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非完整转动相对论系统的 Lindelöf 方程*

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由转动相对论系统的 Hamilton 原理分别建立在广义坐标和准坐标下的 Lindelöf 方程及其改进形式，并从改进的 Lindelöf 方程导出新 Chaplygin 方程. 最后说明由转动系统的相对论分析力学向普通分析力学过渡的方法.

关键词：非完整约束，转动系统，相对论，Lindelöf 方程，Chaplygin 方程

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1 引 言

自旋运动是微观粒子的固有属性. 1979 年，Bengtsson 和 Frauendorf 对 14 种核子的自旋转速最大值做了较为精确的测量，结果表明各核子的自旋转速均有一最大值，且各不相同[1]. 随着科学与技术的进步，证实了越来越多的实验现象与高速转动问题有关. 1985 年，Carmeli 提出了转动相对论[2]，引起了人们的关注，近年来已取得了一些重要的研究成果[3—7].

Lindelöf 方程是非理想非完整系统的动力学方程[8,9]，它比附加 Chetaev 条件的著名 Chaplygin 方程更具有普遍意义，且方程结构形式简单，容易运算. 对相对论分析力学来说，Lindelöf 方程理论是非常有发展前途的. 但目前在此领域内的研究成果还不多. 本文由转动相对论系统的 Hamilton 原理分别建立在广义坐标和准坐标下的 Lindelöf 方程及其改进形式，并从改进的 Lindelöf 方程导出新的 Chaplygin 方程. 最后说明由转动系统的相对论分析力学向普通分析力学过渡的方法.

2 转动相对论系统的 Hamilton 原理

设有 N 个物体构成的力学系统，绕坐标系 $A(oxyz)$ 的 oz 轴转动，在 t 时刻第 i 个物体受到的外力对 oz 轴之矩为 M_i，经典转动惯量为 I_{oi}，相对于 A 系的角坐标为 θ_i，极限角速度 Γ_i. 该系统的位形由 n 个广义坐标 $q_1, q_2, \cdots, q_n$ 确定，第 i 个物体的角坐标可表示为

$$\theta_i = \theta_i(q_1, q_2, \cdots, q_n, t). \tag{1}$$

于是容易得到

$$\delta\theta_i = \sum_{s=1}^{n} \frac{\partial\theta_i}{\partial q_s}\delta q_s, \tag{2}$$

$$\frac{\partial\ddot{\theta}_i}{\partial\ddot{q}_s} = \frac{\partial\dot{\theta}_i}{\partial\dot{q}_s} = \frac{\partial\theta_i}{\partial q_s}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\theta_i}{\partial q_s}\right) = \frac{\partial\dot{\theta}_i}{\partial q_s}. \tag{3}$$

转动相对论力学的动能函数[4]为

$$T_r^* = \sum_{i=1}^{N} I_{oi}\Gamma_i^2\left(1 - \sqrt{1 - \dot{\theta}_i^2/\Gamma_i^2}\right). \tag{4}$$

根据转动系统相对论性基本形式的 d'Alembert 原理

$$\sum_{i=1}^{N}\left[-M_i + \frac{\mathrm{d}}{\mathrm{d}t}(I_i\dot{\theta}_i)\right]\delta\theta_i = 0,$$

$$\left[I_i = \frac{I_{oi}}{\sqrt{1 - \dot{\theta}_i^2/\Gamma_i^2}}\right], \tag{5}$$

我们可得到转动相对论保守系统的 Hamilton 原理

$$\delta\int_{t_0}^{t_1}(T_r^* + U)\mathrm{d}t = 0. \tag{6}$$

3 广义坐标下的 Lindelöf 方程

设有一转动相对论系统，其位形由 n 个广义坐标 $q_1, q_2, \cdots, q_n$ 确定. 该系统由动能 T_r^*、力函数 U 及线性齐次稳定的非完整约束

$$\sum_{s=1}^{n} A_{\beta s}\dot{q}_s = 0, \quad (\beta = 1, 2, \cdots, g) \tag{7}$$

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来描述.

