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约束哈密顿系统在相空间中的精确不变量与绝热不变量

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研究小干扰力作用下约束哈密顿系统对称性的摄动问题.建立了非保守约束哈密顿系统的正则方程,在增广相空间中研究了系统的对称性与精确不变量.基于力学系统的高阶绝热不变量的概念,给出了系统的各阶绝热不变量的形式及存在条件,并建立了绝热不变量与对称变换之间的对应关系.

关键词:约束哈密顿系统,对称性,摄动,不变量

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1. 引 言

用奇异 Lagrange 函数描述的系统,在过渡到相空间中用正则变量描述时,其正则变量间存在固有约束,称之为约束哈密顿系统.由于众多重要的物理系统均属于该系统,例如,描述自然界基本相互作用中的量子电动力学(QED)、量子味动力学(QFD,弱电统一理论)、量子色动力学(QCD,强作用理论)和广义相对论(GR,引力理论),以及超对称、超引力和超弦理论中的 Lagrange 函数均是奇异的,因此约束哈密顿系统的基本理论在理论物理中,特别是现代量子场论中,占有十分重要的地位[1,2].

对称性是力学系统非常重要而又普遍的性质[3—10],小扰动作用下对称性的改变及其不变量与力学系统的可积性密切相关[11],因此对各种系统的对称性的摄动和绝热不变量的研究引起了人们的重视,并已取得了一些重要成果[12—16].

本文研究有限自由度约束哈密顿系统在相空间中的精确不变量和绝热不变量问题,建立了系统的对称性与精确不变量之间的关系.基于力学系统受到小干扰力作用下的高阶绝热不变量的概念,研究给出了约束哈密顿系统在相空间中的高阶绝热不变量,揭示了系统的绝热不变量与无限小对称变换群摄动之间的正反关系.

2. 系统的正则方程

考虑有限自由度系统.假设系统的位形由 n 个广义坐标 q_s($s=1,\cdots,n$)来确定,所受的非势广义力分量为 $Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$,系统的正则动量和正则哈密顿函数定义为

$$p_s=\frac{\partial L}{\partial \dot{q}_s}\quad (s=1,2,\cdots,n),\tag{1}$$

$$H=p_s\dot{q}_s-L.\tag{2}$$

考虑系统的 Lagrange 函数 $L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 是奇异的,即 Hess 矩阵($\partial^2L/\partial\dot{q}_s\partial\dot{q}_l$)的秩 $r<n$.根据隐函数存在定理,设从(1)式可解出 r 个 $\dot{q}_\sigma$ 作为 t,q_s,p_α 和剩下的 $\dot{q}_\rho$ 的函数[1]

$$\dot{q}_\sigma=f_\sigma(t,q_s,p_\alpha,\dot{q}_\rho)\quad (s=1,\cdots,n;\sigma,\alpha=1,\cdots,r;\rho=r+1,\cdots,n).\tag{3}$$

将(3)式代入(1)式,得

$$p_s=g_s(t,q_l,p_\alpha,\dot{q}_\rho)\quad (s,l=1,\cdots,n;\alpha=1,\cdots,r;\rho=r+1,\cdots,n).\tag{4}$$

显然,$g_\alpha=p_\alpha$($\alpha=1,\cdots,r$),而其他 $n-r$ 个 g 将不依赖于 $\dot{q}_\rho$,否则将能解出更多的 $\dot{q}$.于是得到广义坐标和广义动量之间的 $n-r$ 个约束关系[1]

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$$\phi_{\rho-r} = p_\rho - g_\rho(t, q_s, p_\alpha) = 0$$
$$(\rho = r+1, \cdots, n). \tag{5}$$

由(5)式,系统的哈密顿函数(2)式可写为

$$H(t, q_s, p_\alpha, \dot{q}_\rho) = p_\sigma \dot{q}_\sigma + p_\rho \dot{q}_\rho - L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$$
$$= p_\sigma f_\sigma + g_\rho(t, q_s, p_\alpha)\dot{q}_\rho$$
$$- L(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{6}$$

