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Poincaré-Chetaev 变量下变质量非完整动力学系统的运动方程

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研究 Poincaré-Chetaev 变量下,变质量非线性非完整力学系统的运动方程.首先,由变质量力学系统的 D'Alembert-Lagrange 原理导出 Chaplygin 型方程、Nielsen 型方程及 Appell 型方程.其次,研究 Chaplygin 方程与 Appell 方程的等价性问题.最后,举例说明新结果的应用.

关键词:Poincaré-Chetaev 变量,变质量,非完整系统,D'Alembert-Lagrange 原理

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1 引 言

1894 年,德国学者 Hertz 第一次把约束和系统分成完整和非完整两大类,从此开辟了非完整系统力学的新领域.非完整系统力学是指带有不可积分的微分约束系统.它是分析力学的一个重要分支.近 20 年来非完整系统动力学的研究取得重要进展[1—4].

Poincaré 1901 年利用无限小变换的 Lie 可迁群建立了完整力学系统的一类新型运动微分方程[5].Chetaev 于 20 年代将这种思想发展到变换群为非可迁的、约束是非定常的、变量是不独立的情形[6].两位学者建立的方程称为 Poincaré-Chetaev 方程.Hamilton 系统的近代理论推广了经典理论,所用的重要方法之一便是采用非正则坐标,此时运动方程与正则变量下的方程相比较反而特别简单.在这种意义下,对 Hamilton 系统的近代理论来说,Poincaré-Chetaev 方程理论是非常有发展前途的.近代对这类方程的研究已取得重要成果[7—9].但内容局限于常质量系统范围.

随着科学技术的发展,变质量力学理论在喷气、火箭及航天等领域得到广泛应用,因此人们研究变质量系统动力学的兴趣大大提高了,已建立了广义坐标和准坐标表示下的各类运动方程[10—12].本文给出 Poincaré-Chetaev 变量下,变质量非完整系统动力学的 Chaplygin 型方程、Nielsen 型方程及 Appell 型方程,并讨论 Chaplygin 方程与 Appell 方程的等价性问题,最后举例说明新结果的应用.

假设由 N 个质点组成的力学系统,在任意时刻 t 的位置由 Poincaré-Chetaev 变量 $x_1, x_2, \cdots, x_n$ 确定.该系统受有 $n-m$ 个完整约束

$$A_{\sigma s}(x_1, x_2, \cdots, x_n, t)\dot{x}_s + A_\sigma(x_1, x_2, \cdots, x_n, t) = 0, \tag{1}$$

$$(\sigma = m+1, m+2, \cdots, n; s = 1,2,\cdots,n)$$

及 $m-l$ 个 Chetaev 型非线性非完整约束

$$F_\alpha(x_1, x_2, \cdots, x_n; \dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n, t) = 0,$$
$$(\alpha = l+1, \cdots, m). \tag{2}$$

这里及以后重复指标表示求和.

系统具有 l 个自由度.

假设可能位移 $\delta x_1, \delta x_2, \cdots, \delta x_n$ 满足下面关系式:

$$\frac{\partial F_\alpha}{\partial \dot{x}_s}\delta x_s = 0$$

$$(\alpha = l+1, l+2, \cdots, m; s = 1,2,\cdots,n). \tag{3}$$

假设相伴完整系统(暂不考虑非完整约束(2))的实位移参数为 $\eta_1, \eta_2, \cdots, \eta_m$ 和可能位移参数 $\omega_1, \omega_2, \cdots, \omega_m$.其次,定义任意函数 $f(x_1, x_2, \cdots, x_n, t)$ 的实位移 $\mathrm{d}f$ 和可能位移 δf 依此为[8]

$$\mathrm{d}f = [X_0(f) + \eta_p X_p(f)]\mathrm{d}t \quad (p = 1,2,\cdots,m), \tag{4a}$$

$$\delta f = \omega_p X_p(f). \tag{4b}$$

无穷小线性算子的封闭组

$$X_0 = \frac{\partial}{\partial t} + \xi_0^s \frac{\partial}{\partial x_s},$$

$$X_p = \xi_p^s \frac{\partial}{\partial x_s} \quad (p = 1,2,\cdots,m;s = 1,2,\cdots,n). \tag{5}$$

其中 ξ_0^s 及 ξ_p^s 是 $x_1,x_2,\cdots,x_n$ 和 t 的函数.