约束方程(7)又可表示为

$$\dot{q}_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\dot{q}_{\sigma},\tag{8}$$

$(\beta=1,2,\cdots,g;\varepsilon=n-g)$.

此处 $A_{\beta s}$, $B_{\beta\sigma}$, T_r^* 和 U 均与不独立的广义坐标 $q_{\varepsilon+\beta}$ 无关.

约束(8)取变分,得

$$\delta\dot{q}_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta\dot{q}_{\sigma}+\sum_{\sigma=1}^{\varepsilon}\dot{q}_{\sigma}\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\sigma}}{\partial q_{\gamma}}\delta q_{\gamma},\tag{9}$$

引入 Lagrange 乘子 λ_β,将(9)式纳入泛函(6)中,有

$$\int_{t_0}^{t_1}\Big[\delta(T_r^*+U)+\sum_{\beta=1}^{g}\lambda_\beta\Big(\delta\dot{q}_{\varepsilon+\beta}-\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta\dot{q}_{\sigma}-\sum_{\sigma=1}^{\varepsilon}\dot{q}_{\sigma}\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\sigma}}{\partial q_{\gamma}}\delta q_{\gamma}\Big)\Big]\mathrm{d}t=0.\tag{10}$$

现用变分运算简化上式,由于

$$\begin{aligned}&\int_{t_0}^{t_1}\Big[\delta(T_r^*+U)+\sum_{\beta=1}^{g}\lambda_\beta\Big(\delta\dot{q}_{\varepsilon+\beta}-\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta\dot{q}_{\sigma}-\sum_{\sigma=1}^{\varepsilon}\dot{q}_{\sigma}\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\sigma}}{\partial q_{\gamma}}\delta q_{\gamma}\Big)\Big]\mathrm{d}t\\
&=\int_{t_0}^{t_1}\Big\{\sum_{\sigma=1}^{\varepsilon}\Big[\frac{\partial T_r^*}{\partial q_\sigma}+\frac{\partial U}{\partial q_\sigma}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_\sigma}+\sum_{\beta=1}^{g}\dot{\lambda}_\beta B_{\beta\sigma}\\
&\quad+\sum_{\beta=1}^{g}\lambda_\beta\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\dot{q}_\gamma-\sum_{\beta=1}^{g}\lambda_\beta\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}\dot{q}_\gamma\Big]\delta q_\sigma\\
&\quad-\sum_{\beta=1}^{g}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}+\dot{\lambda}_\beta\Big)\delta q_{\varepsilon+\beta}\Big\}\mathrm{d}t\\
&\quad+\sum_{\beta=1}^{g}\Big(\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}+\lambda_\beta\Big)\delta q_{\varepsilon+\beta}\Big|_{t_0}^{t_1}\\
&\quad+\sum_{\sigma=1}^{\varepsilon}\Big(\sum_{\beta=1}^{g}\lambda_\beta B_{\beta\sigma}+\frac{\partial T_r^*}{\partial\dot{q}_\sigma}\Big)\delta q_\sigma\Big|_{t_0}^{t_1}=0.\end{aligned}\tag{11}$$

假设端点条件

$$\begin{aligned}&\delta q_\sigma|_{t=t_0}=\delta q_\sigma|_{t=t_1}=0,\\&\delta q_{\varepsilon+\beta}|_{t=t_0}=\delta q_{\varepsilon+\beta}|_{t=t_1}=0.\end{aligned}\tag{12}$$

由于引入 Lagrange 乘子,使得 δq_σ, $\delta q_{\varepsilon+\beta}$ 相互独立,故由(11)式可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_\sigma}-\frac{\partial T_r^*}{\partial q_\sigma}-\frac{\partial U}{\partial q_\sigma}+\sum_{\beta=1}^{g}\lambda_\beta\Big(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}-\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\Big)\dot{q}_\gamma-\sum_{\beta=1}^{g}\dot{\lambda}_\beta B_{\beta\sigma}=0,\tag{13a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}+\dot{\lambda}_\beta=0,\tag{13b}$$