由(6)式,并考虑到广义动量的定义式(1),易知

$$\frac{\partial H}{\partial \dot{q}_\rho} = 0, \quad \frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma + \frac{\partial g_\rho}{\partial p_\sigma}\dot{q}_\rho,$$
$$\frac{\partial H}{\partial q_s} = -\frac{\partial L}{\partial q_s} + \frac{\partial g_\rho}{\partial q_s}\dot{q}_\rho. \tag{7}$$

根据 Euler-Lagrange 方程,有

$$\dot{p}_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} = \frac{\partial L}{\partial q_s} + Q_s, \tag{8}$$

其中 Q_s 为非势广义力.将(8)式代入(7)式,有[1]

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} - \dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma},$$
$$\dot{p}_s = -\frac{\partial H}{\partial q_s} + \dot{q}_\rho\frac{\partial g_\rho}{\partial q_s} + Q_s$$
$$(\sigma = 1, \cdots, r; \rho = r+1, \cdots, n; s = 1, \cdots, n). \tag{9}$$

(9)式为非保守约束哈密顿系统的正则方程,$\dot{q}_\rho(t)$ 相应于从动量定义式中未能解出的那些 $\dot{q}_\rho(t)$.

3. 系统的对称性与精确不变量

引进时间、广义坐标和广义动量的无限小群变换

$$t^* = t + \Delta t = t + \varepsilon\tau^0(t, \boldsymbol{q}, \boldsymbol{p}),$$
$$q_s^*(t^*) = q_s(t) + \Delta q_s = q_s(t) + \varepsilon\xi_s^0(t, \boldsymbol{q}, \boldsymbol{p}),$$
$$p_s^*(t^*) = p_s(t) + \Delta p_s = p_s(t) + \varepsilon\eta_s^0(t, \boldsymbol{q}, \boldsymbol{p})$$
$$(s = 1, \cdots, n), \tag{10}$$

其中 ε 为无限小参数,$\tau^0, \xi_s^0, \eta_s^0$ 称为无限小变换的生成元.取无限小生成元向量

$$X_0^{(0)} = \tau^0\frac{\partial}{\partial t} + \xi_s^0\frac{\partial}{\partial q_s} + \eta_s^0\frac{\partial}{\partial p_s}, \tag{11}$$

及其一次扩展为

$$X_0^{(1)} = X_0^{(0)} + (\dot{\xi}_s^0 - \dot{q}_s\dot{\tau}^0)\frac{\partial}{\partial \dot{q}_s}$$
$$+ (\dot{\eta}_s^0 - \dot{p}_s\dot{\tau}^0)\frac{\partial}{\partial \dot{p}_s}. \tag{12}$$

定理 1 对于无限小生成元 $\tau^0, \xi_s^0, \eta_s^0$,如果存在规范函数 $G^0 = G^0(t, \boldsymbol{q}, \boldsymbol{p})$满足结构方程

$$-(H - \dot{q}_\rho g_\rho)\dot{\tau}^0 + p_\sigma\dot{\xi}_\sigma^0 + \dot{q}_\sigma\eta_\sigma^0$$
$$- X_0^{(1)}(H - \dot{q}_\rho g_\rho) + Q_s(\xi_s^0 - \dot{q}_s\tau^0) + \dot{G}^0 = 0$$
$$(s = 1, \cdots, n; \sigma = 1, \cdots, r; \rho = r+1, \cdots, n), \tag{13}$$

