第 50 卷 第 7 期 2001 年 7 月
1000-3290/2001/50(07)/1232-05

物 理 学 报
ACTA PHYSICA SINICA

Vol.50, No.7, July, 2001

新的 Lax 可积发展方程族及其无限维双-哈密顿结构*

闫振亚 张鸿庆

(大连理工大学应用数学系, 大连 116024)

(2000 年 11 月 13 日收到)

基于一个新的具有三个位势函数(q, r, s)的等谱问题,获得了一族新的含有一个任意函数的 Lax 可积发展方程. 特别地,当位势函数 s 取不同的函数时,这个方程族约化为若干类型的方程组,进一步利用迹恒等式,给出了这些方程组的双-哈密顿结构,并且证明它们是 Liouville 可积的. 此外,给出了守恒密度和对称.

关键词: 等谱问题, 哈密顿结构, Lax 可积, Liouville 可积

PACC: 0545, 0340

1 引 言

在孤子理论和可积动力系统中,寻找非线性 Lax 可积方程组及其双-哈密顿结构是非常重要却又是很难的课题[1—8]. 众所周知,反散射变换法[4]在精确求解很多非线性发展方程中具有重要的作用,但利用此法求解方程的前提条件是寻找到该方程所对应的等谱问题[1—5]. 最近 Tu[6]提出了从等谱问题出发获得方程族及其哈密顿结构的方法. 对给定的恰当的等谱问题及其辅助谱问题,

$$\phi_x = M(u,\lambda)\phi, \quad \phi_{t_n} = N^{(n)}(u,\lambda)\phi, \tag{1}$$

其中 λ 为谱参数,且 $\lambda_t = 0$. 方程(1)的相容条件($\phi_{xt_n} = \phi_{t_n x}$)为如下零曲率方程:

$$M_{t_n} - N_x^{(n)} + [M, N^{(n)}] = 0,$$
$$[M, N^{(n)}] = MN^{(n)} - N^{(n)}M. \tag{2}$$

一般而言方程(1)为超定的方程族. 如何选择适当的 M 和 $N^{(n)}$ 使得方程(2)恰为某一 Lax 可积方程族,并且利用哈密顿算子及迹恒等式获得该方程族的双-哈密顿结构[7,8],

$$u_{t_n} = J\frac{\delta H_n}{\delta u} = K\frac{\delta H_{n-1}}{\delta u} \tag{3}$$

是非常重要的,其中 J 和 K 为辛算子, $u = (u_1, u_2, \cdots, u_p) \in S^p = S \otimes S \otimes \cdots \otimes S$ (S 为 R 上的 Schwartz 空间)为包含在 M 和 $N^{(n)}$ 中的位势, $\frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_1}, \cdots, \frac{\delta}{\delta u_p}\right)$ 为变分导数

$$\frac{\delta}{\delta u_j} = \sum_{n\geqslant 0}(-\partial)^n \frac{\partial}{\partial u_j^{(n)}},$$
$$u_j^{(n)} = \frac{\partial^n u_j}{\partial x^n}, \quad j = 1, 2, \cdots, p. \tag{4}$$

本文引入如下含有三个位势的等谱问题:

$$\phi_x = M(u,\lambda)\phi,$$
$$M = \begin{bmatrix} \lambda q & \lambda^2 + \lambda r + s \\ -\lambda^2 + \lambda r - s & -\lambda q \end{bmatrix},$$
$$u = \begin{pmatrix} q \\ r \\ s \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{5}$$

通过选择适当的辅助谱问题,给出了含有一个任意函数的 Lax 可积方程族,然后利用迹恒等式,得到它们的双-哈密顿结构.

2 Lax 可积方程族

为了从(5)式推导出 Lax 可积方程族,首先引入(5)式的伴随方程

$$N_x - [M, N] = 0,$$

* 国家重点基础研究发展规划(批准号: G1998030600)和国家自然科学基金(批准号: 10072013)资助的课题.

$$N = N(u,\lambda) = \begin{bmatrix} a & b+c \\ b-c & -a \end{bmatrix}, \tag{6}$$

从(5)和(6)式,可得如下关系式:

$$\begin{aligned} a_x &= 2b\lambda c^2 - 2rc\lambda + 2sb, \\ b_x &= -2a\lambda^2 + 2qc\lambda - 2sa, \\ c_x &= -2qb\lambda - 2ra\lambda. \end{aligned} \tag{7}$$