由于算子构成封闭组,有

$$(X_0,X_p) = C_{0pq}X_q, \tag{6a}$$

$$(X_p,X_q) = C_{pqr}X_r, \quad (p,q,r = 1,2,\cdots,m). \tag{6b}$$

此处 C_{0pq} 和 C_{pqr} 是 $x_1,x_2,\cdots,x_n$ 和 t 的函数.取

$$f(x_1,x_2,\cdots,x_n,t) = x,$$

则关系(4a),(4b)式成为

$$\dot{x}_s = \xi_0^s + \eta_p\xi_p^s, \tag{7a}$$

及

$$\delta x_s = \omega_p\xi_p^s. \tag{7b}$$

利用 (7a),(7b)式约束(2),(3)式可写为

$$\eta_\alpha = \eta_\alpha(x_1,x_2,\cdots,x_n;\eta_1,\eta_2,,\eta_l,t),$$
$$(\alpha = l+1,\cdots,m), \tag{8}$$

$$\omega_\alpha = C_{\alpha i}\omega_i,(i = 1,2,\cdots,l), \tag{9}$$

其中 $C_{\alpha i} = \partial\eta_\alpha/\partial\eta_i$.

根据(8)和(9)式,消去(4a)和(4b)式中不独立的 η_α 和 ω_α,则有

$$df = [Y_0(f) + \eta_i Y_i(f)]dt,(i = 1,2,\cdots,l), \tag{10a}$$

$$\delta f = \omega_i Y_i(f). \tag{10b}$$

其中

$$Y_0 = X_0 + (\eta_\alpha - C_{\alpha i}\eta_i)X_\alpha, \tag{11a}$$

$$Y_i = X_i + C_{\alpha i}X_\alpha$$
$$(i = 1,2,\cdots,l;\alpha = l+1,l+2,\cdots,m), \tag{11b}$$

参数 $Y_0,Y_1,\cdots,Y_l$ 是非线性非完整系统的位移参数.

2 Chaplygin 型方程

设有 N 个质点组成的变质量力学系统,其运动受有约束(1)和(2)的限制.以 u_γ 表示任意质点在固定笛卡尔坐标系 xyz 中的三个坐标.

假设系统的质点上除受有力函数 $U(t,x)$ 的主动力外,还有非势力,它在固定笛卡尔坐标系 xyz 上的投影为 $F_{u_\gamma}(\gamma = 1,2,\cdots,3N)$.由(7a)式可得系统的动能为

$$T(t,x_1,x_2,\cdots,x_n,\eta_1,\eta_2,\cdots,\eta_l),$$

并引入广义 Lagrange 函数

$$L(t,x,\eta) = T(t,x,\eta) + U(t,x), \tag{12}$$

及广义非势力

$$Q_p(t,x,\eta) = F_{u_\gamma}X_p(u_\gamma),$$
$$(p = 1,2,\cdots,l;\gamma = 1,2,\cdots,3N), \tag{13}$$

反推力

$$R_p(t,x,\eta) = R_{u_\gamma}X_p(u_\gamma),R_{u_\gamma} = \dot{m}_\gamma v_\gamma^{(r)}. \tag{14}$$

任意点的质量 $m_\gamma = m_\gamma(t)$.

我们根据 Hamel 法则[8]

$$d\delta f = \delta df, \tag{15}$$

由变质量力学系统的 D'Alembert-Lagrange 原理[13]

$$\left[m_\gamma\ddot{u}_\gamma - F_{u\gamma} - R_{u\gamma} - \frac{\partial U}{\partial u_\gamma}\right]\delta u_\gamma = 0,$$
$$(\gamma = 1,2,\cdots,3N) \tag{16}$$

来导出 Chaplygin 型方程.

