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用格子 Boltzmann 方法模拟 MKDV 方程*

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用精确到 $0(\varepsilon^4)$ 的 5 速格子 Boltzmann 模型模拟 MKDV 方程:$u_t+6u^2u_x+u_{xxx}=0$,并与 MKDV 方程的孤子解比较,二者精确吻合.

关键词:格子 Boltzmann 方法,MKDV 方程,孤子解
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1 引 言

MKDV 方程是非线性物理学的一个重要的方程[1],和一般的非线性偏微分方程一样,只在很特殊初、边值的情况下才有解析解,一般情况只能进行数值求解.近年来用格子 Boltzmann 方法(LBM)数值模拟流体力学[2—4]、反应扩散[5]等方面已经取得成功.由于它具有完全并行和容易施加边界条件等优点,现在已有用它来模拟 Burgers 方程[6]等非线性偏微分方程.本文尝试用格子 Boltzmann 方法模拟 MKDV 方程,并与 MKDV 方程的孤子解比较,两者精确吻合.

2 模 型

引入小参数 $\varepsilon(\varepsilon\ll1)$,作为时间步长和 Knudsen 数,将空间尺度为 L 的一维线段等分为 $\frac{L}{\varepsilon}+1$ 个格点,每个格点上粒子可具有 $-2\varepsilon,-\varepsilon,0,\varepsilon,2\varepsilon$ 五种速度,下一时刻粒子将以此速度运动到附近格点,若遇边界则反弹.

单粒子分布函数 $f_\alpha(x,t)$ 的动力学方程采用单弛豫形式的格子 Boltmann 方程[7]

$$f_\alpha(x+\varepsilon e_\alpha,t+\varepsilon)-f_\alpha(x,t)=-\frac{1}{\tau}[f_\alpha(x,t)-f_\alpha^{(0)}(x,t)],\qquad(1)$$

其中 τ 是弛豫时间,稳定性要求 $\tau>\frac{1}{2}$,$\alpha=0—4$,对应的粒子速度为 $\varepsilon e_\alpha=-2\varepsilon,-\varepsilon,0,\varepsilon,2\varepsilon$.$f_\alpha^{(0)}(x,t)$ 为 t 时刻速度为 εe_α 的粒子的局域平衡分布函数.

将 $f_\alpha(x+\varepsilon e_\alpha,t+\varepsilon)$ 进行 Taylor 展开,并保留到 ε 的 4 次方项:

$$\begin{aligned}f_\alpha(x+\varepsilon e_\alpha,t+\varepsilon)&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\varepsilon\frac{\partial}{\partial t},\varepsilon e_\alpha\frac{\partial}{\partial x}\right)^n f_\alpha(x,t)\\&=f_\alpha(x,t)+\varepsilon\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)f_\alpha(x,t)\\&\quad+\frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)^2 f_\alpha(x,t)\\&\quad+\frac{\varepsilon^3}{6}\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)^3 f_\alpha(x,t)\\&\quad+\frac{\varepsilon^4}{24}\left(\frac{\partial}{\partial t}+e_\alpha\frac{\partial}{\partial x}\right)^4 f_\alpha(x,t)\\&\quad+0(\varepsilon^5).\end{aligned}\qquad(2)$$

利用 Chapman-Enskog 展开将 $f_\alpha(x,t)$ 展至 ε^4 项:

$$\begin{aligned}f_\alpha(x,t)&=\sum_{n=0}^{\infty}\varepsilon^n f_\alpha^{(n)}=f_\alpha^{(0)}+\varepsilon f_\alpha^{(1)}+\varepsilon^2 f_\alpha^{(2)}\\&\quad+\varepsilon^3 f_\alpha^{(3)}+\varepsilon^4 f_\alpha^{(4)}+0(\varepsilon^5).\end{aligned}\qquad(3)$$

引入五个时间尺度,$t_a=\varepsilon^a t\,(a=0—4)$,则

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}+\varepsilon^3\frac{\partial}{\partial t_3}$$

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$$+\varepsilon^4\frac{\partial}{\partial t_4}+0(\varepsilon^5). \tag{4}$$