则非保守约束哈密顿系统存在如下精确不变量:

$$I_0 = -H\tau^0 + p_s\xi_s^0 + G^0 = \mathrm{const}. \tag{14}$$

证明 将 I_0 对时间 t 求导数,并利用结构方程(13)、正则方程(9),以及约束关系(5),有

$$\frac{\mathrm{d}I_0}{\mathrm{d}t} = -\dot{H}\tau^0 - H\dot{\tau}^0 + \dot{p}_s\xi_s^0 + p_s\dot{\xi}_s^0$$
$$+ (H - \dot{q}_\rho g_\rho)\dot{\tau}^0 - p_\sigma\dot{\xi}_\sigma^0 - \dot{q}_\sigma\eta_\sigma^0$$
$$+ X_0^{(1)}(H - \dot{q}_\rho g_\rho) - Q_s(\xi_s^0 - \dot{q}_s\tau^0)$$
$$= \left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \dot{q}_\rho\frac{\partial g_\rho}{\partial q_s} - Q_s\right)(\xi_s^0 - \dot{q}_s\tau^0)$$
$$+ \left(-\dot{q}_\sigma + \frac{\partial H}{\partial p_\sigma} - \dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma}\right)(\eta_\sigma^0 - \dot{p}_\sigma\tau^0)$$
$$= 0.$$

故 I_0 为非保守约束哈密顿系统的精确不变量.

4. 系统对称性的摄动与绝热不变量

假设非保守约束哈密顿系统受到了一个小扰动 $\varepsilon\tilde{Q}_s$ 的作用,则系统的运动微分方程(9)成为

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} - \dot{q}_\sigma\frac{\partial g_\rho}{\partial p_\sigma},$$
$$\dot{p}_s = -\frac{\partial H}{\partial q_s} + \dot{q}_\rho\frac{\partial g_\rho}{\partial q_s} + Q_s + \varepsilon\tilde{Q}_s \tag{15}$$
$$(\sigma = 1, \cdots, r; \rho = r+1, \cdots, n; s = 1, \cdots, n).$$

在小扰动 $\varepsilon\tilde{Q}_s$ 作用下,系统原有的对称性与不变量会发生相应的改变.假设扰动后的时间和空间对应的无限小生成元 τ, ξ_s, η_s 是在系统无扰动的对称变换生成元的基础上发生的小摄动,有

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots,$$
$$\xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots, \tag{16}$$
$$\eta_s = \eta_s^0 + \varepsilon\eta_s^1 + \varepsilon^2\eta_s^2 + \cdots,$$

同时规范函数 G 也发生了小摄动,即

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots \tag{17}$$

无限小生成元向量及其一次扩张为

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}, \tag{18}$$

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\tau})\frac{\partial}{\partial\dot{q}_s}+(\dot{\eta}_s-\dot{p}_s\dot{\tau})\frac{\partial}{\partial\dot{p}_s}. \tag{19}$$

将(16)式代入(19)式,有

$$X^{(1)}=\varepsilon^k X_k^{(1)}\qquad(k=0,1,2,\cdots), \tag{20}$$

其中

$$X_k^{(1)}=\tau^k\frac{\partial}{\partial t}+\xi_s^k\frac{\partial}{\partial q_s}+\eta_s^k\frac{\partial}{\partial p_s}+(\dot{\xi}_s^k-\dot{q}_s\dot{\tau}^k)\frac{\partial}{\partial\dot{q}_s}+(\dot{\eta}_s^k-\dot{p}_s\dot{\tau}^k)\frac{\partial}{\partial\dot{p}_s}. \tag{21}$$

定理 2　对于受到小扰动 $\varepsilon\tilde{Q}_s$ 作用的非保守约束哈密顿系统,如果存在规范函数 $G^k(t,\boldsymbol{q},\boldsymbol{p})$,使无限小变换的生成元 $\tau^k(t,\boldsymbol{q},\boldsymbol{p})$,$\xi_s^k(t,\boldsymbol{q},\boldsymbol{p})$,$\eta_s^k(t,\boldsymbol{q},\boldsymbol{p})$满足结构方程

$$\begin{aligned}&-(H-\dot{q}_\rho g_\rho)\dot{\tau}^k+p_\sigma\dot{\xi}_\sigma^k+\dot{q}_\sigma\eta_\sigma^k\\&-X_k^{(1)}(H-\dot{q}_\rho g_\rho)+Q_s(\xi_s^k-\dot{q}_s\tau^k)\\&+\tilde{Q}_s(\xi_s^{k-1}-\dot{q}_s\tau^{k-1})+\dot{G}^k=0\\&(s=1,\cdots,n;\sigma=1,\cdots,r;\rho=r+1,\cdots,n;\\&k=0,1,2,\cdots),\end{aligned} \tag{22}$$