及

$$\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}+\lambda_\beta=0.\tag{13c}$$

(对于 $t=t_0$ 和 $t=t_1$)

联立解(13b)和(13c),得到

$$\lambda_\beta=-\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}},\tag{14}$$

将(13b)和(14)代入(13a),有

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_\sigma}-\frac{\partial T_r^*}{\partial q_\sigma}-\frac{\partial U}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}\cdot\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}-\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}\Big)\dot{q}_\gamma+\sum_{\beta=1}^{g}B_{\beta\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}=0.\tag{15}$$

现继续变换(15)式

令 $\tilde{T}_r^*(q_\sigma,\dot{q}_\sigma)=T_r^*\Big((q_\sigma,\dot{q}_\sigma,\sum_{\gamma=1}^{\varepsilon}B_{\beta\gamma}\dot{q}_\gamma\Big)$,有

$$\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_\sigma}=\frac{\partial T_r^*}{\partial\dot{q}_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}B_{\beta\sigma},\tag{16a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_\sigma}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_\sigma}+\sum_{\beta=1}^{g}B_{\beta\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}+\sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\dot{q}_\gamma,\tag{16b}$$

$$\begin{aligned}\frac{\partial\tilde{T}_r^*}{\partial q_\sigma}&=\frac{\partial T_r^*}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}\frac{\partial\dot{q}_{\varepsilon+\beta}}{\partial q_\sigma}\\&=\frac{\partial T_r^*}{\partial q_\sigma}+\sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial\dot{q}_{\varepsilon+\beta}}\sum_{\gamma=1}^{\varepsilon}\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}\dot{q}_\gamma,\end{aligned}\tag{16c}$$

将(16b)和(16c)代入(15)式,则得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{T}_r^*}{\partial q_\sigma}-\frac{\partial U}{\partial q_\sigma}=0,\tag{17}$$

$(\sigma=1,2,\cdots,\varepsilon)$.

(17)式就是广义坐标下非完整转动相对论系统的 Lindelöf 方程.

由(13)式可见,设系统的广义约束力可表示为

$$R_\sigma=-\sum_{\beta=1}^{g}\dot{\lambda}_\beta B_{\beta\sigma}+\sum_{\beta=1}^{g}\lambda_\beta\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}-\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\Big)\dot{q}_\gamma,\tag{18a}$$

$$R_{\varepsilon+\beta}=\dot{\lambda}_\beta.\tag{18b}$$

广义约束力在相应的广义虚位移上所做的虚功为

$$\begin{aligned}&\sum_{\sigma=1}^{\varepsilon}R_\sigma\delta q_\sigma+\sum_{\beta=1}^{g}R_{\varepsilon+\beta}\delta q_{\varepsilon+\beta}\\&=\sum_{\sigma=1}^{\varepsilon}\Big[-\sum_{\beta=1}^{g}\dot{\lambda}_\beta B_{\beta\sigma}+\sum_{\beta=1}^{g}\lambda_\beta\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}-\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\Big)\dot{q}_\gamma\Big]\delta q_\sigma+\sum_{\beta=1}^{g}\dot{\lambda}_\beta\delta q_{\varepsilon+\beta},\end{aligned}\tag{19}$$

由(9)式可得

$$\delta\dot{q}_{\varepsilon+\beta}-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma\Big)=\sum_{\gamma=1}^{\varepsilon}\dot{q}_\gamma\sum_{\sigma=1}^{\varepsilon}\Big(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}-\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\Big)\delta q_\sigma. \tag{20}$$

将(20)式代入(19)式,得

$$\sum_{\sigma=1}^{\varepsilon}R_\sigma\delta q_\sigma+\sum_{\beta=1}^{g}R_{\varepsilon+\beta}\delta q_{\varepsilon+\beta}=\sum_{\beta=1}^{g}\frac{\mathrm{d}}{\mathrm{d}t}\lambda_\beta\Big(\delta q_{\varepsilon+\beta}-\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma\Big), \tag{21}$$

一般说来,

$$\sum_{\sigma=1}^{\varepsilon}R_\sigma\delta q_\sigma+\sum_{\beta=1}^{g}R_{\varepsilon+\beta}\delta q_{\varepsilon+\beta}\neq 0. \tag{22}$$

这类约束称为非理想约束,所以,Lindelöf 方程(17)是广义坐标下非理想非完整转动相对论系统的动力学方程.

