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高维增广相空间中广义力学系统的对称性和不变量

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依据广义增广相空间中的 Hamilton 作用量在无穷小变换群作用下的不变性,给出广义完整保守和非保守力学系统的对称性和不变量及有关结论的逆命题,最后举一实例说明.

关键词:增广相空间, 广义力学, 对称性, 不变量

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1　引　　言

力学系统的对称性和不变量是其本质特征的反映. 目前,人们在位形空间和相空间中关于力学系统的对称性和不变量进行了有益的探讨,但大都限于研究普通分析力学和量子力学中的问题[1—3]. 本文给出高维增广相空间中广义完整保守和非保守力学系统的对称性和不变量及有关结论的逆命题,并举例说明新理论的应用.

2　广义完整保守力学系统在高维增广相空间中的不变量

对于广义完整保守力学系统,在高维增广相空间中的 Hamilton 原理为作用量

$$I=\int_{t_0}^{t_1}\Big[\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m)}{q}_k-H(\overset{(m-1)}{q}_k,p_{k/m},t)\Big]\mathrm{d}t,\tag{1}$$

在运动正路上取驻值.

广义完整保守力学系统,在高维增广相空间 $(\overset{(m-1)}{q}_k,p_{k/m},t)$ 中运动正路由 Hamilton 正则方程[4]描述:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overset{(m-1)}{q}_k=\frac{\partial H}{\partial p_{k/m}},\quad \dot p_{k/m}=-\frac{\partial H}{\partial \overset{(m-1)}{q}_k},\tag{2}$$

$$(k=1,2,\cdots,n;m=1,2,\cdots,\omega).$$

假设在高维增广相空间中的无穷小多参数变换群为

$$\begin{aligned}
\bar t&=t+\varepsilon^i\tau^i(t,q_k,\dot q_k,\cdots,\overset{(\omega)}{q}_k,p_{k/1},p_{k/2},\cdots,p_{k/w}),\\
\bar q_k&=q_k+\varepsilon^i\overset{(m-1)}{\xi}{}^i_k(t,q_k,\dot q_k,\cdots,\overset{(\omega)}{q}_k,p_{k/1},p_{k/2},\cdots,p_{k/w}),\\
\overset{(m)}{\bar q}_k&=\overset{(m)}{q}_k+\varepsilon^i\Big[\overset{(m)}{\xi}{}^i_k-\overset{(m)}{q}_k\dot\tau^i+\overset{(m-1)}{\varphi}{}^i_k(t,q_k,\dot q_k,\cdots,\\
&\qquad\overset{(\omega)}{q}_k,p_{k/1},p_{k/2},\cdots,p_{k/w})\Big],\\
\bar p_{k/m}&=p_{k/m}+\varepsilon^i\overset{(m-1)}{\eta}{}^i_k(t,q_k,\dot q_k,\cdots,\overset{(\omega)}{q}_k,\\
&\qquad p_{k/1},p_{k/2},\cdots,p_{k/w}).
\end{aligned}\tag{3}$$

其中 ε^i 为小参数,τ^i, $\overset{(m-1)}{\xi}{}^i_k$, $\overset{(m-1)}{\eta}{}^i_k$, $\overset{(m-1)}{\varphi}{}^i_k$ 均为(1)式所含变量的生成函数,在广义增广相空间中所有变量均自由变化. 在(3)式的作用下,(1)式为

$$I(\varepsilon)=\int_{t_0}^{t_1}\Big[\sum_{k=1}^{n}\sum_{m=1}^{\omega}\bar p_{k/m}\overset{(m)}{\bar q}_k-\bar H\Big]\mathrm{d}t,\tag{4}$$

则变换后作用量之差为

$$\begin{aligned}
\Delta I&=I(\varepsilon)-I\\
&=\varepsilon^i\int_{t_0}^{t_1}\Big\{\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}(\overset{(m)}{\xi}{}^i_k+\overset{(m-1)}{\varphi}{}^i_k)-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\overset{(m-1)}{\xi}{}^i_k\\
&\quad+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\Big(\overset{(m)}{q}_k-\frac{\partial H}{\partial p_{k/m}}\Big)\overset{(m-1)}{\eta}{}^i_k-\frac{\partial H}{\partial t}\tau^i-H\dot\tau^i\Big\}\mathrm{d}t,
\end{aligned}\tag{5}$$