其中 $k=0$ 时,约定 $\xi_s^{-1}=\tau^{-1}=0$,则

$$I_z=\sum_{k=0}^{z}\varepsilon^k[-H\tau^k+p_s\xi_s^k+G^k] \tag{23}$$

为该系统的一个 z 阶绝热不变量.

证明　将 I_z 对时间 t 求导数,并利用结构方程(22)、方程(15),以及约束关系(5),有

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}&=\sum_{k=0}^{z}\varepsilon^k[-\dot{H}\tau^k-H\dot{\tau}^k+\dot{p}_s\xi_s^k+p_s\dot{\xi}_s^k\\&\quad+(H-\dot{q}_\rho g_\rho)\dot{\tau}^k-p_\sigma\dot{\xi}_\sigma^k-\dot{q}_\sigma\eta_\sigma^k\\&\quad+X_k^{(1)}(H-\dot{q}_\rho g_\rho)-Q_s(\xi_s^k-\dot{q}_s\tau^k)\\&\quad-\tilde{Q}_s(\xi_s^{k-1}-\dot{q}_s\tau^{k-1})]\\&=\sum_{k=0}^{z}\varepsilon^k\Bigg[\left(\dot{p}_s+\frac{\partial H}{\partial q_s}-\dot{q}_\rho\frac{\partial g_\rho}{\partial q_s}-Q_s\right)(\xi_s^k-\dot{q}_s\tau^k)\\&\quad+\left(-\dot{q}_\sigma+\frac{\partial H}{\partial p_\sigma}-\dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma}\right)(\eta_\sigma^k-\dot{p}_\sigma\tau^k)\\&\quad+\dot{q}_\rho\tau^k\left(\dot{p}_\rho-\frac{\partial g_\rho}{\partial q_s}\dot{q}_s-\frac{\partial g_\rho}{\partial p_\sigma}\dot{p}_\sigma-\frac{\partial g_\rho}{\partial t}\right)\\&\quad-\tilde{Q}_s(\xi_s^{k-1}-\dot{q}_s\tau^{k-1})\Bigg]\\&=\sum_{k=0}^{z}\varepsilon^k[\varepsilon\tilde{Q}_s(\xi_s^k-\dot{q}_s\tau^k)-\tilde{Q}_s(\xi_s^{k-1}-\dot{q}_s\tau^{k-1})]\\&=\varepsilon^{z+1}\tilde{Q}_s(\xi_s^z-\dot{q}_s\tau^z).\end{aligned} \tag{24}$$

根据高阶绝热不变量的定义[12],I_z 为非保守约束哈密顿系统的一个 z 阶绝热不变量.

5. 系统对称性摄动的逆问题

假设受有小扰动 $\varepsilon\tilde{Q}_s$ 作用的非保守约束哈密顿系统存在一个一阶绝热不变量

$$I_1=\phi_0(t,\boldsymbol{q},\boldsymbol{p})+\varepsilon\phi_1(t,\boldsymbol{q},\boldsymbol{p}), \tag{25}$$

由于其运动正轨应满足方程(15),因此有

$$\begin{aligned}&\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}+\dot{q}_\rho\frac{\partial g_\rho}{\partial q_s}+Q_s+\varepsilon\tilde{Q}_s\right)(\xi_s-\dot{q}_s\tau)\\&+\left(-\dot{q}_\sigma+\frac{\partial H}{\partial p_\sigma}-\dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma}\right)(\eta_\sigma-\dot{p}_\sigma\tau)=0.\end{aligned} \tag{26}$$