为使非理想约束(9)理想化,现令

$$\frac{\partial B_{\beta\gamma}}{\partial q_\sigma}-\frac{\partial B_{\beta\sigma}}{\partial q_\gamma}=0, \tag{23}$$

则由(20)式便得

$$\delta\dot{q}_{\varepsilon+\beta}-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma\Big)=0, \tag{24}$$

积分上式,有

$$\delta q_{\varepsilon+\beta}=\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma. \tag{25}$$

(25)式就是理想约束,由此可知,(23)式是使非理想约束(9)理想化的必要条件,这与文献[7]所得结果一致.

4　准坐标下的 Lindelöf 方程

设转动相对论系统受有 g 个形式如(8)式的非完整约束,借助下列线性方程引入准速度

$$\omega_\mu=\sum_{\sigma=1}^{\varepsilon}a_{\mu\sigma}\dot{q}_\sigma,\quad(\mu=1,2,\cdots,\varepsilon). \tag{26}$$

利用(26)及约束方程(8)可将所有广义速度 $\dot{q}_1,\dot{q}_2,\cdots,\dot{q}_n$ 用准速度 ω_μ 表出

$$\dot{q}_s=\sum_{\mu=1}^{\varepsilon}b_{s\mu}\omega_\mu,\ (s=1,2,\cdots,n), \tag{27a}$$

因此有

$$\delta q_s=\sum_{\mu=1}^{\varepsilon}b_{s\mu}\delta\pi_\mu. \tag{27b}$$

此处 $a_{\mu\sigma}$, $b_{s\mu}$, T_r^* 及力函数 U 均与不独立的广义坐标 $q_{\varepsilon+\beta}$ 无关.

将约束(27a)变分,可得

$$\delta\dot{q}_s=\sum_{\mu=1}^{\varepsilon}b_{s\mu}\delta\dot{\pi}_\mu+\sum_{\mu=1}^{\varepsilon}\omega_\mu\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}\sum_{\gamma=1}^{\varepsilon}b_{\sigma\gamma}\delta\pi_\gamma, \tag{28}$$

引入 Lagrange 乘子 λ_s^*,将(28)式纳入泛函(6)中,有

$$\int_{t_0}^{t_1}\Big\{\delta(T_r^*+U)+\sum_{s=1}^{n}\lambda_s^*\Big(\delta\dot{q}_s-\sum_{\mu=1}^{\varepsilon}b_{s\mu}\delta\dot{\pi}_\mu-\sum_{\mu=1}^{\varepsilon}\omega_\mu\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}\sum_{\gamma=1}^{\varepsilon}b_{\sigma\gamma}\delta\pi_\gamma\Big\}\mathrm{d}t=0. \tag{29}$$

