetilde{t}=t+\Delta t,$$
$$\widetilde{\pi}_s(\widetilde{t})=\pi_s(t)+\Delta\pi_s\quad(s=1,2,\cdots,n), \tag{23}$$

在一次近似的情况下,其展开式为

$$\widetilde{t}=t+\varepsilon\xi_0(q,\omega,\dot{\omega},\ddot{\omega},\cdots,\overset{(\delta-1)}{\omega},t),$$
$$\widetilde{\pi}_s=\pi_s+\varepsilon\xi_s(q,\omega,\dot{\omega},\ddot{\omega},\cdots,\overset{(\delta-1)}{\omega},t),$$
$$(s=1,2,\cdots,n). \tag{24}$$

注意到,(23)式中 $\pi_s,\widetilde{\pi}_s$ 只是一种记号,而 $\Delta\pi_s$ 是有意义的,这里 ε 为小参数,ξ_0,ξ_s 为无限小单参数群变换的生成元. 引入无限小变换生成元向量

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\sum_{s=1}^{n}\xi_s\frac{\partial}{\partial\pi_s}, \tag{25}$$

它的一次扩展

$$X^{(1)}=\xi_0\frac{\partial}{\partial t}+\sum_{s=1}^{n}\xi_s\frac{\partial}{\partial\pi_s}+\sum_{s=1}^{n}(\dot{\xi}_s-\omega_s\dot{\xi}_0)\frac{\partial}{\partial\omega_s}$$
$$=X^{(0)}+\sum_{s=1}^{n}\eta^{(1)}\frac{\partial}{\partial\omega_s}, \tag{26}$$

其中　$\eta^{(1)}=\dot{\xi}_s-\omega_s\dot{\xi}_0.$

二次扩展

$$X^{(2)}=X^{(1)}+\sum_{s=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\eta^{(1)}-\dot{\xi}_0\dot{\omega}_s\right]\frac{\partial}{\partial\dot{\omega}_s}$$
$$=X^{(1)}+\sum_{s=1}^{n}\eta^{(2)}\frac{\partial}{\partial\dot{\omega}_s}, \tag{27}$$
$$\eta^{(2)}=\frac{\mathrm{d}}{\mathrm{d}t}\eta^{(1)}-\dot{\xi}_0\dot{\omega}_s.$$

三次扩展

$$X^{(3)}=X^{(2)}+\sum_{s=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\eta^{(2)}-\dot{\xi}_0\ddot{\omega}_s\right]\frac{\partial}{\partial\ddot{\omega}_s}$$
$$=X^{(2)}+\sum_{s=1}^{n}\eta^{(3)}\frac{\partial}{\partial\ddot{\omega}_s}, \tag{28}$$
$$\eta^{(3)}=\frac{\mathrm{d}}{\mathrm{d}t}\eta^{(2)}-\dot{\xi}_0\ddot{\omega}_s,$$

……

k 次扩展

$$X^{(k)}=X^{(k-1)}+\sum_{s=1}^{n}\eta^{(k)}\frac{\partial}{\partial\overset{(k-1)}{\omega_s}},$$
$$\eta^{(k)}=\frac{\mathrm{d}}{\mathrm{d}t}\eta^{k-1}-\dot{\xi}_0\overset{(k-1)}{\omega_s} \tag{29}$$

或

$$X^{(k)}=X^{(0)}+\sum_{s=1}^{n}\eta^{(1)}\frac{\partial}{\partial\omega_s}+\sum_{s=1}^{n}\eta^{(2)}\frac{\partial}{\partial\dot{\omega}_s}$$
$$+\cdots+\sum_{s=1}^{n}\eta^{(k)}\frac{\partial}{\partial\overset{(k-1)}{\omega_s}}$$
$$=X^{(0)}+\sum_{s=1}^{n}\sum_{i=1}^{k}\eta^{(i)}\frac{\partial}{\partial\overset{(i-1)}{\omega_s}}. \tag{30}$$

为了与方程(21)相结合,令 $k=2\delta$,于是(29)式可写为

$$X^{(2\delta)}=X^{(2\delta-1)}+\sum_{s=1}^{n}\eta^{(2\delta)}\frac{\partial}{\partial\overset{(2\delta-1)}{\omega_s}}. \tag{31}$$