令 Д 为把质量当作常数时的偏导数记号,$\frac{\mathrm{D}}{\mathrm{D}t}$ 为把质量当作常数时对时间的导数.于是原理(16)可写为

$$\frac{D}{Dt}[m_\gamma\dot{u}_\gamma\delta u_\gamma] - m_\gamma\dot{u}_\gamma\frac{d}{dt}\delta u_\gamma$$
$$- F_{u_\gamma}\delta u_\gamma - R_{u_\gamma}\delta u_\gamma - \delta U = 0. \tag{17}$$

令 $f = u_\gamma$,由(4a)和(4b)式,得

$$\dot{u}_\gamma = X_0(u_\gamma) + \eta_p X_p(u_\gamma), \tag{18a}$$

$$\delta u_\gamma = \omega_p X_p(u_\gamma),(p = 1,2,\cdots,m). \tag{18b}$$

将(18b)代入(17)式,有

$$\frac{\mathrm{D}}{\mathrm{D}t}[\omega_p m_\gamma\dot{u}_\gamma X_p(u_\gamma)] - m_\gamma\dot{u}_\gamma\delta\dot{u}_\gamma$$
$$- \omega_p F_{u_\gamma}X_p(u_\gamma) - \omega_p R_{u_\gamma}X_p(u_\gamma) - \delta U = 0, \tag{19}$$

由(4a)和(4b)式, 得

$$d\delta f = d\omega_p X_p(f) + \omega_p[X_0X_p(f)$$
$$+ \eta_q X_q X_p(f)]dt, \tag{20}$$

$$\delta df = \omega_p X_p X_0(f)dt + \delta\eta_p X_p(f)dt$$
$$+ \eta_p\omega_q X_q X_p(f)dt, \tag{21}$$

将(20)和(21)式代入(15)式,有

$$\delta\eta_p X_p(f) = \frac{d\omega_p}{dt}X_p(f) + \omega_p(X_0,X_p)f$$
$$+ \eta_p\omega_q(X_p,X_q)f,(p,q,r = 1,2,\cdots,m).$$

根据(6a)式可化为

$$\delta\eta_p X_p(f) = \frac{d\omega_p}{dt}X_p(f) + \omega_p C_{0pq}X_q(f)$$
$$+ \eta_p\omega_q C_{pqr}X_r(f), \tag{22}$$

使(22)式中等式两边 $X_p(f)$ 的系数相等，得

$$\delta\eta_p = \frac{\mathrm{d}\omega_p}{\mathrm{d}t} + \omega_p C_{0pq} + \eta_q\omega_r C_{qrp}. \tag{23}$$

变质量系统的动能

$$T = \frac{1}{2}m_\gamma\dot{u}_\gamma^2, \tag{24a}$$

$$\frac{\partial T}{\partial\eta_p} = m_\gamma\dot{u}_\gamma\frac{\partial\dot{u}_\gamma}{\partial\eta_p}, \tag{24b}$$

$$\delta T = m_\gamma\dot{u}_\gamma\delta\dot{u}_\gamma. \tag{24c}$$

又由(18a)式,得

$$\frac{\partial\dot{u}_\gamma}{\partial\eta_p} = X_p(u_\gamma). \tag{25}$$

将(24b),(24c)和(25)式同时代入原理(19)，有

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\omega_p\frac{\text{Д}T}{\text{Д}\eta_p}\right] - \delta(T+U) - \omega_p Q_p - \omega_p R_p = 0. \tag{26}$$

(26)式就是在 Poincaré-Chetaev 变量下,变质量力学系统的一般方程.

现继续变换(26)式，它可写为

$$\frac{\mathrm{d}\omega_p}{\mathrm{d}t}\frac{\text{Д}T}{\text{Д}\eta_p} + \omega_p\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_p} - \frac{\text{Д}T}{\text{Д}\eta_p}\delta\eta_p - \omega_p X_p(T+U) - \omega_p Q_p - \omega_p R_p = 0, \tag{27}$$

将(23)代入(27)式，有

$$\frac{\mathrm{d}\omega_p}{\mathrm{d}t}\frac{\text{Д}T}{\text{Д}\eta_p} + \omega_p\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_p} - \frac{\text{Д}T}{\text{Д}\eta_p}\left[\frac{\mathrm{d}\omega_p}{\mathrm{d}t} + \omega_q C_{0qp} + \eta_q\omega_r C_{qrp}\right] - \omega_p X_p(T+U) - \omega_p Q_p - \omega_p R_p = 0, \tag{28}$$