现用变分运算来简化上式,并注意(27b)式得

$$\begin{aligned}&\int_{t_0}^{t_1}\Big\{\delta(T_r^*+U)+\sum_{s=1}^{n}\lambda_s^*\Big(\delta\dot{q}_s-\sum_{\mu=1}^{\varepsilon}b_{s\mu}\delta\dot{\pi}_\mu-\sum_{\mu=1}^{\varepsilon}\omega_\mu\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}\sum_{\gamma=1}^{\varepsilon}b_{\sigma\gamma}\delta\pi_\gamma\Big\}\mathrm{d}t\\
&=\int_{t_0}^{t_1}\Big\{\sum_{\mu=1}^{\varepsilon}\Big[\sum_{\sigma=1}^{\varepsilon}\frac{\partial T_r^*}{\partial q_\sigma}b_{\sigma\mu}+\sum_{\sigma=1}^{\varepsilon}\frac{\partial U}{\partial q_\sigma}b_{\sigma\mu}\\
&\quad+\sum_{s=1}^{n}\dot{\lambda}_s^*b_{s\mu}+\sum_{s=1}^{n}\lambda_s^*\sum_{\gamma=1}^{\varepsilon}\Big(\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}b_{\sigma\gamma}\\
&\quad-\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}b_{\sigma\mu}\Big)\omega_\gamma\Big]\delta\pi_\mu\\
&\quad+\sum_{s=1}^{n}\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_s}-\dot{\lambda}_s^*\Big)\delta q_s\Big\}\mathrm{d}t\\
&\quad+\sum_{s=1}^{n}\Big(\frac{\partial T_r^*}{\partial\dot{q}_s}+\lambda_s^*\Big)\delta q_s\Big|_{t_0}^{t_1}\\
&\quad-\sum_{s=1}^{n}\lambda_s^*\sum_{\mu=1}^{\varepsilon}b_{s\mu}\delta\pi_\mu\Big|_{t_0}^{t_1}=0,\end{aligned} \tag{30}$$

假设端点条件

$$\delta\pi_\mu\mid_{t=t_0}=\delta\pi_\mu\mid_{t=t_1}=0, \tag{31a}$$

考虑(27b)式,有

$$\delta q_s\mid_{t=t_0}=\delta q_s\mid_{t=t_1}=0, \tag{31b}$$

由于引入 Lagrange 乘子,使得 δq_s, $\delta\pi_\mu$ 相互独立,于是,由(30)式可得

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial T_r^*}{\partial\dot{q}_\sigma}b_{\sigma\mu}+\sum_{\sigma=1}^{\varepsilon}\frac{\partial U}{\partial q_\sigma}b_{\sigma\mu}+\sum_{s=1}^{n}\lambda_s^*b_{s\mu}+\sum_{s=1}^{n}\lambda_s^*\sum_{\gamma=1}^{\varepsilon}\Big(\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}b_{s\gamma}-\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}b_{\sigma\mu}\Big)\omega_\gamma=0, \tag{32a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_s}+\lambda_s^*=0, \tag{32b}$$

及

$$\frac{\partial T_r^*}{\partial\dot{q}_s}+\lambda_s^*=0, \tag{32c}$$

(对于 $t=t_0$ 及 $t=t_1$).

由(32b)和(32c)式得到

$$\lambda_s^* = -\frac{\partial T_r^*}{\partial \dot{q}_s}, \tag{33}$$

将(32b)和(33)式代入(32a)式,可有

$$\sum_{s=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial \dot{q}_s}b_{s\mu} - \sum_{\sigma=1}^{\varepsilon}\frac{\partial T_r^*}{\partial q_\sigma}b_{\sigma\mu} - \sum_{\sigma=1}^{\varepsilon}\frac{\partial U}{\partial q_\sigma}b_{\sigma\mu}$$
$$+ \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial \dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\left(\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}b_{\sigma\gamma} - \sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}b_{\sigma\mu}\right)\omega_\gamma$$
$$= 0. \tag{34}$$

现继续变换(34)式,

令 $\tilde{T}_r^*(q_\sigma,\omega_\gamma) = T_r^*\left(q_\sigma,\sum_{\mu=1}^{\varepsilon}b_{s\mu}\omega_\mu\right)$,

有

$$\frac{\partial \tilde{T}_r^*}{\partial q_\sigma} = \frac{\partial T_r^*}{\partial q_\sigma} + \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial \dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}\omega_\gamma, \tag{35}$$

所以

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial T_r^*}{\partial q_\sigma}b_{\sigma\mu} = \sum_{\sigma=1}^{\varepsilon}\frac{\partial \tilde{T}_r^*}{\partial q_\sigma}b_{\sigma\mu} - \sum_{\sigma=1}^{\varepsilon}\left(\sum_{s=1}^{n}\frac{\partial T_r^*}{\partial \dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}\omega_\gamma\right)b_{\sigma\mu}. \tag{36}$$