根据常微分方程不变性的判据,在变换(24)下运动微分方程(21)不变性由下式表示,

$$X^{(2\delta)}[\overset{(2\delta-1)}{\omega_s}-g_s(q,\omega,\dot{\omega},\ddot{\omega},\cdots,\overset{(2\delta-2)}{\omega},t)]=0$$
$$(s=1,2,\cdots,n), \tag{32}$$

其中

$$\overset{(2\delta-1)}{\omega_s} = g_s(q, \omega, \dot{\omega}, \ddot{\omega}, \cdots, \overset{(2\delta-2)}{\omega}, t).$$

展开(32)式,得

$$X^{(2\delta)}\left[\overset{(2\delta-1)}{\omega_s} - g_s(q, \omega, \dot{\omega}, \ddot{\omega}, \cdots, \overset{(2\delta-2)}{\omega}, t)\right]$$

$$= - X^{(2\delta-1)}(g_s) + \left[\sum_{s=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\eta^{(2\delta-1)} - \dot{\xi}_0 \overset{(2\delta-1)}{\omega_s}\right]\Bigg|_{g_s = \overset{(2\delta-1)}{\omega_s}} = 0. \qquad (33)$$

由上可得

$$\left[\sum_{s=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\eta^{(2\delta-1)} - \dot{\xi}_0 \overset{(2\delta-1)}{\omega_s}\right]\Bigg|_{g_s = \overset{(2\delta-1)}{\omega_s}} = X^{(2\delta-1)}(g_s), \qquad (34)$$

当 $\delta = 1$ 时,有 $X^{(1)}(g_s) = \left(\sum_{s=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\eta^{(1)} - \dot{\xi}_0 \dot{\omega}_s\right)\Big|_{g_s = \dot{\omega}_s}$

称(34)式为 Lie 对称变换的确定方程.

定义 如果无限小变换(24)的生成元 ξ_0, ξ_s 满足确定方程(34),则称相应变换为 Lie 对称变换.

4 结构方程与守恒量

定理 1 对于满足确定方程(34)的无限小生成元 ξ_0, ξ_s,如果存在规范函数

$$\lambda = \lambda(q, \omega, \dot{\omega}, \ddot{\omega}, \cdots, \overset{(\delta-1)}{\omega}, t)$$

满足结构方程

$$X^{(\delta)}(\widetilde{L}) + \widetilde{L}\,\dot{\xi}_0 + \sum_{s=1}^{n} \widetilde{R}_s \cdot (\xi_s - \omega_s\xi_0) + \dot{\lambda} = 0, \qquad (35)$$

则准坐标下广义完整力学系统的运动方程存在如下 Lie 对称性守恒量:

$$I = \widetilde{L}\,\xi_0 + \sum_{s=1}^{n}\sum_{\alpha=1}^{\delta} \widetilde{p}_{s/\alpha} \frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}(\xi_s - \omega_s\xi_0) + \lambda = \mathrm{const}. \qquad (36)$$

证明

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \widetilde{L}\,\dot{\xi}_0 + \dot{\widetilde{L}}\,\xi_0 + \sum_{s=1}^{n}\sum_{\alpha=1}^{\delta} \widetilde{p}_{s/\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}(\xi_s - \omega_s\xi_0)$$

$$+ \sum_{s=1}^{n}\sum_{\alpha=1}^{\delta} \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{p}_{s/\alpha}\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}(\xi_s - \omega_s\xi_0)$$

$$- X^{(\delta)}(\widetilde{L}) - \widetilde{L}\,\dot{\xi}_0 - \sum_{s=1}^{n} \widetilde{R}_s(\xi_s - \omega_s\xi_0)$$

$$= \sum_{s=1}^{n}(\xi_s - \omega_s\xi_0)\left[\frac{\partial\widetilde{L}}{\partial\pi_s} + \sum_{\alpha=1}^{\delta}(-1)^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\frac{\partial\widetilde{L}}{\partial\overset{(\alpha-1)}{\omega_s}} + \widetilde{R}_s\right] = 0, \qquad (37)$$

Lie 对称正问题的解法如下:首先,对给定广义力学系统建立确定方程(34),并求出生成元 ξ_0, ξ_s. 将生成元代入结构方程(35)求得 $\dot{\lambda}$. 如果 $\dot{\lambda}$ 为零或为某函数的全导数,则可求出规范函数 λ. 最后将所得的 ξ_0, ξ_s 和 λ 代入(36)式可求得 Lie 对称的守恒量.

5 Lie 对称性逆问题

所谓逆问题,就是由已知守恒量来求相应的 Lie 对称性.