将

$$a = \sum_{j\geqslant 0} a_n\lambda^{-j}, \quad b = \sum_{j\geqslant 0} b_n\lambda^{-j}, \quad c = \sum_{j\geqslant 0} c_n\lambda^{-j} \tag{8}$$

代入方程(7),得

$$\begin{aligned} a_{2j} &= b_{2j} = c_{2j+1} = 0, \\ a_{2j+1} &= -\frac{1}{2}b_{2j-1,x} + qc_{2j} - sa_{2j-1}, \\ b_{2j+1} &= \frac{1}{2}a_{2j-1,x} + rc_{2j} - sb_{2j-1}, \\ c_{2j,x} &= 2qb_{2j+1} - 2ra_{2j+1} = qa_{2j-1,x} \\ &\quad + rb_{2j-1,x} + 2s(ra_{2j-1} - qb_{2j-1}). \end{aligned} \tag{9}$$

从方程(9),可推得

$$\begin{aligned} &a_0 = b_0 = 0, \quad c_0 = 1, \quad a_1 = q, \\ &b = r, \quad c_2 = \frac{1}{2}(q^2 + r^2), \\ &a_3 = -\frac{1}{2}r_x + \frac{1}{2}q(q^2 + r^2) - qs, \\ &b_3 = \frac{1}{2}q_x + \frac{1}{2}r(q^2 + r^2) - rs, \\ &c_4 = \frac{1}{2}rq_x - \frac{1}{2}qr_x + \frac{3}{8}(q^2 + r^2)^2 \\ &\quad - s(q^2 + r^2), \cdots \end{aligned} \tag{10}$$

考虑如下的辅助谱问题:

$$\phi_{t_n} = N^{(n)}(u,\lambda)\phi,$$

$$N^{(n)} = (\lambda^n N)_+ + \begin{pmatrix} 0 & \Delta_n \\ \Delta_n & 0 \end{pmatrix}, \tag{11}$$

则(5)和(11)式的相容条件($\phi_{xt_n} = \phi_{t_n x}$)为如下零曲率方程 $M_{t_n} - N_x^{(n)} + [M, N^{(n)}] = 0$,即

$$q_{t_n} = a_{2n+1,x} - 2sb_{2n+1} + 2r\Delta_{2n}, \tag{12a}$$

$$r_{t_n} = b_{2n+1,x} + 2sa_{2n+1} - 2q\Delta_{2n}, \tag{12b}$$

$$s_{t_n} = \Delta_{2n,x}, \tag{12c}$$

其中 Δ_{2n} 为任意可微函数.(12)式为所得 Lax 可积方程族.

3　双-哈密顿结构和 Liouville 可积性

下面考虑方程族(12)的几种特例:

情况 1　当 $s = 0$,且 $\Delta_{2n} = 0$ 时,方程族(12)约化为

$$q_{t_n} = a_{2n+1,x}, \quad r_{t_n} = b_{2n+1,x}, \tag{13}$$

特别地,当 $n = 1$ 时,(13)式化为

$$\begin{aligned} q_t &= -\frac{1}{2}r_{xx} + \frac{1}{2}[q(q^2 + r^2)]_x, \\ r_t &= \frac{1}{2}q_{xx} + \frac{1}{2}[r(q^2 + r^2)]_x, \end{aligned} \tag{14}$$

其为广义的导数非线性 Schrödinger 方程组.

为了研究(13)式的双-哈密顿结构,由(9)式将(13)式改写为

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = J\begin{pmatrix} a_{2n+1} \\ b_{2n+1} \end{pmatrix} = K\begin{pmatrix} a_{2n-1} \\ b_{2n-1} \end{pmatrix} = JL^n\begin{pmatrix} q \\ r \end{pmatrix}, \tag{15}$$

其中

$$\begin{pmatrix} a_{2n+1} \\ b_{2n+1} \end{pmatrix} = L\begin{pmatrix} a_{2n-1} \\ b_{2n-1} \end{pmatrix},$$

$$L = \begin{pmatrix} q\partial^{-1}q\partial & -\frac{1}{2}\partial + q\partial^{-1}r\partial \\ \frac{1}{2}\partial + r\partial^{-1}q\partial & r\partial^{-1}r\partial \end{pmatrix}, \tag{16a}$$

$$J = \begin{pmatrix} \partial & 0 \\ 0 & \partial \end{pmatrix},$$

$$K = JL = \begin{pmatrix} \partial q\partial^{-1}qc & -\frac{1}{2}\partial^2 + \partial q\partial^{-1}r\partial \\ \frac{1}{2}\partial^2 + \partial r\partial^{-1}q\partial & \partial r\partial^{-1}r\partial \end{pmatrix}, \tag{16b}$$