又

$$\frac{\partial \tilde{T}_r^*}{\partial \omega_\mu} = \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial \dot{q}_s}b_{s\mu},\quad \frac{\partial}{\partial \pi_\mu} = \sum_{\sigma=1}^{\varepsilon}\frac{\partial}{\partial q_\sigma}b_{\sigma\mu}. \tag{37}$$

所以由(37)式的第一式,得

$$\sum_{s=1}^{n}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_r^*}{\partial \dot{q}_s}\right)b_{s\mu}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \omega_\mu} - \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial \dot{q}_s}\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}\sum_{\gamma=1}^{\varepsilon}b_{\sigma\gamma}\omega_\gamma. \tag{38}$$

由对准坐标的偏导数定义,有

$$\sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\mu}}{\partial q_\sigma}b_{\sigma\gamma} = \frac{\partial b_{s\mu}}{\partial \pi_\gamma},\quad \sum_{\sigma=1}^{\varepsilon}\frac{\partial b_{s\gamma}}{\partial q_\sigma}b_{\sigma\mu} = \frac{\partial b_{s\gamma}}{\partial \pi_\mu}. \tag{39}$$

将(36)和(38)式代入(34)式,并考虑(39)式,便得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \omega_\mu} - \frac{\partial \tilde{T}_r^*}{\partial \pi_\mu} - \frac{\partial U}{\partial \pi_\mu} = 0, \tag{40}$$
$$(\mu = 1,2,\cdots,\varepsilon).$$

(40)式就是准坐标下非完整转动相对论系统的 Lindelöf 方程.

5 广义坐标下 Lindelöf 方程的改进形式

我们将约束方程(8)变分,稍加变换,可得

$$\delta\dot{q}_{\varepsilon+\beta} = \sum_{\gamma=1}^{\varepsilon}\dot{q}_\gamma\sum_{\sigma=1}^{\varepsilon}\left(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma} - \frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\right)\delta q_\sigma$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma\right), \tag{41}$$

将 Hamilton 原理的表达式(6)写成展开形式,有

$$\delta\int_{t_0}^{t_1}(T_r^* + U)\mathrm{d}t$$
$$= \int_{t_0}^{t_1}\left\{\sum_{\sigma=1}^{\varepsilon}\left[\frac{\partial T_r^*}{\partial q_\sigma}\delta q_\sigma + \frac{\partial T_r^*}{\partial \dot{q}_\sigma}\delta\dot{q}_\sigma + \frac{\partial U}{\partial q_\sigma}\delta q_\sigma\right]\right.$$
$$\left.+ \sum_{\beta=1}^{g}\frac{\partial T_r^*}{\partial \dot{q}_{\varepsilon+\beta}}\delta\dot{q}_{\varepsilon+\beta}\right\}\mathrm{d}t = 0. \tag{42}$$

现将(8)和(41)式代入(42)式,并在时域边界取 $\delta q_\sigma = 0$,可得

$$\int_{t_0}^{t_1}\sum_{\sigma=1}^{\varepsilon}\left\{\frac{\partial \tilde{T}_r^*}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_\sigma} + \frac{\partial U}{\partial q_\sigma} + \sum_{\beta=1}^{g}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}}\right.$$
$$\left.\cdot\sum_{\gamma=1}^{\varepsilon}\left(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma} - \frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\right)\dot{q}_\gamma - \sum_{\beta=1}^{g}B_{\beta\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}}\right\}\delta q_\sigma\mathrm{d}t$$
$$+ \sum_{\beta=1}^{g}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}}\sum_{\sigma=1}^{\varepsilon}B_{\beta\sigma}\delta q_\sigma\Big|_{t_0}^{t_1} = 0, \tag{43}$$