或写为

$$\omega_p\left[\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_p} - C_{0pq}\frac{\text{Д}T}{\text{Д}\eta_q} - \eta_p C_{qpr}\frac{\text{Д}T}{\text{Д}\eta_r} - Q_p - R_p - X_p(T+U)\right] = 0,$$
$$(p, q, r = 1,2,\cdots,m). \tag{29}$$

利用(9)式，从(29)式中消去不独立的 ω_α,得

$$\left\{\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_i} - C_{0iq}\frac{\text{Д}T}{\text{Д}\eta_q} - \eta_q C_{qir}\frac{\text{Д}T}{\text{Д}\eta_r} - Q_i - R_i - X_i(T+U) + C_{\alpha i}\left[\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_\alpha} - C_{0\alpha q}\frac{\text{Д}T}{\text{Д}\eta_q} - \eta_q C_{q\alpha r}\frac{\text{Д}T}{\text{Д}\eta_r} - Q_\alpha - R_\alpha - X_\alpha(T+U)\right]\right\}\omega_i = 0, \tag{30}$$

由于 ω_i 是彼此独立的,于是，由(30)式有

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_i} - C_{0iq}\frac{\text{Д}T}{\text{Д}\eta_q} - \eta_q C_{qir}\frac{\text{Д}T}{\text{Д}\eta_r} - Q_i - R_i - X_i(T+U) + C_{\alpha i}\left[\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_\alpha} - C_{0\alpha q}\frac{\text{Д}T}{\text{Д}\eta_q} - \eta_q C_{q\alpha r}\frac{\text{Д}T}{\text{Д}\eta_r} - Q_\alpha - R_\alpha - X_\alpha(T+U)\right] = 0,$$
$$(i = 1,2,\cdots,l;\ \alpha = l+1, l+2,\cdots,m;\ q, r = 1,2,\cdots,m). \tag{31}$$

方程(31)就是在 Poincaré-Chetaev 变量下,变质量非完整系统动力学的 Chaplygin 型方程.

3 Nielsen 型方程

根据(4a)式,现将变质量系统的动能对时间 t 求导数，得

$$\frac{\mathrm{D}T}{\mathrm{D}t} = X_0(T) + \eta_p X_p(T) + \dot{\eta}_p\frac{\text{Д}T}{\text{Д}\eta_p}, \tag{32}$$

由文献[14],不难得到

$$\frac{\text{Д}}{\text{Д}\eta_p}\left(\frac{\mathrm{D}T}{\mathrm{D}t}\right) = X_0\left(\frac{\text{Д}T}{\text{Д}\eta_p}\right) + \eta_q X_q\left(\frac{\text{Д}T}{\text{Д}\eta_p}\right) + \dot{\eta}_q\frac{\text{Д}^2 T}{\text{Д}\eta_p\text{Д}\eta_q} + X_p(T), \tag{33}$$

而

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\text{Д}T}{\text{Д}\eta_p} = X_0\left(\frac{\text{Д}T}{\text{Д}\eta_p}\right) + \eta_q X_q\left(\frac{\text{Д}T}{\text{Д}\eta_p}\right) + \dot{\eta}_q\frac{\text{Д}^2 T}{\text{Д}\eta_p\text{Д}\eta_q}. \tag{34}$$

比较(33)与(34)式，有

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\text{Д}T}{\text{Д}\eta_p}\right) = \frac{\text{Д}}{\text{Д}\eta_p}\left(\frac{\mathrm{D}T}{\mathrm{D}t}\right) - X_p(t). \tag{35}$$

令

$$P_{0p} = C_{0pq}\frac{\text{Д}T}{\text{Д}\eta_q} + \eta_q C_{qpr}\frac{\text{Д}T}{\text{Д}\eta_r},$$
$$(p, q, r = 1,2,\cdots,m). \tag{36}$$