$\partial = \partial/\partial x$,$\partial\partial^{-1} = \partial^{-1}\partial = 1$.很显然,$J$,$K$ 为斜对称算子,进一步可验证 J,K 为辛算子.取 Killing-Cartan 标准型为 $\langle X, Y\rangle = \mathrm{Tr}(XY)$,直接计算,得(注意 $s = 0$)

$$\left\langle N, \frac{\partial M}{\partial q}\right\rangle = 2\lambda a, \quad \left\langle N, \frac{\partial M}{\partial r}\right\rangle = 2\lambda b,$$

$$\left\langle N, \frac{\partial M}{\partial \lambda}\right\rangle = -4\lambda c + 2qa + 2rb. \tag{17}$$

利用迹恒等式[6],得

$$\frac{\delta}{\delta u}(-4\lambda c + 2qa + 2rb) = \lambda^{-\beta}\frac{\partial}{\partial\lambda}\lambda^{\beta}(2\lambda a, 2\lambda b)^{\mathrm{T}}, \tag{18}$$

其中 β 为待定常数.

将 $a = \sum_{n\geqslant 0} a_{2n+1}\lambda^{-(2n+1)}$,$b = \sum_{n\geqslant 0} b_{2n+1}\lambda^{-(2n+1)}$,$c$

$= \sum_{n\geqslant 0} c_{2n}\lambda^{-2n}$ 代入(18)式,并且比较其两端 $\lambda^{-(2n+1)}$ 的系数,得

$$\frac{\delta}{\delta u}(-4c_{2n+2}+2qa_{2n+1}+2rb_{2n+1})$$
$$=(\beta-2n)(2a_{2n+1},2b_{2n+1})^{\mathrm{T}}. \qquad (19)$$

为了确定 β,取 $n=0$,则从(19)式可得 $\beta=0$.因此从(16a)和(19)式,有

$$\begin{pmatrix} a_{2n+1} \\ b_{2n+1} \end{pmatrix} = \frac{\delta H_{2n}}{\delta u} = L\frac{\delta H_{2n-2}}{\delta u} = L\begin{pmatrix} a_{2n-1} \\ b_{2n-1} \end{pmatrix}, \qquad (20)$$

其中哈密顿函数为

$$H_0 = \frac{1}{2}(q^2+r^2),$$

$$H_{2n} = \frac{2c_{2n+2}-qa_{2n+1}-rb_{2n+1}}{2n}, \quad n\geqslant 1. \qquad (21)$$

因此推得方程族(13)或(15)的双-哈密顿结构为

$$u_{t_1} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n = J\frac{\delta H_{2n}}{\delta u} = K\frac{\delta H_{2n-2}}{\delta u}, \qquad (22)$$

特别地,当 $n=1$ 时,得到广义导数非线性 Schrödinger 方程组(14)的双-哈密顿结构为

$$u_{t_n} = \begin{pmatrix} q_{t_1} \\ r_{t_1} \end{pmatrix} = X_1 = J\frac{\delta H_2}{\delta u} = K\frac{\delta H_0}{\delta u},$$

其中哈密顿函数 H_0,H_2 为

$$H_0 = \frac{1}{2}(q^2+r^2),$$

$$H_2 = \frac{1}{4}(rq_x-qr_x)+\frac{1}{8}(q^2+r^2)^2,$$

因为 $JL=K=-K^*=-(JL)^*=-L^*J^*=L^*J$,所以直接计算,可得

$$\begin{aligned}\{H_{2n},H_{2m}\}_J &= \int <\frac{\delta H_{2n}}{\delta u}, J\frac{\delta H_{2m}}{\delta u}> \mathrm{d}x \\ &= \int <\frac{\delta H_{2n}}{\delta u}, JL\frac{\delta H_{2m-2}}{\delta u}> \mathrm{d}x \\ &= \int <\frac{\delta H_{2n}}{\delta u}, L^*J\frac{\delta H_{2m-2}}{\delta u}> \mathrm{d}x \\ &= \int <L\frac{\delta H_{2n}}{\delta u}, J\frac{\delta H_{2m-2}}{\delta u}> \mathrm{d}x \\ &= \int <\frac{\delta H_{2n+2}}{\delta u}, J\frac{\delta H_{2m-2}}{\delta u}> \mathrm{d}x \\ &= \{H_{2n+2},H_{2m-2}\}_J = \cdots \\ &= \{H_{2m},H_{2n}\}_J, \qquad m,n\geqslant 0, \end{aligned} \qquad (23)$$