假设已知广义力学系统(18)有初积分

$$I = I(q_s, \omega_s, \dot{\omega}_s, \ddot{\omega}_s, \cdots, \overset{(2\delta-2)}{\omega_s}, t) = \mathrm{const}, \qquad (38)$$

于是,有

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \sum_{s=1}^{n}\frac{\partial I}{\partial\pi_s}\omega_s + \sum_{s=1}^{n}\frac{\partial I}{\partial\omega_s}\dot{\omega}_s + \cdots + \sum_{s=1}^{n}\frac{\partial I}{\partial\overset{(2\delta-2)}{\omega_s}}\overset{(2\delta-1)}{\omega_s} = 0, \qquad (39)$$

将系统的运动微分方程(18)的两端乘以 $\bar{\xi}_s = \xi_s - \omega_s\xi_0$,并对 s 求和,得

$$\sum_{s=1}^{n}\left[\frac{\partial\widetilde{L}}{\partial\pi_s} + \sum_{\alpha=1}^{\delta}(-1)^{\alpha}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left(\frac{\partial\widetilde{L}}{\partial\overset{(\alpha-1)}{\omega_s}}\right) - \widetilde{R}_s\right]\bar{\xi}_s = 0, \qquad (40)$$

将(39)与(40)式相加,分出含 $\overset{(2\delta-1)}{\omega_k}$ 的项,令其系数为零,得到

$$\sum_{s=1}^{n}\frac{\partial^2\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}\,\partial\overset{(\delta-1)}{\omega_k}}\bar{\xi}_s - \frac{\partial I}{\partial\overset{(2\delta-2)}{\omega_k}} = 0 \quad (k = 1, 2, \cdots, n), \qquad (41)$$

由此解得

$$\bar{\xi}_s = \sum_{k=1}^{n}\widetilde{h}_{sk}\frac{\partial I}{\partial\overset{(2\delta-2)}{\omega_k}}, \qquad (42)$$

其中 $\widetilde{h}_{sk}$ 由下式确定

$$\sum_{k=1}^{n}\widetilde{h}_{sk}h_{kr} = \delta_{sr},$$

$$h_{sk} = \frac{\partial^2\widetilde{L}}{\partial\overset{(\delta-1)}{\omega_s}\,\partial\overset{(\delta-1)}{\omega_k}}, \qquad (43)$$

为使(24)式为 Lie 对称变换,现令

$$I = \widetilde{L}\,\xi_0 + \sum_{s=1}^{n}\sum_{\alpha=1}^{\delta}\widetilde{p}_{s/\alpha}\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}(\xi_s - \omega_s\xi_0) + \lambda, \qquad (44)$$

则有

$$\xi_0 = \frac{1}{\widetilde{L}}\left[I - \sum_{s=1}^{n}\sum_{\alpha=1}^{\delta}\widetilde{p}_{s/\alpha}\frac{\mathrm{d}^{\alpha-1}}{\mathrm{d}t^{\alpha-1}}(\xi_s - \omega_s\xi_0) - \lambda\right]. \tag{45}$$

定理 2　如果由(42)和(45)式确定的无限小生成元 ξ_0，ξ_s 满足确定方程(34)，则无限小变换(24)是 Lie 对称的.

6　举　　列

已知广义力学系统的 Lagrange 函数

$$L = \frac{1}{2}\dot{q}_1^{\,2} + \frac{1}{2}q_1^{\,2}\dot{q}_2^{\,2} + \frac{1}{2}\ddot{q}_1^{\,2}, \tag{46}$$

并设非势广义力为零. 试研究系统的对称性与守恒量.

解　首先研究 Lie 对称性正问题.

第一步 列写准坐标下系统的运动微分方程.

选取准速度

$$\omega_1 = \dot{q}_1,\ \omega_2 = \frac{1}{2}q_1^{\,2}\dot{q}_2, \tag{47}$$

则有

$$\dot{q}_1 = \omega_1,\ \dot{q}_2 = \frac{2\omega_2}{q_1^{\,2}}, \tag{48}$$

于是

$$a_{11} = 1,\ a_{12} = 0,\ a_{21} = 0,\ a_{22} = \frac{q_1^{\,2}}{2}, \tag{49}$$

$$b_{11} = 1,\ b_{12} = 0,\ b_{21} = 0,\ b_{22} = \frac{2}{q_1^{\,2}}. \tag{50}$$