根据(11b)式，得

$$Y_i(U) = X_i(U) + C_{\alpha i}X_\alpha(U). \tag{37}$$

现将(35),(36)和(37)式同时代入 Chaplygin 方程(31),得到

$$\frac{\text{Д}}{\text{Д}\eta_i}\left(\frac{\mathrm{D}T}{\mathrm{D}t}\right) - 2X_i(T) - Q_i - R_i - P_{0i} + C_{\alpha i}\left[\frac{\text{Д}}{\text{Д}\eta_\alpha}\left(\frac{\mathrm{D}T}{\mathrm{D}t}\right) - 2X_\alpha(T) - Q_\alpha - R_\alpha - P_{0\alpha}\right] = Y_i(U). \tag{38}$$

方程(38)就是在 Poincaré-Chetaev 变量下,变质量非完整系统的 Nielsen 型方程.

4 Appell 型方程

令 $f = u_\gamma$,则由(10a)和(10b)式,得

$$\dot{u}_\gamma = Y_0(u_r) + \eta_i Y_i(u_\gamma), \tag{39a}$$

$$\delta u_\gamma = \omega_i Y_i(u_\gamma),(i=1,2,\cdots,l), \quad (39b)$$

于是，原理(16)可改写为

$$\left(m_\gamma \ddot{u}_\gamma - F_{u_\gamma} - R_{u_\gamma} - \frac{\partial U}{\partial u_\gamma}\right)\delta u_\gamma = \left(m_\gamma \ddot{u}_\gamma - F_{u_\gamma} - R_{u_\gamma} - \frac{\partial U}{\partial u_\gamma}\right)\omega_i Y_i(u_\gamma) = \left[m_\gamma \ddot{u}_\gamma Y_i(u_\gamma) - F_{u_\gamma} Y_i(u_\gamma) - R_{u_\gamma} Y_i(u_\gamma) - \frac{\partial U}{\partial u_\gamma} Y_i(u_\gamma)\right]\omega_i = 0,$$

$$(i=1,2,\cdots,l;\gamma=1,2,\cdots,3N). \quad (40)$$

由于 ω_i 是彼此独立的，(40)式成为

$$m_\gamma \ddot{u}_\gamma Y_i(u_\gamma) - F_{u_\gamma} Y_i(u_\gamma) - R_{u_\gamma} Y_i(u_\gamma) = Y_i(U),$$

$$(i=1,2,\cdots,l). \quad (41)$$

现将(39a)式对时间 t 求导数，有

$$\ddot{u}_\gamma = \ddot{\eta}_i Y_i(u_\gamma) + \text{不含 } \ddot{\eta} \text{ 的项}, \quad (42)$$

于是，得

$$\frac{\partial \ddot{u}_\gamma}{\partial \ddot{\eta}_i} = Y_i(u_\gamma), \quad (43)$$

将(43)代入(41)式，有

$$m_\gamma \ddot{u}_\gamma \frac{\partial \ddot{u}_\gamma}{\partial \ddot{\eta}_i} - F_{u_\gamma} Y_i(u_\gamma) - R_{u_\gamma} Y_i(u_\gamma) = Y_i(U). \quad (44)$$

变质量系统的加速度动能[11]

$$S = \frac{1}{2} m_\gamma \ddot{u}_\gamma^2, \quad (45)$$

令 $\tilde{S}$ 为 S 中通过(42)式消去不独立的 $\ddot{\eta}_{l+1},\ddot{\eta}_{l+2},\cdots,\ddot{\eta}_m$ 所得到的表达式. 因此 $\tilde{S}$ 是 $x_1 . x_2,\cdots,x_n,\ddot{\eta}_1,\ddot{\eta}_2,\cdots,\ddot{\eta}_l$ 和 t 的函数.

