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可积开边界条件下 XXX-$\frac{1}{2}$自旋链模型的 Gaudin 公式

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利用量子空间可因式化 F 算子,在量子反散射的框架内计算出了可积开边界条件下 XXX-$\frac{1}{2}$ 自旋链模型的 Bethe 态的标量积和模,得到了用谱参量函数的行列式表达的开边界条件下的 Gaudin 公式.

关键词:可积模型, 关联函数, 开边界

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1.引 言

对于量子可积系统,人们最为关心的是哈密顿算子的谱函数和物理算子的关联函数.利用 Bethe Ansatz 方法,许多可积模型的谱和本征态被构造出来[1—4].但是,除了少数在自由费米点的模型以及处理临界现象或无质量情形的共形场,对许多可积模型而言,例如 XXZ-$\frac{1}{2}$ 自旋链模型[5],物理算子的关联函数的计算依然是一个复杂问题.在周期性边界条件下,Korepin 等人利用量子反散射方法和畴壁条件下二维统计模型的配分函数,给出了 XXZ-$\frac{1}{2}$自旋链模型 Bethe 态的标量积,并把它表示成由谱参量的函数构成的行列式,进而证明了 Gaudin 假设,即 Bethe 态的模可由一谱参量构成的行列式表示[6].由于 Bethe 态产生算子的交换复杂性,必须引入辅助量子对偶场,这就使计算结果中包含着辅助量子对偶场的真空期待值.仅在长程渐近情形时,关联函数的精确表达才能得到.最近,Maillet 等人利用 Drinfel'd 的扭转子,构造出了可因式化的 F 算子,并运用于周期性边界条件下的 XXZ-$\frac{1}{2}$ 自旋链模型[7,8].在 F 算子的作用下,物理算子所作用的量子空间变为完全对称的,这就克服了由于对称群所引起的交换复杂性,避免了辅助量子对偶场的引入.另一方面,自从低维可积系统由周期性边界条件被推广到独立的可积开边界条件后[9],许多模型的谱和本征态被 Bethe Ansatz 方法构造出来[10—14].类似于周期性边界的情形,大量的工作被用来研究体系形式因子和关联函数的计算.计算形式因子和关联函数的关键在于对量子本征态的研究.本文将以具有可积开边界条件的 XXX-$\frac{1}{2}$ 自旋链模型为例,利用可因式化 F 算子,计算体系的 Bethe 态的标量积和模,并给出用谱参量函数的行列式表达的开边界条件下的 Gaudin 公式.

2.周期性边界条件下的 XXX-$\frac{1}{2}$ 自旋链模型和可因式化 F 算子

为便于后面的讨论,首先给出周期性边界条件下的 XXX-$\frac{1}{2}$ 自旋链模型的一些主要结果,详细的讨论可参阅文献[4].模型所对应的 R 矩阵为

$$R(u)=\begin{pmatrix} a(u) & 0 & 0 & 0\\ 0 & b(u) & c(u) & 0\\ 0 & c(u) & b(u) & 0\\ 0 & 0 & 0 & a(u)\end{pmatrix},\quad(1)$$

其中 $a(u)=1$,$b(u)=\frac{u}{u+\eta}$,$c(u)=\frac{\eta}{u+\eta}$.上述 R 矩阵满足杨-Baxter 方程(YBE),

$$R_{12}(u_1-u_2)R_{13}(u_1-u_3)R_{23}(u_2-u_3)$$
$$=R_{23}(u_2-u_3)R_{13}(u_1-u_3)R_{12}(u_1-u_2).\quad(2)$$

对长度为 N 的自旋链,若定义 $T_{i_1 i_2 \cdots i_N}(u) = L_{i_N}(u) \cdots L_{i_2}(u) L_{i_1}(u)$,则单值矩阵可表示为 $T(u) = T_{12\cdots N}(u)$. 这里 $L_n(u)$是结合代数

$$R_{12}(u_1 - u_2) L_n^{(1)}(u_1) L_n^{(2)}(u_2) = L_n^{(2)}(u_2) L_n^{(1)}(u_1) R_{12}(u_1 - u_2) \tag{3}$$

的基本表示 $L_n(u) = R_{0n}(u - \frac{1}{2}\eta - \xi_n)$,$L_n^{(1)} \equiv L_n \otimes 1$,$L_n^{(2)} \equiv 1 \otimes L_n$. $T(u)$在辅助空间可以表示为

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_{[0]}. \tag{4}$$

由 $T(u)$的定义及(3)式可知 $T(u)$亦满足基本关系式(3). 在量子反散射的框架内,赝真空态由 $A(u)$和 $D(u)$的本征态确定,$B(u)$和 $C(u)$分别作为产生和湮没算子,

$$\begin{aligned} &A(u)|0\rangle = |0\rangle, \\ &D(u)|0\rangle = \delta(u)|0\rangle, \\ &\delta(u) = \prod\nolimits_{i=1}^{N} b\left(u - \frac{1}{2}\eta - \xi_i\right), \\ &C(u)|0\rangle = 0, \\ &B(u)|0\rangle \neq 0. \end{aligned} \tag{5}$$

赝真空态 $|0\rangle$ 对应于所有自旋向上的完全铁磁态 $|0\rangle = \prod\nolimits_{i=1}^{N} \uparrow_i$. 可因式化 F 算子定义为[7]

$$F_{\sigma(1)\cdots\sigma(N)}(\xi_{\sigma(1)},\cdots,\xi_{\sigma(N)}) R^{\sigma}_{1\cdots N}(\xi_1,\cdots,\xi_N) = F_{1\cdots N}(\xi_1,\cdots,\xi_N), \tag{6}$$