把(3),(4)式代入(2)式,比较两边系数得到不同时间尺度的格子 Boltzmann 方程

$$\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(1)}, \tag{5}$$

$$\frac{\partial}{\partial t_1}f_\alpha^{(0)}-\tau\left(1-\frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^2f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(2)}, \tag{6}$$

$$\frac{\partial}{\partial t_2}f_\alpha^{(0)}+\left(2-\frac{1}{\tau}\right)\frac{\partial}{\partial t_1}f_\alpha^{(1)}+\left(\tau^2-\tau+\frac{1}{6}\right)\cdot\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^3f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(3)}, \tag{7}$$

$$\frac{\partial}{\partial t_3}f_\alpha^{(0)}+\left(2\tau^2-2\tau+\frac{1}{4}\right)\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^2\frac{\partial}{\partial t_1}f_\alpha^{(0)}+(1-2\tau)\frac{\partial}{\partial t_2}\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)f_\alpha^{(0)}+\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^4f_\alpha^{(0)}+\left(1-\frac{1}{2\tau}\right)\frac{\partial}{\partial t_1}f_\alpha^{(2)}=-\frac{1}{\tau}f_\alpha^{(4)}. \tag{8}$$

将(5)+ε×(6)+ε²×(7)+ε³×(8)并求和得

$$\frac{\partial}{\partial t}\sum_\alpha f_\alpha^{(0)}+\frac{\partial}{\partial x}\sum_\alpha e_\alpha f_\alpha^{(0)}+\varepsilon\left(\frac{1}{2}-\tau\right)\cdot\sum_\alpha\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^2f_\alpha^{(0)}+\varepsilon^2\left(\tau^2-\tau+\frac{1}{6}\right)\cdot\sum_\alpha\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^3f_\alpha^{(0)}+\varepsilon^3\left[\left(2\tau^2-2\tau+\frac{1}{4}\right)\cdot\frac{\partial}{\partial t_1}\sum_\alpha\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^2f_\alpha^{(0)}+\left(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\right)\sum_\alpha\left(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\right)^4f_\alpha^{(0)}\right]+0(\varepsilon^4)=-\frac{1}{\tau}\sum_{j=1}^{4}\varepsilon^{j-1}\sum_\alpha f_\alpha^{(j)}. \tag{9}$$

定义宏观量

$$u=\sum_\alpha f_\alpha(x,t). \tag{10}$$

质量守恒要求

$$u=\sum_\alpha f_\alpha^{(0)}(x,t). \tag{11}$$

则有 $\sum_\alpha f_\alpha^{(j)}=0(j=1—4)$,因此(9)式右边等于零.

选择局域平衡分布函数的各阶矩的形式为

$$\sum_\alpha e_\alpha f_\alpha^{(0)}=2u^3,$$
$$\sum_\alpha e_\alpha^2f_\alpha^{(0)}=\frac{36}{5}u^5,$$
$$\sum_\alpha e_\alpha^3f_\alpha^{(0)}=\frac{216}{7}u^7+\beta u,$$
$$\sum_\alpha e_\alpha^4f_\alpha^{(0)}=144u^9+8\beta u^3. \tag{12}$$

其中 $\beta=\dfrac{1}{\varepsilon^2(\tau^2-\tau+1/6)}$,则(9)式化为

$$u_t+6u^2u_x+u_{xxx}+0(\varepsilon^4)=0. \tag{13}$$

(13)式即 MKDV 方程.至此,我们从 5 速单弛豫形式的格子 Boltzmann 方程出发,利用 Chapman-Enskog 展开和多尺度技术,精确到 $0(\varepsilon^4)$,导出了宏观量 u 所满足的 MKDV 方程.