现在变换(44)式的等式两边各项. 由于

$$m_\gamma \ddot{u}_\gamma \frac{\partial \ddot{u}_\gamma}{\partial \ddot{\eta}_i} = \frac{\partial \tilde{S}}{\partial \ddot{\eta}_i} = \frac{\partial S}{\partial \ddot{\eta}_i} + C_{\alpha i} \frac{\partial S}{\partial \ddot{\eta}_\alpha}, \quad (46)$$

将(37)和(46)式代入(44)式，得

$$\frac{\partial S}{\partial \ddot{\eta}_i} + C_{\alpha i} \frac{\partial S}{\partial \ddot{\eta}_\alpha} - F_{u_\gamma}(X_i + C_{\alpha i} X_\alpha) - R_{u_\gamma}(X_i + C_{\alpha i} X_\alpha) = X_i(U) + C_{\alpha i} X_\alpha(U),$$

或写为

$$\frac{\partial S}{\partial \ddot{\eta}_i} - Q_i - R_i - X_i(U) + C_{\alpha i}\left(\frac{\partial S}{\partial \ddot{\eta}_\alpha} - Q_\alpha - R_\alpha - X_\alpha(U)\right) = 0,$$

$$(i=1,2,\cdots,l;\alpha=l+1,l+2,\cdots,m). \quad (47)$$

方程(47)就是在 Poincaré-Chetaev 变量下，变质量非完整系统的 Appell 型方程.

(47)式还可写为另一种形式，即

$$\frac{\partial \tilde{S}}{\partial \ddot{\eta}_i} = \tilde{Q}_i + \tilde{R}_i + Y_i(U). \quad (48)$$

此处 $\tilde{S} = \tilde{S}(x_1,x_2,\cdots,x_n,\ddot{\eta}_1,\ddot{\eta}_2,\cdots,\ddot{\eta}_l,t)$,

$\tilde{Q}_i = Q_i + C_{\alpha i} Q_\alpha$, $\tilde{R}_i = R_i + C_{\alpha i} R_\alpha$.

5 Chaplygin 型方程与 Appell 型方程等价

由 Appell 型方程(47)来推导 Chaplygin 型方程. 现将(18a)式对时间 t 求导数，有

$$\ddot{u}_\gamma = \ddot{\eta}_p X_p(u_\gamma) + \text{不含 } \ddot{\eta} \text{ 之项}, \quad (49)$$

由此可得

$$\frac{\partial \ddot{u}_\gamma}{\partial \ddot{\eta}_p} = \frac{\partial \dot{u}_\gamma}{\partial \eta_p} = X_p(u_\gamma). \quad (50)$$

根据(45)式，有

$$\frac{\partial S}{\partial \ddot{\eta}_p} = m_\gamma \ddot{u}_\gamma \frac{\partial \ddot{u}_\gamma}{\partial \ddot{\eta}_p} = m_\gamma \ddot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p} = \frac{\mathrm{D}}{\mathrm{D}t}\left(m_\gamma \dot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p}\right) - m_\gamma \dot{u}_\gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{u}_\gamma}{\partial \eta_p}\right) = \frac{\mathrm{D}}{\mathrm{D}t}\left(m_\gamma \dot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p}\right) - m_\gamma \dot{u}_\gamma \frac{\mathrm{d}}{\mathrm{d}t}[X_p(u_\gamma)],$$

利用(4a),(6)及(18a)式可简化为

$$\frac{\partial S}{\partial \ddot{\eta}_p} = \frac{\mathrm{D}}{\mathrm{D}t}\left(m_\gamma \dot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p}\right) - m_\gamma \dot{u}_\gamma\left(X_p(\dot{u}_\gamma) + C_{0pq}\frac{\partial \dot{u}_\gamma}{\partial \eta_p} + \eta_q C_{qpr}\frac{\partial \dot{u}_\gamma}{\partial \eta_r}\right). \quad (51)$$

由于

$$\frac{\text{д} T}{\text{д} \eta_p} = m_\gamma \dot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p}, \quad (52)$$

将(52)代入(51)式，得

$$\frac{\partial S}{\partial \ddot{\eta}_p} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\text{д} T}{\text{д} \eta_p}\right) - m_\gamma \dot{u}_\gamma X_p(\dot{u}_\gamma) - C_{0pq}\frac{\text{д} T}{\text{д} \eta_q} - \eta_q C_{qpr}\frac{\text{д} T}{\text{д} \eta_r}, \quad (53)$$