因此得 $\{H_{2n},H_{2m}\}_J=0$.另外还得到

$$\begin{aligned}[X_n,X_m] &= \left[J\frac{\delta H_{2n}}{\delta u}, J\frac{\delta H_{2m}}{\delta u}\right] \\ &= J\frac{\delta}{\delta u}\{H_{2n},H_{2m}\}_J = 0, \\ & m,n\geqslant 0. \end{aligned} \qquad (24)$$

这表明方程族(15)具有无穷多对合的对称.

因此可得如下结论:

结论 1 (i)方程族(15)在 Liouville 意义下是可积的;(ii)方程族(15)拥有一串守恒密度 $\{H_{2n}\}_{n=0}^{\infty}$,并且这些守恒密度在 Poisson 括号作用下是两两对合的.

情况 2 当 $s=q^2+r^2$,$\Delta_{2n}=2qa_{2n+1,x}+2rb_{2n+1,x}-4q(q^2+r^2)b_{2n+1}+4r(q^2+r^2)a_{2n+1}$ 时,(12)式约化为

$$\begin{aligned}u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = X_n &= J\begin{pmatrix} 2a_{2n+1}-4qc_{2n} \\ 2b_{2n+1}-4rc_{2n} \end{pmatrix} \\ &= K\begin{pmatrix} 2a_{2n-1}-4qc_{2n-2} \\ 2b_{2n-1}-4rc_{2n-2} \end{pmatrix}, \end{aligned} \qquad (25)$$

其中 $J=L_3L_2L_3^*$,$K=L_3L_2L_1L_2L_3^*$,

$$L_1 = \begin{pmatrix} \frac{1}{2}q\partial^{-1}q\partial & -\frac{1}{4}+\frac{1}{2}q\partial^{-1}r\partial \\ \frac{1}{4}+\frac{1}{2}r\partial^{-1}q\partial & \frac{1}{2}r\partial^{-1}r\partial \end{pmatrix},$$

$$L_2 = \begin{pmatrix} 2\partial & -4(q^2+r^2) \\ 4(q^2+r^2) & 2\partial \end{pmatrix},$$

$$L_3 = \begin{pmatrix} \frac{1}{2}+2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & \frac{1}{2}-2q\partial^{-1}r \end{pmatrix},$$

L_1,L_2 为斜对称算子,即 $L_1^*=-L_1$,$L_2^*=-L_2$,因此可证得 J,K 也为斜对称算子,即

$$\begin{aligned}J^* &= (L_3L_2L_3^*)^* = L_3L_2^*L_3^* \\ &= -(L_3L_2L_3^*) = -J, \\ K^* &= (L_3L_2L_1L_2L_3^*)^* = L_3L_2^*L_1^*L_2^*L_3^* \\ &= -(L_3L_2L_1L_2L_3^*) = -K.\end{aligned}$$

类似于情况 1,可得(25)式的双-哈密顿结构为

$$u_{t_n} = \begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = \overline{X}_n = J\frac{\delta\overline{H}_{2n}}{\delta u} = K\frac{\delta\overline{H}_{2n-2}}{\delta u}, \qquad (26)$$

其中哈密顿函数为

$$\overline{H}_0 = \frac{1}{2}(q^2+r^2),$$

$$\overline{H}_2 = \frac{1}{2}(rq_x-qr_x)-\frac{3}{4}(q^2+r^2)^2, \qquad (27a)$$

$$\overline{H}_{2n} = \frac{2c_{2n+2}-qa_{2n+1}-rb_{2n+1}}{n}, \qquad n\geqslant 2. \qquad (27b)$$

又因为 $J(J^{-1}K)=K=-K^{*}=-[J(J^{-1}K)]^{*}=-(J^{-1}K)^{*}J^{*}=(J^{-1}K)^{*}J$，且 $\frac{\delta\overline{H}_{2n}}{\delta u}=J^{-1}K\frac{\delta\overline{H}_{2n-2}}{\delta u}$，所以可证明 $\{\overline{H}_{2n},\overline{H}_{2m}\}_J=0$，并且 $[\overline{X}_n,\overline{X}_m]=0(m,n\geqslant 0)$.