由于

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}t}&=\frac{\partial\phi_0}{\partial t}+\frac{\partial\phi_0}{\partial q_s}\dot{q}_s+\frac{\partial\phi_0}{\partial p_s}\dot{p}_s\\&\quad+\varepsilon\left(\frac{\partial\phi_1}{\partial t}+\frac{\partial\phi_1}{\partial q_s}\dot{q}_s+\frac{\partial\phi_1}{\partial p_s}\dot{p}_s\right),\end{aligned} \tag{27}$$

根据(24)式,综合(26)和(27)式,有

$$\begin{aligned}&\frac{\partial\phi_0}{\partial t}+\frac{\partial\phi_0}{\partial q_s}\dot{q}_s+\frac{\partial\phi_0}{\partial p_s}\dot{p}_s\\&+\varepsilon\left(\frac{\partial\phi_1}{\partial t}+\frac{\partial\phi_1}{\partial q_s}\dot{q}_s+\frac{\partial\phi_1}{\partial p_s}\dot{p}_s\right)\\&+\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}+\dot{q}_\rho\frac{\partial g_\rho}{\partial q_s}+Q_s+\varepsilon\tilde{Q}_s\right)(\xi_s-\dot{q}_s\tau)\\&+\left(-\dot{q}_\sigma+\frac{\partial H}{\partial p_\sigma}-\dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma}\right)(\eta_\sigma-\dot{p}_\sigma\tau)\\&=\varepsilon^2\tilde{Q}_s(\xi_s^1-\dot{q}_s\tau^1).\end{aligned} \tag{28}$$

经整理后可得

$$\begin{aligned}&\frac{\partial\phi_0}{\partial t}+\frac{\partial\phi_0}{\partial q_s}\dot{q}_s+\frac{\partial\phi_0}{\partial p_s}\dot{p}_s-\dot{p}_s(\xi_s^0-\dot{q}_s\tau^0)\\&+\left(-\frac{\partial H}{\partial q_s}+\dot{q}_\rho\frac{\partial g_\rho}{\partial q_s}+Q_s+\varepsilon\tilde{Q}_s\right)(\xi_s^0-\dot{q}_s\tau^0)\\&+\varepsilon\left(-\dot{p}_s-\frac{\partial H}{\partial q_s}+\dot{q}_\rho\frac{\partial g_\rho}{\partial q_s}+Q_s\right)(\xi_s^1-\dot{q}_s\tau^1)\\&+\varepsilon\left(\frac{\partial\phi_1}{\partial t}+\frac{\partial\phi_1}{\partial q_s}\dot{q}_s+\frac{\partial\phi_1}{\partial p_s}\dot{p}_s\right)\\&+\left(-\dot{q}_\sigma+\frac{\partial H}{\partial p_\sigma}-\dot{q}_\rho\frac{\partial g_\rho}{\partial p_\sigma}\right)\\&\times(\eta_\sigma-\dot{p}_\sigma\tau)+o(\varepsilon)=0,\end{aligned} \tag{29}$$

其中分离出不含小参数 ε 的项,令 $\dot{p}_s$ 的系数为零,有

$$\xi_s^0-\dot{q}_s\tau^0-\frac{\partial\phi_0}{\partial p_s}=0, \tag{30}$$

并假设

$$\phi_0 = -H\tau^0 + p_s\xi_s^0 + G^0, \tag{31}$$

于是有

$$\tau^0 = (p_s\dot{q}_s - H)^{-1}\left(\phi_0 - G^0 - p_s\frac{\partial\phi_0}{\partial p_s}\right),$$

$$\xi_l^0 = \frac{\partial\phi_0}{\partial p_l} + \dot{q}_l(p_s\dot{q}_s - H)^{-1}\left(\phi_0 - G^0 - p_s\frac{\partial\phi_0}{\partial p_s}\right)$$