因为

$$m_\gamma \dot{u}_\gamma X_p(\dot{u}_\gamma) = m_\gamma \dot{u}_\gamma \frac{\partial \dot{u}_\gamma}{\partial \eta_p} = X_p(T). \quad (54)$$

利用(54)式，可将(53)式改写为

$$\frac{\partial S}{\partial \ddot{\eta}_p} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\text{д} T}{\text{д} \eta_p}\right) - X_p(T) - C_{0pq}\frac{\text{д} T}{\text{д} \eta_q} - \eta_q C_{qpr}\frac{\text{д} T}{\text{д} \eta_r}, \quad (55)$$

由此可有

$$\frac{\partial S}{\partial \dot{\eta}_i} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{д T}{д \eta_i}\right) - X_i(T) - C_{0iq}\frac{д T}{д \eta_q} - \eta_q C_{qir}\frac{д T}{д \eta_r}, \tag{56a}$$

$$\frac{\partial S}{\partial \dot{\eta}_\alpha} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{д T}{д \eta_\alpha}\right) - X_\alpha(T) - C_{0\alpha q}\frac{д T}{д \eta_q} - \eta_q C_{q\alpha r}\frac{д T}{д \eta_r}. \tag{56b}$$

现将(56a)和(56b)代入(47)式，即得 Chaplygin 型方程(31).因此在 Poincaré-Chetaev 变量下，变质量力学系统的 Appell 方程与 Chaplygin 方程等价.

6 举　　列

Appell-Hamel 列[13].已知质量为 $m = m(t)$ 的质点，其运动受有非完整约束

$$\dot{z} = a\sqrt{\dot{x}^2 + \dot{y}^2}$$

的限制.试建立系统的运动方程.

解　应用 Appell 方程(47)来求解此题.
此处该方程给出

$$\frac{\partial S}{\partial \dot{\eta}_1} = Q_1 - R_1 - X_1(U) + C_{31}\left[\frac{\partial S}{\partial \dot{\eta}_3} - Q_3 - R_3 - X_3(U)\right] = 0,$$

$$\frac{\partial S}{\partial \dot{\eta}_2} = Q_2 - R_2 - X_2(U) + C_{32}\left[\frac{\partial S}{\partial \dot{\eta}_3} - Q_3 - R_3 - X_3(U)\right] = 0, \tag{57}$$

选取 x, y, z 为 Poincaré-Chetaev 变量，由于缺少完整约束方程，选取相伴完整系统的实位移参数为

$$\eta_1 = \dot{x}, \eta_2 = \dot{y}, \eta_3 = \dot{z}. \tag{58}$$

所以位移算子为

$$X_0 = \frac{\partial}{\partial t},\quad X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial y},\quad X_3 = \frac{\partial}{\partial z}. \tag{59}$$

约束方程可改写为

$$\eta_3 = a\sqrt{\eta_1^2 + \eta_2^2}, \tag{60}$$

于是

$$C_{31} = \frac{\partial \eta_3}{\partial \eta_1} = \frac{a\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}},$$

$$C_{32} = \frac{\partial \eta_3}{\partial \eta_2} = \frac{a\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}}. \tag{61}$$

这里 $i = 1, 2, \alpha = 3$.

非完整系统的位移算子

$$Y_0 = X_0 = \frac{\partial}{\partial t},$$

$$Y_1 = X_1 + \frac{a\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}}X_3 = \frac{\partial}{\partial x} + \frac{a\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}}\frac{\partial}{\partial z},$$

$$Y_2 = X_2 + \frac{a\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}}X_3 = \frac{\partial}{\partial y} + \frac{a\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}}\frac{\partial}{\partial z}. \tag{62}$$

系统的加速度动能为

$$S = \frac{1}{2}m(t)(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) = \frac{1}{2}m(t)(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2), \tag{63a}$$

$$\frac{\partial S}{\partial \dot{\eta}_1} = m(t)\dot{\eta}_1,\quad \frac{\partial S}{\partial \dot{\eta}_2} = m(t)\dot{\eta}_2,\quad \frac{\partial S}{\partial \dot{\eta}_3} = m(t)\dot{\eta}_3. \tag{63b}$$