因此有如下结论：

结论 2　(i)方程族(25)在 Liouville 意义下是可积的；(ii)哈密顿函数 $\{\overline{H}_{2n}\}_{n=0}^{\infty}$ 是方程族(25)的一串守恒密度，并且这些守恒密度在 Poisson 括号作用下是两两对合的. 特别地，当 $n=1$ 时，(25)式约化为

$$q_{t_1}=-\frac{1}{2}r_{xx}-\frac{1}{2}[q(q^2+r^2)]_x-(q^2+r^2)q_x-2r(qr_x-rq_x)-\frac{3}{2}r(q^2+r^2)^2, \tag{28a}$$

$$r_{t_1}=\frac{1}{2}q_{xx}-\frac{1}{2}[r(q^2+r^2)]_x-(q^2+r^2)r_x+2q(qr_x-rq_x)+\frac{3}{2}q(q^2+r^2)^2, \tag{28b}$$

其为新的高次广义导数非线性 Schrödinger 方程组. 它的双-哈密顿结构为

$$u_{t_1}=\begin{pmatrix}q_{t_1}\\ r_{t_1}\end{pmatrix}=J\frac{\delta\overline{H}_2}{\delta u}=K\frac{\delta\overline{H}_0}{\delta u}, \tag{29}$$

其中哈密顿函数为(27a)式.

4　结　　论

从一个等谱问题出发，获得一族新的含有任意函数的 Lax 可积方程. 当位势函数 s 取两种特例时，利用迹恒等式，给出了这两个方程组的双-哈密顿结构，并且证明它们是 Liouville 可积的. 另外，当位势函数 $s=f(q^2+r^2)$($f(\cdot)$为任意可微函数)时，可以得到一族方程，进一步或许能够像情况 1 和 2 一样，也能研究它的双-哈密顿结构，这将以后给出.

作者之一(闫振亚)感谢复旦大学范恩贵博士后的指导和帮助.

[1] M. J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transformation (SIAM, Philadephia, 1981).

[2] A. C. Newell, Solitons in Mathematical and Physics (SIAM, Philadephia, 1985).

[3] M. J. Ablowitz, P. A. Clarkson, Solitons, Nonlinear Evolution Equations and the Inverse Scattering Transformation (Cambridge University Press, Cambridge, 1990).

[4] C. S. Gardner *et al.*, *Phys. Rev. Lett.*, **19**(1967), 1095.

[5] M. Ito, *Phys. Lett.*, **A91**(1982), 335.

[6] G. Z. Tu, *J. Math. Phys.*, **30**(1989), 330; *J. Phys.*, **A22**(1989), 2375.

[7] F. Magri, *J. Math. Phys.*, **19**(1978), 1156.

[8] E. Fan, H. Zhang, *J. Math. Phys.*, **41**(2000), 2058.

NEW LAX INTEGRABLE HIERARCHY OF EVOLUTION EQUATIONS AND ITS INFINITE-DIMENSIONAL BI-HAMILTONIAN STRUCTURE*

YAN ZHEN-YA ZHANG HONG-QING
(*Department of Applied Mathematics*, *Dalian University of Technology*, *Dalian* 116024, *China*)
(Received 13 November 2000)

ABSTRACT

Based on a new isospectral problem with three potential functions (q, r, s), a new Lax integrable hierarchy of evolution equations with an arbitrary function is obtained in this paper. When the potential, s, is put into differential functions, the hierarchy of equations can reduce to several kinds of systems of equations. By using the trace identity, their bi-Hamiltonian structures are given, and it is shown that they are integrable in the Liouville's sense. Moreover, the conserved densities and symmetries are also found.

Keywords: isospectral problem, Hamiltonian structure, Lax integrable, Liouville integrable
PACC: 0545, 0340

* Project supported by the State Key Development Program for Basic Research of China (Grant No. G1998030600), and the National Natural Science Foundation of China (Grant No. 10072013).