经整理可得

$$\begin{aligned}
\Delta I=\varepsilon^i\int_{t_0}^{t_1}\Big\{&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\xi}{}^i_k-H\tau^i\Big)\\
&-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\Big(\dot p_{k/m}+\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\Big)\Big(\overset{(m-1)}{\xi}{}^i_k-\overset{(m)}{q}_k\tau^i\Big)\\
&+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\Big(\overset{(m)}{q}_k-\frac{\partial H}{\partial p_{k/m}}\Big)\Big(\overset{(m-1)}{\eta}{}^i_k-\dot p_{k/m}\tau^i\Big)
\end{aligned}$$

$$+\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\varphi}{}_k^i\Big\}\mathrm{d}t. \tag{6}$$

进而更一般地有

$$\Delta I=\varepsilon^i\int_{t_0}^{t_1}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\xi}{}_k^i-H\tau^i-\overset{(m-1)}{P}{}^i\Big)$$
$$+\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\varphi}{}_k^i+\overset{(m)}{P}{}^i-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\left(\dot{p}_{k/m}+\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\right)$$
$$\cdot\Big(\overset{(m-1)}{\xi}{}_k^i-\overset{(m)}{q}_k\tau^i\Big)+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\left(\overset{(m)}{q}_k-\frac{\partial H}{\partial p_{k/m}}\right)$$
$$\cdot\Big(\overset{(m-1)}{\eta}{}_k^i-\dot{p}_{k/m}\tau^i\Big)\Big\}\mathrm{d}t. \tag{7}$$

若力学系统对称,则(7)式在广义力学系统的正路上恒为零.因此,若有

$$\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\varphi}{}_k^i+\overset{(m)}{P}{}^i=0, \tag{8}$$

则广义力学系统在高维增广相空间中的不变量为

$$\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\xi}{}_k^i-H\tau^i-\overset{(m-1)}{P}{}^i=\mathrm{const}. \tag{9}$$

其中 $\overset{(m-1)}{P}{}^i=\overset{(m-1)}{P}{}^i(t,\overset{(m-1)}{q}_k,p_{k/m})$,称为规范函数.

当 $m=1$ 时,有

$$\sum_{k=1}^{n}p_k\xi_k^i-H\tau^i-P^i=\mathrm{const}, \tag{10}$$

这与文献[3]的结果一致.

究竟什么样的无穷小变换群能使 Hamilton 作用量(1)保持不变呢?将(8)式代入(5)式,则得 $\Delta I=0$ 的条件为

$$\overset{(m)}{P}{}^i=\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m)}{\xi}{}_k^i-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\overset{(m-1)}{\xi}{}_k^i$$
$$-\frac{\partial H}{\partial t}\tau^i-H\dot{\tau}^i. \tag{11}$$

3 广义完整非保守力学系统在高维增广相空间中的不变量

广义完整非保守系统的 Hamilton 正则方程[4,5]为

$$\frac{\mathrm{d}}{\mathrm{d}t}\overset{(m-1)}{q}_k=\frac{\partial H}{\partial p_{k/m}},\ \dot{p}_{k/m}=-\frac{\partial H}{\partial\overset{(m-1)}{q}_k}+Q_k, \tag{12}$$
$$(k=1,2,\cdots,n;m=1,2,\cdots,\omega).$$

此处广义非保守力 $Q_k=Q_k(q_i,\dot{q}_i,\cdots,\overset{(\omega)}{q}_i,t)$,对于广义完整保守力学系统,若高维增广相空间中的无穷小变换群(3)可致使与系统对应的保守部分的 Hamilton 作用量(1)保持不变,且满足

$$\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\varphi}{}_k^i+\overset{(m)}{P}{}^i-\sum_{k=1}^{n}\sum_{m=1}^{\omega}Q_k$$
$$\cdot\Big(\overset{(m-1)}{\xi}{}_k^i-\overset{(m)}{q}_k\tau^i\Big)=0, \tag{13}$$

则该系统的动力学不变量为

$$\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\xi}{}_k^i-H\tau^i-\overset{(m-1)}{P}{}^i=\mathrm{const}. \tag{14}$$

此处生成函数和规范函数满足条件:

$$\overset{(m)}{P}{}^i=\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m)}{\xi}{}_k^i-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\overset{(m-1)}{\xi}{}_k^i$$
$$-\frac{\partial H}{\partial t}\tau^i-H\dot{\tau}^i-\sum_{k=1}^{n}\sum_{m=1}^{\omega}Q_k\Big(\overset{(m-1)}{\xi}{}_k^i-\overset{(m)}{q}_k\tau^i\Big). \tag{15}$$

4 逆 命 题

在这一部份中,力学系统的动力学不变量也一定对应一组无穷小对称变换.