由于作用质点上的主动力仅有重力

$$U = -mgz,$$
$$X_1(U) = 0, X_2(U) = 0, X_3(U) = -mg, \tag{64}$$
$$Q_1 = Q_2 = Q_3 = 0.$$

反推力在 x, y, z 轴上的投影依次为

$$R_1 = \dot{m}v_1^{(r)}, R_2 = \dot{m}v_2^{(r)}, R_3 = \dot{m}v_3^{(r)}. \tag{65}$$

于是 Appell 方程(57)成为

$$m\dot{\eta}_1 - R_1 + \frac{a\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}}[m\dot{\eta}_3 - R_3 + mg] = 0,$$

$$m\dot{\eta}_2 - R_2 + \frac{a\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}}[m\dot{\eta}_3 - R_3 + mg] = 0. \tag{66}$$

利用(60)式简化上式，得

$$m(a^2 + 1)\dot{\eta}_1 - ma^2\eta_2\left[\frac{\dot{\eta}_1\eta_2 - \eta_1\dot{\eta}_2}{\eta_1^2 + \eta_2^2}\right] = -\frac{mga\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}} + \tilde{R}_1,$$

$$m(a^2 + 1)\dot{\eta}_2 - ma^2\eta_1\left[\frac{\dot{\eta}_1\eta_2 - \eta_1\dot{\eta}_2}{\eta_1^2 + \eta_2^2}\right] = -\frac{mga\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}} + \tilde{R}_2. \tag{67}$$

其中

$$\tilde{R}_1 = R_1 + \frac{a\eta_1}{\sqrt{\eta_1^2 + \eta_2^2}}R_3, \tilde{R}_2 = R_2 + \frac{a\eta_2}{\sqrt{\eta_1^2 + \eta_2^2}}R_3.$$

于是，应用代入法，由(67)式得

$$m(\dot{\eta}_1\eta_2-\dot{\eta}_2\eta_1)=\tilde{R}_1\eta_2-\tilde{R}_2\eta_1,$$

$$m(\dot{\eta}_1\eta_1+\dot{\eta}_2\eta_2)=-\frac{mag\sqrt{\eta_1^2+\eta_2^2}}{a^2+1}+\frac{\tilde{R}_1\eta_1+\tilde{R}_2\eta_2}{a^2+1}. \qquad (68)$$

解(68)式，得到

$$\dot{\eta}_1=-\frac{ag\eta_1}{(a^2+1)\sqrt{\eta_1^2+\eta_2^2}}+\frac{a^2(\tilde{R}_1\eta_2^2-\tilde{R}_2\eta_1\eta_2)+\tilde{R}_1(\eta_1^2+\eta_2^2)}{m(a^2+1)(\eta_1^2+\eta_2^2)},$$

$$\dot{\eta}_2=-\frac{ag\eta_2}{(a^2+1)\sqrt{\eta_1^2+\eta_2^2}}+\frac{\tilde{R}_2(\eta_1^2+\eta_2^2)-a^2(\tilde{R}_1\eta_1\eta_2-\tilde{R}_2\eta_1^2)}{m(a^2+1)(\eta_1^2+\eta_2^2)}. \qquad (69)$$

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EQUATIONS OF MOTION OF VARIABLE MASS NONHOLONOMIC DYNAMICAL SYSTEMS IN POINCARÉ-CHETAEV VARIABLES

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ABSTRACT

The equations of motion of variable-mass nonlinear nonholonomic dynamical systems in Poincaré – Chetaev variables have been studied. Firstly, the Poincaré – Chetaev variables $x_1, x_2, \cdots, x_n$ and more with n-m holonomic constraints and $m-l$ nonlinear nonholonomic constraints of Chetaev type were introduced. Secondly, the equations of Chaplygin 's form, Nielsen 's form and Appell 's form were derived from the D 'Alembert-Lagrange principle for a variable-mass mechanical system. Finally, the problem of equivalence between the Chaplygin 's equations and the Appell 's equations was discussed. Then the theory is illustrated by an example due to Appell.

Keywords: Poincaré – Chetaev variable, variable mass, nonholonomic system, D 'Alembert-Lagrange principle

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