其中$\frac{\partial \tilde{T}_r^*}{\partial q_\sigma}$,$\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_\sigma}$和$\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}}$分别为将(8)式代入$\frac{\partial T_r^*}{\partial q_\sigma}$,$\frac{\partial T_r^*}{\partial \dot{q}_\sigma}$和$\frac{\partial T_r^*}{\partial \dot{q}_{\varepsilon+\beta}}$后的表达式,由于 δq_σ 的任意性,由(43)式得

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_\sigma} - \frac{\partial \tilde{T}_r^*}{\partial q_\sigma} - \frac{\partial U}{\partial q_\sigma}$$
$$- \sum_{\beta=1}^{g}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}}\sum_{\gamma=1}^{\varepsilon}\left(\frac{\partial B_{\beta\gamma}}{\partial q_\sigma} - \frac{\partial B_{\beta\sigma}}{\partial q_\gamma}\right)\dot{q}_\gamma$$
$$+ \sum_{\beta=1}^{g}B_{\beta\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}} = 0, \tag{44}$$

这就是广义坐标下 Lindelöf 方程的改进形式.

若$\frac{\partial B_{\beta\gamma}}{\partial q_\sigma} - \frac{\partial B_{\beta\sigma}}{\partial q_\gamma} = 0$,则(44)式成为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_\sigma} - \frac{\partial \tilde{T}_r^*}{\partial q_\sigma} - \frac{\partial U}{\partial q_\sigma} + \sum_{\beta=1}^{g}B_{\beta\sigma}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{T}_r^*}{\partial \dot{q}_{\varepsilon+\beta}} = 0. \tag{45}$$

(45)式就是广义坐标下 Chaplygin 方程的改进形式.

6 准坐标下 Lindelöf 方程的改进形式

将约束(27a)取变分,稍加变换,可得

$$\delta\dot{q}_s = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{\mu=1}^{\varepsilon} b_{s\mu}\delta\pi_\mu\Big) + \sum_{\gamma=1}^{\varepsilon}\omega_\gamma\sum_{\mu=1}^{\varepsilon}\Big(\frac{\partial b_{s\gamma}}{\partial\pi_\mu} - \frac{\partial b_{s\mu}}{\partial\pi_\gamma}\Big)\delta\pi_\mu, \tag{46}$$

将 Hamilton 原理的表达式(6)写成展开形式,有

$$\delta\int_{t_0}^{t_1}(T_r^* + U)\mathrm{d}t = \int_{t_0}^{t_1}\Big(\sum_{\sigma=1}^{\varepsilon}\frac{\partial T_r^*}{\partial q_\sigma}\delta q_\sigma + \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial\dot{q}_s}\delta\dot{q}_s + \sum_{\sigma=1}^{\varepsilon}\frac{\partial U}{\partial q_\sigma}\delta q_\sigma\Big)\mathrm{d}t = 0. \tag{47}$$

现将(46)式代入(47)式,经分部积分法,并在时域边界处取 $\delta\pi_\mu = 0$,可得

$$\int_{t_0}^{t_1}\sum_{\mu=1}^{\varepsilon}\Big\{\frac{\partial T_r^*}{\partial\pi_\mu} + \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial\dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial b_{s\gamma}}{\partial\pi_\mu} - \frac{\partial b_{s\mu}}{\partial\pi_\gamma}\Big)\omega_\gamma + \frac{\partial U}{\partial\pi_\mu} - \sum_{s=1}^{n} b_{s\mu}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_s}\Big\}\delta\pi_\mu\mathrm{d}t + \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial\dot{q}_s}\sum_{\mu=1}^{\varepsilon} b_{s\mu}\delta\pi_\mu\Big|_{t_0}^{t_1} = 0. \tag{48}$$

由于 $\delta\pi_\mu$ 的任意性,由(48)式得

$$\frac{\partial T_r^*}{\partial\pi_\mu} + \sum_{s=1}^{n}\frac{\partial T_r^*}{\partial\dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial b_{s\gamma}}{\partial\pi_\mu} - \frac{\partial b_{s\mu}}{\partial\pi_\gamma}\Big)\omega_\gamma + \frac{\partial U}{\partial\pi_\mu} - \sum_{s=1}^{n} b_{s\mu}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_s} = 0, \tag{49}$$