4.1 广义完整保守力学系统

假设有一广义完整保守系统,它存在一个动力学不变量 $G(\overset{(m-1)}{q}_k,p_{k/m},t)$,则

$$\frac{\mathrm{d}G}{\mathrm{d}t}=\frac{\partial G}{\partial t}+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial\overset{(m-1)}{q}_k}\overset{(m)}{q}_k+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial p_{k/m}}\dot{p}_{k/m}$$
$$=\frac{\partial G}{\partial t}+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial\overset{(m-1)}{q}_k}\left(\overset{(m)}{q}_k-\frac{\partial H}{\partial p_{k/m}}\right)+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial p_{k/m}}$$
$$\cdot\left(\dot{p}_{k/m}+\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\right)+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial\overset{(m-1)}{q}_k}\frac{\partial H}{\partial p_{k/m}}$$
$$-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial p_{k/m}}\frac{\partial H}{\partial\overset{(m-1)}{q}_k}. \tag{16}$$

现研究对于动力学不变量 $G(\overset{(m-1)}{q}_k,p_{k/m},t)$是否对应有一组无穷小对称变换以使力学系统的 Hamilton 作用量保持不变,将(16)式与(7)式结合,则得

$$\Delta I(\varepsilon)=\varepsilon^i\int_{t_0}^{t_1}\Big\{\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\xi}{}_k^i-H\tau^i-\overset{(m-1)}{P}{}^i\Big)$$
$$-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\left(\dot{p}_{k/m}+\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\right)\left(\overset{(m-1)}{\xi}{}_k^i-\overset{(m)}{q}_k\tau^i-\frac{\partial G}{\partial p_{k/m}}\right)$$
$$+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\left(\overset{(m)}{q}_k-\frac{\partial H}{\partial p_{k/m}}\right)\left(\overset{(m-1)}{\eta}{}_k^i-\dot{p}_{k/m}\tau^i+\frac{\partial G}{\partial\overset{(m-1)}{q}_k}\right)$$
$$+\sum_{k=1}^{n}\sum_{m=1}^{\omega}p_{k/m}\overset{(m-1)}{\varphi}{}_k^i+\overset{(m)}{P}{}^i-\frac{\mathrm{d}G}{\mathrm{d}t}+\frac{\partial G}{\partial t}$$
$$+\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial\overset{(m-1)}{q}_k}\frac{\partial H}{\partial p_{k/m}}-\sum_{k=1}^{n}\sum_{m=1}^{\omega}\frac{\partial G}{\partial p_{k/m}}\frac{\partial H}{\partial\overset{(m-1)}{q}_k}\Big\}\mathrm{d}t,$$
$$\tag{17}$$

去掉规范函数 $\overset{(m)}{P^i}$ 后,(17)式可以写为

$$\Delta I(\varepsilon) = \varepsilon^i \int_{t_0}^{t_1} \Big\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\xi^i_k} - H\tau^i - G \Big) + \sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\varphi^i_k} - \sum_{k=1}^{n} \sum_{m=1}^{\omega} \Big(\dot{p}_{k/m} + \frac{\partial H}{\partial \overset{(m-1)}{q_k}} \Big) \cdot \Big(\overset{(m-1)}{\xi^i_k} - \overset{(m)}{q_k}\tau^i - \frac{\partial G}{\partial p_{k/m}} \Big) + \sum_{k=1}^{n} \sum_{m=1}^{\omega} \Big(\overset{(m)}{q_k} - \frac{\partial H}{\partial p_{k/m}} \Big) \cdot \Big(\overset{(m-1)}{\eta^i_k} - \dot{p}_{k/m}\tau^i + \frac{\partial G}{\partial \overset{(m-1)}{q_k}} \Big) + \frac{\partial G}{\partial t} + \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial \overset{(m-1)}{q_k}} \cdot \frac{\partial H}{\partial p_{k/m}} - \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial p_{k/m}} \frac{\partial H}{\partial \overset{(m-1)}{q_k}} \Big\} \mathrm{d}t . \tag{18}$$