计算 Boltzmann 三标记号，得

$$\gamma_{12}^{\,2} = -\gamma_{21}^{\,2} = \frac{2}{q_1}, \tag{51}$$

其余为零. 又

$$\widetilde{L} = \frac{1}{2}\omega_1^{\,2} + \frac{2\omega_2^{\,2}}{q_1^{\,2}} + \frac{1}{2}\dot{\omega}_1^2, \tag{52}$$

$$\frac{\partial\widetilde{L}}{\partial\pi_1} = \sum_{s=1}^{2}\frac{\partial\widetilde{L}}{\partial q_s}b_{s1} = -\frac{4\omega_2^{\,2}}{q_1^{\,3}},$$

$$\frac{\partial\widetilde{L}}{\partial\pi_2} = \sum_{s=1}^{2}\frac{\partial\widetilde{L}}{\partial q_s}b_{s2} = 0, \tag{53}$$

$$\frac{\partial\widetilde{L}}{\partial\omega_1} = \omega_1,\ \frac{\partial\widetilde{L}}{\partial\omega_2} = \frac{4\omega_2}{q_1^{\,2}}, \tag{54}$$

$$\frac{\partial\widetilde{L}}{\partial\dot{\omega}_1} = \dot{\omega}_1,\ \frac{\partial\widetilde{L}}{\partial\dot{\omega}_2} = 0, \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\omega_1} = \ddot{\omega}_1,\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\omega_2} = \frac{4q_1\dot{\omega}_2 - 8\omega_1\omega_2}{q_1^{\,3}}, \tag{56}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\widetilde{L}}{\partial\dot{\omega}_1}\right) = \dddot{\omega}_1,\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial\widetilde{L}}{\partial\dot{\omega}_2}\right) = 0. \tag{57}$$

已知非势广义力为零，即

$$\widetilde{Q}_1 = \widetilde{Q}_2 = 0, \tag{58}$$

$$\widetilde{R}_1 = -\widetilde{Q}_1 + \sum_{l=1}^{2}\sum_{k=1}^{2}\gamma_{k1}^{l}\left(\frac{\partial\widetilde{L}}{\partial\omega_l} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_l}\right)\omega_k = -\frac{8\omega_2^2}{q_1^{\,3}},$$

$$\widetilde{R}_2 = -\widetilde{Q}_2 + \sum_{l=1}^{2}\sum_{k=1}^{2}\gamma_{k2}^{l}\left(\frac{\partial\widetilde{L}}{\partial\omega_l} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_l}\right)\omega_k = \frac{8\omega_1\omega_2}{q_1^{\,3}}. \tag{59}$$

方程(18)给出

$$\frac{\partial\widetilde{L}}{\partial\pi_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\omega_1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_1} = \sum_{l=1}^{2}\sum_{k=1}^{2}\gamma_{k1}^{l}\left(\frac{\partial\widetilde{L}}{\partial\omega_l} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_l}\right)\omega_k, \tag{60}$$

$$\frac{\partial\widetilde{L}}{\partial\pi_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\omega_2} = \sum_{l=1}^{2}\sum_{k=1}^{2}\gamma_{k2}^{l}\frac{\partial\widetilde{L}}{\partial\omega_l}\omega_k.$$

现将(53)—(59)式代入(60)式中，则得准坐标下系统的运动方程

$$\dddot{\omega}_1 = \dot{\omega}_1 - \frac{4\omega_2^2}{q_1^3},\ \dot{\omega}_2 = 0. \tag{61}$$

第二步：建立确定方程并求解.

对方程(61)，确定方程(34)给出

$$\overset{(4)}{\xi}_1 - 6\ddot{\xi}_0\dddot{\omega}_1 - 4\dot{\xi}_0\overset{(4)}{\omega}_1 - 4\dddot{\xi}_0\dot{\omega}_1 - \dot{\omega}_1\overset{(4)}{\xi}_0 = -(\dot{\xi}_2 - \omega_2\dot{\xi}_0)\frac{8\omega_2}{q_1^{\,3}} + \left[\ddot{\xi}_1 - \omega_1\ddot{\xi}_0 - 2\dot{\xi}_0\left(\dot{\omega}_1 - \frac{4\omega_2^{\,2}}{q_1^{\,3}}\right)\right],$$

$$\ddot{\xi}_2 - 2\dot{\omega}_2\dot{\xi}_0 - \omega_2\ddot{\xi}_0 = \ddot{\xi}_2 - \omega_2\ddot{\xi}_0 = 0. \tag{62}$$

取无限小变换生成元如下：

$$\xi_0 = -1,\ \xi_1 = 0,\ \xi_2 = 0, \tag{63}$$

它们满足确定方程(62)，所以，对应的变换是 Lie 对称的.