将(27a)式代入上式,则有

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{T}_r^*}{\partial\omega_\mu} - \frac{\partial\tilde{T}_s^*}{\partial\pi_\mu} - \frac{\partial U}{\partial\pi_\mu} - \sum_{s=1}^{n}\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_s}\sum_{\gamma=1}^{\varepsilon}\Big(\frac{\partial b_{s\gamma}}{\partial\pi_\mu} - \frac{\partial b_{s\mu}}{\partial\pi_\gamma}\Big)\omega_\gamma - \sum_{s=1}^{n}\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_s}\tilde{\dot{b}}_{s\mu} = 0. \tag{50}$$

其中 $\frac{\partial\tilde{T}_r^*}{\partial\omega_\mu}$,$\frac{\partial\tilde{T}_r^*}{\partial\pi_\mu}$ 和 $\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_s}$ 分别为将(27a)式代入 $\frac{\partial T_r^*}{\partial\omega_\mu}$,$\frac{\partial T_r^*}{\partial\pi_\mu}$ 和 $\frac{\partial T_r^*}{\partial\dot{q}_s}$ 后的表达式,(50)式就是准坐标下 Lindelöf 方程的改进形式.

若 $\frac{\partial b_{s\gamma}}{\partial\pi_\mu} - \frac{\partial b_{s\mu}}{\partial\pi_\gamma} = 0$,则(50)式成为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{T}_r^*}{\partial\omega_\mu} - \frac{\partial\tilde{T}_r^*}{\partial\pi_\mu} - \frac{\partial U}{\partial\pi_\mu} - \sum_{s=1}^{n}\frac{\partial\tilde{T}_r^*}{\partial\dot{q}_s}\tilde{\dot{b}}_{s\mu} = 0. \tag{51}$$

(51)式就是准坐标下 Chaplygin 方程的改进形式.

由上可知,Chaplygin 方程(45),(51)分别是 Lindelöf 方程(44),(50)的特例.

7 讨　　论

在 $\dot{\theta}_i \ll \Gamma_i$ 的经典近似下,转动惯量

$$I_i = \frac{I_{0i}}{\sqrt{1 - \dot{\theta}_i^2/\Gamma_i^2}} \approx I_{0i},$$

取 $\sqrt{1 - \dot{\theta}_i^2/\Gamma_i^2}$ 关于 $\dot{\theta}_i/\Gamma_i$ 幂级数展开式的前两项,转动相对论系统的动能函数,

$$T_r^* \approx \sum_{i=1}^{N} I_{0i}\Gamma_i^2 - \sum_{i=1}^{N} I_{0i}\Gamma_i^2\Big(1 - \frac{\dot{\theta}_i^2}{2\Gamma_i^2}\Big) \approx \frac{1}{2}\sum_{i=1}^{N} I_{0i}\dot{\theta}_i^2 = T_r.$$

化为经典转动系统的动能函数,随之,本文的理论蜕化为经典转动系统力学理论.

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LINDELöF'S EQUATIONS OF NONHOLONOMIC ROTATIONAL RELATIVISTIC SYSTEMS*

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ABSTRACT

In this paper, the Lindelöf's equations of nonholonomic rotational relativistic systems are studied. First, the quasi-velocities of nonholonomic systems and the Hamilton's principle of rotational relativistic systems are introduced. Next, the Lindelöf's equations and their improvable expressions in terms of generalized coordinats and quasi-corrdinates are obtained by using Hamilton's principle. Finally, by means of inprovable Lindelöf's equations, the new form of Chaplygin's equations is derived and the transitional method from relativistic analytical mechanics of the rotational systems to general analytical mechanics is illustrated.

Keywords: nonholonomic constraint, rotational system, relativity, Lindelöf's equation, Chaplygin's equation
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