因此,若生成函数满足如下关系式:

$$\begin{aligned} \tau^i &= \frac{1}{H}\Big(\sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\xi^i_k} - G \Big), \\ \overset{(m-1)}{\xi^i_k} &= \frac{\partial H}{\partial p_{k/m}}\tau^i + \frac{\partial G}{\partial p_{k/m}}, \\ \overset{(m-1)}{\eta^i_k} &= -\frac{\partial H}{\partial \overset{(m-1)}{q_k}}\tau^i - \frac{\partial G}{\partial \overset{(m-1)}{q_k}}, \\ & \sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\varphi^i_k} + \frac{\partial G}{\partial t} - \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial p_{k/m}} \frac{\partial H}{\partial \overset{(m-1)}{q_k}} + \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial \overset{(m-1)}{q_k}} \frac{\partial H}{\partial p_{k/m}} = 0 . \end{aligned} \tag{19}$$

则有 $\Delta I(\varepsilon) = 0$,因此广义完整保守力学系统,若 $G(\overset{(m-1)}{q_k}, p_{k/m}, t)$是该系统的一个守恒量,则由(19)式确定的高维增广相空间中的无穷小多参数变换群使得广义力学系统在该空间的 Hamilton 作用量保持不变.

4.2 广义完整非保守力学系统

若 $G(\overset{(m-1)}{q_k}, p_{k/m}, t)$是广义完整非保守力学系统的一个动力学不变量,则由关系式

$$\begin{aligned} \tau^i &= \frac{1}{H}\Big(\sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\xi^i_k} - G \Big), \\ \overset{(m-1)}{\xi^i_k} &= \frac{\partial H}{\partial p_{k/m}}\tau^i + \frac{\partial G}{\partial p_{k/m}}, \\ \overset{(m-1)}{\eta^i_k} &= \Big(-\frac{\partial H}{\partial \overset{(m-1)}{q_k}} + Q_k \Big)\tau^i - \frac{\partial G}{\partial \overset{(m-1)}{q_k}}, \\ & \sum_{k=1}^{n} \sum_{m=1}^{\omega} p_{k/m} \overset{(m-1)}{\varphi^i_k} + \frac{\partial G}{\partial t} - \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial p_{k/m}} \frac{\partial H}{\partial \overset{(m-1)}{q_k}} + \sum_{k=1}^{n} \sum_{m=1}^{\omega} \frac{\partial G}{\partial \overset{(m-1)}{q_k}} \frac{\partial H}{\partial p_{k/m}} - \sum_{k=1}^{n} \sum_{m=1}^{\omega} Q_k \cdot \Big(\overset{(m-1)}{\xi^i_k} - \overset{(m)}{q_k}\tau^i \Big) = 0 \end{aligned} \tag{20}$$

确定的高维增广相空间中的无穷小多参数变换群使得广义力学系统在该空间中的 Hamilton 作用量保持不变,即 $\Delta I(\varepsilon) = 0$.

5 举 例

在圆柱孔 A 及 B 内插入圆柱形的均匀而有弹性重梁的端点,把 A,B 看成是一个水平安置的圆柱部分.梁的大小、密度及弹性系数都认为是已知的.试研究此系统的对称性和不变量.

解 这是一个完整保守系统,首先求其广义动量.

利用原理 如果一个力学系统处于稳定平衡状态,那么对于此系统的任何可能的移动,都使系统的位能增加.

以 $2l$ 表示二支点间的距离,ρ 表示梁的密度,$\mathrm{d}s$ 表示梁轴弯曲的弧元素,设 ox 轴联接二支点,原点在 AB 的中点,oy 轴的方向垂直向上,设弹性轴的方程是 $y = y(x)$,而计算梁的位能.当梁弯曲时由弹性力所形成的位能为

$$E_1 = \frac{1}{2}\mu \int_0^L \Big(\frac{\mathrm{d}\varphi}{\mathrm{d}s} \Big)^2 \mathrm{d}s , \tag{21}$$

其中 $L = 2l$,φ 是切线与 ox 轴的交角,而 μ 是常系数,它依赖于弹性系数及梁的横断面的转动惯量.