第三步 建立结构方程，并求守恒量.

结构方程(35)给出

$$X^{(2)}(\widetilde{L}) + \widetilde{L}\ \dot{\xi}_0 + \sum_{s=1}^{2}\widetilde{R}_s(\xi_s - \omega_s\xi_0) + \dot{\lambda} = 0, \tag{64}$$

由于

$$X^{(2)}(\widetilde{L}) = (\dot{\xi}_1 - \omega_1\dot{\xi}_0)\omega_1 + (\dot{\xi}_2 - \omega_2\dot{\xi}_0)\frac{4\omega_2}{q_1^{\ 2}} + (\ddot{\xi}_1 - 2\dot{\omega}_1\dot{\xi}_0 - \omega_1\ddot{\xi}_0)\dot{\omega}_1, \tag{65}$$

$$\begin{aligned}&\sum_{s=1}^{2}\widetilde{R}_s(\xi_s - \omega_s\xi_0)\\ &= R_1(\xi_1 - \omega_1\xi_0) + R_2(\xi_2 - \omega_2\xi_0)\\ &= -\frac{8\omega_2^2}{q_1^{\ 3}}(\xi_1 - \omega_1\xi_0) + \frac{8\omega_2\omega_1}{q_1^{\ 3}}(\xi_2 - \omega_2\xi_0),\end{aligned} \tag{66}$$

将(63),(65)和(66)代入(64)式,求得结构方程为

$$\begin{aligned}&(\dot{\xi}_1 - \omega_1\dot{\xi}_0)\omega_1 + (\dot{\xi}_2 - \omega_2\dot{\xi}_0)\frac{4\omega_2}{q_1^{\ 2}}\\ &+ (\ddot{\xi}_1 - 2\dot{\omega}_1\dot{\xi}_0 - \omega_1\ddot{\xi}_0)\dot{\omega}_1 + \widetilde{L}\ \dot{\xi}_0\\ &- \frac{8\omega_2^{\ 2}}{q_1^{\ 3}}(\xi_1 - \omega_1\xi_0) + \frac{8\omega_2\omega_1}{q_1^{\ 3}}\\ &\cdot(\xi_2 - \omega_2\xi_0) + \dot{\lambda} = 0,\end{aligned} \tag{67}$$

对于生成元(63),我们由(67)式得到规范函数导数

$$\dot{\lambda} = 0, \text{取 } \lambda = 0, \tag{68}$$

再利用(22)式求系统的广义动量

$$\begin{aligned}\widetilde{p}_{1/1} &= \frac{\partial\widetilde{L}}{\partial\omega_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\widetilde{L}}{\partial\dot{\omega}_1} = \omega_1 - \ddot{\omega}_1,\\ \widetilde{p}_{1/2} &= \frac{\partial\widetilde{L}}{\partial\dot{\omega}_1} = \dot{\omega}_1,\\ \widetilde{p}_{2/1} &= \frac{\partial\widetilde{L}}{\partial\omega_2} = \frac{4\omega_2}{q_1^{\ 2}}.\end{aligned} \tag{69}$$

(36)式给出守恒量

$$I = \widetilde{L}\ \xi_0 + \widetilde{p}_{1/1}(\xi_1 - \omega_1\xi_0) + \widetilde{p}_{1/2}\frac{\mathrm{d}}{\mathrm{d}t}(\xi_1 - \omega_1\xi_0) + \widetilde{p}_{2/1}(\xi_2 - \omega_2\xi_0) + \lambda = \text{const}, \tag{70}$$

将(63),(68)和(69)式同时代入(70)式,得到系统运动方程的守恒量为

$$I = \frac{1}{2}\omega_1^{\ 2} + \frac{1}{2}\dot{\omega}_1^{\ 2} - \omega_1\ddot{\omega}_1 + \frac{2\omega_2^{\ 2}}{q_1^{\ 2}} = \text{const}, \tag{71}$$

其次,研究 Lie 对称性逆问题,假设系统有积分

$$I = \frac{1}{2}\omega_1^{\ 2} + \frac{1}{2}\dot{\omega}_1^{\ 2} - \omega_1\ddot{\omega}_1 + \frac{2\omega_2^{\ 2}}{q_1^{\ 2}} = \text{const},$$