$$(l = 1,\cdots,n). \tag{32}$$

进一步分析可得

$$\tau^1 = (p_s\dot{q}_s - H)^{-1}\left(\phi_1 - G^1 - p_s\frac{\partial\phi_1}{\partial p_s}\right),$$

$$\xi_l^1 = \frac{\partial\phi_1}{\partial p_l} + \dot{q}_l(p_s\dot{q}_s - H)^{-1}\left(\phi_1 - G^1 - p_s\frac{\partial\phi_1}{\partial p_s}\right)$$

$$(l = 1,\cdots,n). \tag{33}$$

综合以上讨论,得到下述定理.

定理 3 如果受有小扰动 $\varepsilon\tilde{Q}_s$ 作用的非保守约束哈密顿系统存在形如(25)式的一个一阶绝热不变量,则系统存在无限小对称变换,其无限小生成元的未摄动项和一阶摄动项分别由(32)和(33)式确定.

6. 算 例

例[1] 系统的 Lagrange 函数和非势广义力分别为

$$L = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + \frac{1}{2}(q_1 - q_2)^2,$$

$$Q_1 = -\dot{q}_2, \quad Q_2 = \dot{q}_1, \tag{34}$$

试研究系统对称性的摄动与绝热不变量.

系统的广义动量和哈密顿函数为

$$p_1 = \dot{q}_1 + q_2, \quad p_2 = 0,$$

$$H = \frac{1}{2}p_1^2 - p_1 q_2 - \frac{1}{2}q_1^2 + q_1 q_2. \tag{35}$$

Lagrange 函数的 Hess 矩阵的秩为 $r = 1$,从而系统正则变量之间存在约束

$$\phi_1 = p_2 = 0. \tag{36}$$

系统的运动微分方程(9)可写为

$$\dot{q}_1 = p_1 - q_2, \quad \dot{p}_1 = q_1 - q_2 - \dot{q}_2,$$

$$\dot{p}_2 = p_1 - q_1 + p_1 - q_2. \tag{37}$$

系统的结构方程(13)给出

$$-\left(\frac{1}{2}p_1^2 - p_1 q_2 - \frac{1}{2}q_1^2 + q_1 q_2\right)\dot{\tau}^0$$

$$+ p_1\dot{\xi}_1^0 + \dot{q}_1\eta_1^0 - \xi_1^0(q_2 - q_1) - \xi_2^0(q_1 - p_1)$$

$$- \eta_1^0(p_1 - q_s) - \dot{q}_2(\xi_1^0 - \dot{q}_1\tau^0)$$

$$+ \dot{q}_1(\xi_2^0 - \dot{q}_2\tau^0) + \dot{G}^0 = 0, \tag{38}$$

方程(38)有解

$$\tau^0 = 0, \quad \xi_1^0 = \xi_2^0 = \eta_1^0 = \eta_2^0 = 1,$$

$$G^0 = -2q_1 + q_2. \tag{39}$$

由定理 1,系统有如下精确不变量:

$$I_0 = p_1 + p_2 - 2q_1 + q_2 = \text{const}. \tag{40}$$

下面研究系统对称性的摄动与绝热不变量.假设系统受到的小扰动为

$$\varepsilon\tilde{Q}_1 = \varepsilon\dot{q}_1, \quad \varepsilon\tilde{Q}_2 = \varepsilon\dot{q}_2, \tag{41}$$

系统的结构方程(22)给出

$$-\left(\frac{1}{2}p_1^2 - p_1 q_2 - \frac{1}{2}q_1^2 + q_1 q_2\right)\dot{\tau}^1 + p_1\dot{\xi}_1^1$$

$$+ \dot{q}_1\eta_1^1 - \xi_1^1(q_2 - q_1) - \xi_2^1(q_1 - p_1) - \eta_1^1(p_1 - q_2)$$

$$- \dot{q}_2(\xi_1^1 - \dot{q}_1\tau^1) + \dot{q}_1(\xi_2^1 - \dot{q}_2\tau^1)$$

$$+ \dot{q}_1(\xi_1^0 - \dot{q}_1\tau^0) + \dot{q}_2(\xi_2^0 - \dot{q}_2\tau^0) + \dot{G}^1 = 0, \tag{42}$$