引力场所形成的位能为

$$E_2 = \int_0^L \rho y \,\mathrm{d}s , \tag{22}$$

于是,得到梁的总位能为

$$E = E_1 + E_2 = \int_0^L \Big[\frac{1}{2}\mu \Big(\frac{\mathrm{d}\varphi}{\mathrm{d}s} \Big)^2 + \rho y \Big] \mathrm{d}s . \tag{23}$$

由于

$$\mathrm{d}s = \sqrt{1 + \dot{y}^2}\,\mathrm{d}x , \quad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}} ,$$

于是有

$$E = \int_{-l}^{l} \Big\{ \frac{1}{2}\mu \frac{\ddot{y}^2}{(1 + \dot{y}^2)^{5/2}} + \rho y \sqrt{1 + \dot{y}^2} \Big\} \mathrm{d}x . \tag{24}$$

令 Lagrange 函数等于积分号下的表达式,即

$$L = \frac{1}{2}\mu \frac{\ddot{y}^2}{(1 + \dot{y}^2)^{5/2}} + \rho y \sqrt{1 + \dot{y}^2} , \tag{25}$$

则,(24)式可写为

$$E = \int_{-l}^{l} L \,\mathrm{d}x . \tag{26}$$

此处 y，x 依次相当于本文中各公式里的 q，t. 位能 E 取极值，必须满足

$$\delta E = 0. \tag{27}$$

设端点条件为 $\delta y|_{x=-l} = \delta y|_{x=l} = 0$，$\delta\dot{y}|_{x=-l} = \delta\dot{y}|_{x=l} = 0$. 根据文献[4]，由(27)式不难求得系统的广义动量为

$$p_{1/1} = \frac{\partial L}{\partial \dot{y}} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial \ddot{y}} = \frac{5}{2}\frac{\mu\dot{y}\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}}{(1+\dot{y}^2)^{1/2}} - \frac{\mu\dddot{y}}{(1+\dot{y}^2)^{5/2}}, \tag{28}$$

$$p_{1/2} = \frac{\partial L}{\partial \ddot{y}} = \frac{\mu\ddot{y}}{(1+\dot{y}^2)^{5/2}}. \tag{29}$$

其次，研究正问题，求系统的动力学不变量.

系统的 Hamilton 函数为

$$H = p_{1/1}\dot{y} + p_{1/2}\ddot{y} - L = \frac{5}{2}\frac{\mu\dot{y}^2\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}^2}{(1+\dot{y}^2)^{1/2}} - \frac{\mu\dot{y}\dddot{y}}{(1+\dot{y}^2)^{5/2}} + \frac{1}{2}\frac{\mu\ddot{y}^2}{(1+\dot{y}^2)^{5/2}} - \rho y\sqrt{1+\dot{y}^2}. \tag{30}$$

取无限小变换的生成元如下：

$$\tau = -1, \xi_1 = 0, \eta_1 = 0, \varphi_1 = 0. \tag{31}$$

将(30)和(31)代入(11)式，求得规范函数

$$\overset{(m)}{P} = 0, \text{可令} \overset{(m-1)}{P} = 0, (m = 1,2). \tag{32}$$

由于(31)和(32)式满足条件(8)，因此不变量(9)给出

$$\frac{5}{2}\frac{\mu\dot{y}^2\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}^2}{(1+\dot{y}^2)^{1/2}} - \frac{\mu\dot{y}\dddot{y}}{(1+\dot{y}^2)^{5/2}} + \frac{1}{2}\frac{\mu\ddot{y}^2}{(1+\dot{y}^2)^{5/2}} - \rho y\sqrt{1+\dot{y}^2} = \text{const}. \tag{33}$$

最后，研究逆问题，求系统的对称变换.