求与其相应的 Lie 对称性,由(43)式得

$$h_{11} = 1, h_{22} = \frac{4}{q_1^{\ 2}}, h_{12} = h_{21} = 0. \tag{72}$$

故有 $\widetilde{h}_{11} = 1, \widetilde{h}_{22} = \frac{q_1^{\ 2}}{4}$,其余为零. (73)

利用(42)式,求得

$$\begin{aligned}\bar{\xi}_1 &= \widetilde{h}_{11}\frac{\partial I}{\partial \omega_1^{(2\delta-2)}} + \widetilde{h}_{12}\frac{\partial I}{\partial \omega_2^{(2\delta-2)}} = \omega_1,\\ \bar{\xi}_2 &= \widetilde{h}_{21}\frac{\partial I}{\partial \omega_1^{(2\delta-2)}} + \widetilde{h}_{22}\frac{\partial I}{\partial \omega_2^{(2\delta-2)}} = \omega_2,\end{aligned} \tag{74}$$

亦即

$$\bar{\xi}_1 = \omega_1 = \xi_1 - \omega_1\xi_0, \bar{\xi}_2 = \omega_2 = \xi_2 - \omega_2\xi_0, \tag{75}$$

又由(45)式,得

$$\begin{aligned}\xi_0 &= \frac{1}{\frac{1}{2}\omega_1^{\ 2} + \frac{1}{2}\dot{\omega}_1^{\ 2} + \frac{2\omega_2^{\ 2}}{q_1^{\ 2}}}\Big[\frac{1}{2}\omega_1^{\ 2} + \frac{1}{2}\dot{\omega}_1^{\ 2} - \omega_1\ddot{\omega}_1\\ &\quad + \frac{2\omega_2^{\ 2}}{q_1^{\ 2}} - (\omega_1 - \ddot{\omega}_1)\omega_1 - \dot{\omega}_1^{\ 2} - \frac{4\omega_2^{\ 2}}{q_1^{\ 2}}\Big] = -1,\end{aligned} \tag{76}$$

考虑到(75)和(76)式,我们得到无限小变换的生成元

$$\xi_0 = -1, \xi_1 = 0, \xi_2 = 0.$$

由于生成元满足确定方程(34),因此相应的无限小变换

$$\widetilde{t} = t - \varepsilon, \widetilde{\pi}_1 = \pi_1, \widetilde{\pi}_2 = \pi_2 \tag{77}$$

是 Lie 对称的.

[1] M. Lutzky, *J. Phys A, Math Gen.*, **12**(1979), 973.

[2] Y. Y. Zhao, F. X. Mei, *Advances in Mechanics*, **21**(1991), 83 (in Chinese)[赵跃宇、梅凤翔,力学进展,**21**(1991),83].

[3] Y. C. Li, J. H. Liang, *J. Shangqiu Teachers College*, **15**(1999), 21 (in Chinese)[李元成、梁景辉,商丘师专学报,**15**(1999),21].

[4] J. L. Fu, R. W. Liu, F. X, Mei, *J. BIT*, **7**(1998), 215 (in Chinese)[傅景礼、刘荣万、梅凤翔,北京理工大学学报,**7**(1998),215].

[5] Y. F. Qiao, *Acta Mechanics Sinica*, **22**(1990), 99 (in Chinese)[乔永芬,力学学报,**22**(1990),99].

[6] F. X. Mei, Foundations of mechanics of nonholonomic systems (Beijing Institute of Technology Press, Beijing, 1985), p. 50 (in Chinese) [梅凤翔,非完整力学基础(北京工业学院出版社,北京,1985), p. 50].

LIE SYMMETRIES THEOREM AND ITS INVERSE OF GENERALIZED MECHANICAL SYSTEMS IN TERMS OF QUASI-COORDINATES*

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ABSTRACT

In this paper, Lie symmetries and conserved quantities of generalized mechanical systems in terms of quasi-coordinates were studied. First, the definition of an infinitesimal generator for the generalized mechanical systems in terms of quasi-coordinates were given, then the determining equations of the Lie symmetries were established by using Lie's method of invariance of ordinary differential equations under infinitesimal transformations. Next, the structure equation and the form of conserved quantities were obtained. Finally, the inverse problem of Lie symmetries systems were discussed and an example to illustrate the application of the result was given.

Keywords: generalized mechanics, quasi-coordinates, Lie symmetries, determining equation, structure equation, conserved quantity
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