方程(42)有解

$$\tau^1 = -1, \quad \xi_1^1 = \xi_2^1 = \eta_1^1 = \eta_2^1 = 0,$$

$$G^1 = -q_1 - q_2. \tag{43}$$

由定理 2,系统有如下形式的一阶绝热不变量:

$$I_1 = p_1 + p_2 - 2q_1 + q_2$$

$$+ \varepsilon\left(\frac{1}{2}p_1^2 - p_1 q_2 - \frac{1}{2}q_1^2 + q_1 q_2 - q_1 - q_2\right). \tag{44}$$

进一步可求得系统的更高阶绝热不变量.

最后研究系统对称性摄动的逆问题.假设系统受小扰动(41)式的作用,且存在一个一阶绝热不变量

$$I_1 = p_1 + p_2 - 2q_1 + q_2 + \varepsilon(p_1 - 3q_1), \tag{45}$$

(30)和(31)式给出

$$\xi_1^0 - \dot{q}_1\tau^0 - 1 = 0, \quad \xi_2^0 - \dot{q}_2\tau^0 - 1 = 0,$$

$$p_1 + p_2 - 2q_1 + q_2$$

$$= -\left(\frac{1}{2}p_1^2 - p_1 q_2 - \frac{1}{2}q_1^2 + q_1 q_2\right)\tau^0$$

$$+ p_1\xi_1^0 + p_2\xi_2^0 + G^0, \tag{46}$$

解得

$$\tau^0 = \frac{2(-2q_1 + q_2 - G^0)}{p_1^2 + q_1^2 - 2q_1 q_2},$$

$$\xi_1^0 = 1 + \dot{q}_1\tau^0, \quad \xi_2^0 = 1 + \dot{q}_2\tau^0. \tag{47}$$

取

$$G^0 = -2q_1 + q_2, \tag{48}$$

则有

$$\tau^0 = 0, \quad \xi_1^0 = \xi_2^0 = 1. \tag{49}$$

同理,有

$$\xi_1^1 - \dot{q}_1\tau^1 - 1 = 0, \quad \xi_2^1 - \dot{q}_2\tau^1 = 0,$$
$$p_1 - 3q_1 = -H\tau^1 + p_1\xi_1^1 + p_2\xi_2^1 + G^1, \tag{50}$$

解得

$$\tau^1 = \frac{2(-3q_1 - G^1)}{p_1^2 + q_1^2 - 2q_1q_2}, \xi_1^1 = 1 + \dot{q}_1\tau^1,$$
$$\xi_2^1 = \dot{q}_2\tau^1. \tag{51}$$

取

$$G^1 = -3q_1, \tag{52}$$

则有

$$\tau^1 = 0, \xi_1^1 = 1, \xi_2^1 = 0, \tag{53}$$

即相应于一阶绝热不变量(47)式,系统存在无限小对称变换,其未摄动项和一阶摄动项的时间和空间的生成元由(49)和(53)式给出.

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Exact invariants and adiabatic invariants of constrained Hamiltonian systems in phase space *

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Abstract

The perturbation problem of symmetries for constrained Hamiltonian systems under the action of small excitation is studied. Firstly, the canonical equations of nonconservative constrained Hamiltonian systems are established, and the symmetries and exact invariants of the systems in the extended phase space are studied. Secondly, according to the concept of high-order adiabatic invariants of dynamical systems, this paper gives the form of adiabatic invariants and the conditions for their existence and establishes the relationship between adiabatic invariants and symmetrical transformations. Finally, an example is given to illustrate the application of the results.

Keywords: constrained Hamiltonian system, symmetry, perturbation, invariant

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