假设已知系统有不变量

$$G(\overset{(m-1)}{y_k}, p_{k/m}, x) = p_{1/1}\dot{y} + p_{1/2}\ddot{y} - L = \frac{5}{2}\frac{\mu\dot{y}^2\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}^2}{(1+\dot{y}^2)^{1/2}} - \frac{\mu\dot{y}\dddot{y}}{(1+\dot{y}^2)^{5/2}} + \frac{1}{2}\frac{\mu\ddot{y}^2}{(1+\dot{y}^2)^{5/2}} - \rho y\sqrt{1+\dot{y}^2}, \tag{34}$$

根据(19)式，则可得

$$\begin{aligned}\tau &= \frac{1}{H}\Big(\sum_{m=1}^{2} p_{1/m}\overset{(m-1)}{\xi_1} - G\Big) \\ &= \Big[\frac{5}{2}\frac{\mu\dot{y}^2\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}^2}{(1+\dot{y}^2)^{1/2}} - \frac{\mu\dot{y}\dddot{y}}{(1+\dot{y}^2)^{5/2}} + \frac{1}{2}\frac{\mu\ddot{y}^2}{(1+\dot{y}^2)^{5/2}} - \rho y\sqrt{1+\dot{y}^2}\Big]^{-1} \\ &\cdot \Big[\frac{5}{2}\frac{\mu\dot{y}\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} + \frac{\rho y\dot{y}}{\sqrt{1+\dot{y}^2}} - \frac{\mu\dddot{y}}{(1+\dot{y}^2)^{5/2}}\Big]\xi_1 + \frac{\mu\ddot{y}}{(1+\dot{y}^2)^{5/2}}\dot{\xi}_1 - \frac{5}{2}\frac{\mu\dot{y}^2\ddot{y}^2}{(1+\dot{y}^2)^{7/2}} - \frac{\rho y\dot{y}^2}{(1+\dot{y}^2)^{1/2}} \\ &+ \frac{\mu\dot{y}\dddot{y}}{(1+\dot{y})^{5/2}} - \frac{1}{2}\frac{\mu\ddot{y}^2}{(1+\dot{y}^2)^{5/2}} + \rho y\sqrt{1+\dot{y}^2},\end{aligned} \tag{35}$$

$$\xi_1 = \frac{\partial H}{\partial p_{1/1}}\tau + \frac{\partial G}{\partial p_{1/1}} = (1+\tau)\dot{y}, \tag{36}$$

$$\dot{\xi}_1 = \frac{\partial H}{\partial p_{1/2}}\tau + \frac{\partial G}{\partial p_{1/2}} = (1+\tau)\ddot{y}. \tag{37}$$

由(35)，(36)和(37)式解得

$$\begin{aligned}&\tau = -1, \xi_1 = 0, \dot{\xi}_1 = 0, \\ &\eta_1 = -\frac{\partial H}{\partial y}\tau - \frac{\partial G}{\partial y} = \Big(-\frac{\rho\dot{y}^2}{\sqrt{1+\dot{y}^2}} + \rho\sqrt{1+\dot{y}^2}\Big)(1+\tau) = 0, \\ &\dot{\eta}_1 = -\frac{\partial H}{\partial \dot{y}}\tau - \frac{\partial G}{\partial \dot{y}} = -p_{1/1}(1+\tau) = 0, \\ &\varphi_1 = \frac{1}{p_{1/1}}\Big[-\frac{\partial G}{\partial p_{1/1}}\frac{\partial H}{\partial y} + \frac{\partial G}{\partial y}\frac{\partial H}{\partial p_{1/1}}\Big] = 0, \\ &\dot{\varphi}_1 = \frac{1}{p_{1/2}}\Big[-\frac{\partial G}{\partial p_{1/2}}\frac{\partial H}{\partial \dot{y}} + \frac{\partial G}{\partial \dot{y}}\frac{\partial H}{\partial p_{1/2}}\Big] = 0.\end{aligned} \tag{38}$$

于是，所求的对称变换为

$$\bar{x} = x - \varepsilon, \bar{y} = y, \bar{\dot{y}} = \dot{y}, \bar{\ddot{y}} = \ddot{y}, \bar{p}_{1/1} = p_{1/1}, \bar{p}_{1/2} = p_{1/2}.$$

由上可知，本题中自变量 x 的平移变换是该系统的对称变换.

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SYMMETRY AND INVARIANT IN GENERALIZED MECHANICAL SYSTEMS IN THE HIGH-DIMENSIONAL EXTENDED PHASE SPACE

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ABSTRACT

By means of the invariance of Hamilton's action under infinitesimal transformation in the high-dimensional extended phase space, the symmetry and invariant of the generalized holonomic conservative and nonconservative mechanical systems and some related propositions are obtained. Finally one example is given.

Keywords: extended phase space, generalized mechanics, symmetry, invariant
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