3 MKDV 方程的数值模拟

用(1)式数值模拟 MKDV 方程,首先由(11),(12)式求出局域平衡分布函数:

$$f_0^{(0)}=\frac{1}{12}\left[-\beta u+2(1+2\beta)u^3-\frac{18}{5}u^5-\frac{216}{7}u^7+72u^9\right],$$
$$f_1^{(0)}=\frac{1}{6}\left[-\beta u+8(1-\beta)u^3+\frac{144}{5}u^5-\frac{216}{7}u^7-144u^9\right],$$
$$f_2^{(0)}=u+\frac{2}{3}\beta u^3-\frac{51}{5}u^5+\frac{72}{7}u^7+60u^9,$$
$$f_3^{(0)}=\frac{1}{6}\left[\beta u-8(1+\beta)u^3+\frac{144}{5}u^5-\frac{216}{7}u^7-144u^9\right],$$
$$f_4^{(0)}=\frac{1}{12}\left[\beta u-2(1-2\beta)u^3-\frac{18}{5}u^5+\frac{216}{7}u^7+72u^9\right]. \tag{14}$$

将(14)式代入(1)式则可由 t 时刻的单粒子分布函数 $f_\alpha(x,t)$ 求出 $t+\varepsilon$ 时刻的单粒子分布函数 $f_\alpha(x,t+\varepsilon)$,再由(10)式可以求出 $t+\varepsilon$ 时刻的 u,由此循环迭代,u 的变化即满足 MKDV 方程.为检验数值模拟的有效性,我们选择问题:

$$u_t+6u^2u_x+u_{xxx}=0,x\in(x_1,x_2),$$
$$u(x,0)=a\,\mathrm{sech}[a(x-x_0)],$$
$$u(x_1,t)=a\,\mathrm{sech}[a(x_1-a^2t-x_0)],$$
$$u(x_2,t)=a\,\mathrm{sech}[a(x_2-a^2t-x_0)].$$

它的解为

$$u(x,t)=a\,\mathrm{sech}[a(x-a^2t-x_0)].$$

其中 a 为孤子高度,x_0 为孤子的初始位置.

用(1)式模拟上述问题时，取 $\varepsilon=0.01$，$\beta=0.002501$，相应 $\tau=2000$，数值模拟结果如图(1)所示．从模拟的结果我们可以看出：

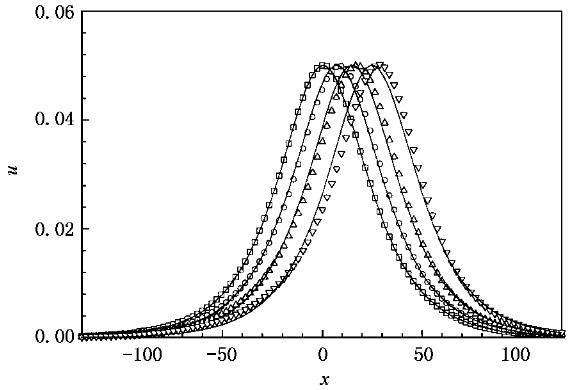

图 1　MKDV 方程孤子解(其中 $x_0=0$，$a=0.05$，实线为解析解，□，○，△，▽分别是 $t=200$，3000，6000，10000 的结果)

1．$t=6000$ 以前，模拟结果与解析解精确吻合．但随着时间的进一步增加，数值模拟结果与精确解之间会有偏离，这是由于存在微扰 $O(\varepsilon^4)$，它对孤子的高度、传播速度以及形状均有影响[8]．对于 5 速模型，只能做到 $O(\varepsilon^4)$．原则上，7 速模型可以做到 $O(\varepsilon^6)$．另一方面，也可以通过减小 ε 来减小微扰，但这意味着要增加计算时间．

2．该模型适用于孤子高度 $a\leqslant0.05$ 的情况，$a>0.05$ 时，偏离会增大．

3．计算表明，β 不能比 0.002501 大太多(相应的弛豫时间 τ 要增大)，否则与解析解的偏离会增大，这是由于局域平衡分布函数变负造成的．

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SIMULATION OF THE MKDV EQUATION WITH LATTICE BOLTZMANN METHOD*

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ABSTRACT

The MKDV equation ($u_t + 6u^2 u_x + u_{xxx} = 0$) is simulated by the 5-speed lattice Boltzmann method of $0(\varepsilon^4)$ precision, then the result is compared with the soliton solution of the MKDV equation. They inosculate with each other.

Keywords: lattice Boltzmann method, MKDV equation, soliton solution
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