其中 σ 为 S_N 对称群中的任一元素,$R^{\sigma}_{1\cdots N}(\xi_1,\cdots,\xi_N)$为基本 R 矩阵(1)的有序乘积,其顺序取决于 σ. 它把 $V_1 \otimes \cdots \otimes V_N$ 空间转变为 $V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(N)}$ 空间. 由 YBE 和基本关系式(3)可知,可因式化 F 算子具有如下特征:

$$\begin{aligned} &F_{1\cdots N}(\xi_1,\cdots,\xi_N) T_{0,1\cdots N}(\xi_1,\cdots,\xi_N) F^{-1}_{1\cdots N}(\xi_1,\cdots,\xi_N) \\ &= F_{\sigma(1)\cdots\sigma(N)}(\xi_{\sigma(1)},\cdots,\xi_{\sigma(N)}) T_{0,\sigma(1)\cdots\sigma(N)}(\xi_{\sigma(1)},\cdots, \\ &\quad \xi_{\sigma(N)}) F^{-1}_{\sigma(1)\cdots\sigma(N)}(\xi_{\sigma(1)},\cdots,\xi_{\sigma(N)}). \end{aligned} \tag{7}$$

因此,若定义 F 基下的单值矩阵为

$$\begin{aligned} &\tilde{T}_{0,1\cdots N}(\xi_1,\cdots,\xi_N) \\ &= F_{1\cdots N}(\xi_1,\cdots,\xi_N) T_{0,1\cdots N}(\xi_1,\cdots,\xi_N) \\ &\quad \times F^{-1}_{1\cdots N}(\xi_1,\cdots,\xi_N), \end{aligned} \tag{8}$$

则由(7)式可知

$$\tilde{T}_{0,1\cdots N}(\xi_1,\cdots,\xi_N) = \tilde{T}_{0,\sigma(1)\cdots\sigma(N)}(\xi_{\sigma(1)},\cdots,\xi_{\sigma(N)}),$$

表明在格点和参量 ξ_i 同时交换的情况下,单值矩阵 $\tilde{T}_{0,1,\cdots N}(\xi_1,\cdots,\xi_N)$对量子空间 $V^{\otimes N}$是完全对称的. 在 F 基下,单值矩阵的矩阵元表示得非常简洁[7],

$$\begin{aligned} &\tilde{D}_{1\cdots N}(\xi_1,\cdots,\xi_N) = \bigotimes_{i=1}^{N} \begin{pmatrix} b\left(u - \frac{1}{2}\eta - \xi_i\right) & 0 \\ 0 & 1 \end{pmatrix}_{[i]}, \\ &\tilde{B}_{1\cdots N}(\xi_1,\cdots,\xi_N) = \sum_{i=1}^{N} c\left(u - \frac{1}{2}\eta - \xi_i\right)\sigma_i^- \\ &\bigotimes_{j\neq i} \begin{pmatrix} b\left(u - \frac{1}{2}\eta - \xi_j\right) & 0 \\ 0 & B^{-1}(\xi_j - \xi_i) \end{pmatrix}_{[j]}, \\ &\tilde{C}_{1\cdots N}(\xi_1,\cdots,\xi_N) = \sum_{i=1}^{N} c\left(u - \frac{1}{2}\eta - \xi_i\right)\sigma_i^+ \\ &\bigotimes_{j\neq i} \begin{pmatrix} b\left(u - \frac{1}{2}\eta - \xi_j\right) b_{-1}(\xi_i - \xi_j) & 0 \\ 0 & 1 \end{pmatrix}_{[j]}, \end{aligned} \tag{9}$$

其中 $\sigma_i^{\pm}$ 为作用在第 i 个格点的局域算子,$\sigma^{\pm} = \frac{1}{2}(\sigma^1 \pm \mathrm{i}\sigma^2)$,$\{\sigma^k, k = 1,2,3\}$为泡利矩阵. 由于 $\tilde{T}_{0,1\cdots N}(\xi_1,\cdots,\xi_N)$亦满足基本关系式(3),因此 $\tilde{T}$ 中矩阵元所满足的交换关系与 T 中矩阵元之间的交换关系一样. 同时可以证明,赝真空态在 F 算子的作用下不变,即 $F_{1\cdots N}|0\rangle = |0\rangle$. 因此,系统本征态的标积可方便地在 F 基下计算,

$$\begin{aligned} &\langle 0| C(u_1)\cdots C(u_n) B(v_1)\cdots B(v_n)|0\rangle \\ &= \langle 0| \tilde{C}(u_1)\cdots \tilde{C}(u_n) \tilde{B}(v_1)\cdots \tilde{B}(v_n)|0\rangle. \end{aligned} \tag{10}$$

3. 可积开边界条件下 XXX-$\frac{1}{2}$自旋链模型的本征态

在开边界条件下,为了保证系统的可积性,除了 R 矩阵,还需要反射矩阵 K[9],

$$\begin{aligned} &K_+(u) = K\left(u + \frac{1}{2}\eta, \xi_+\right), \\ &K_-(u) = K\left(u - \frac{1}{2}\eta, \xi_-\right), \\ &K(u,\xi) = \begin{pmatrix} u + \xi & \\ & -u + \xi \end{pmatrix}, \end{aligned} \tag{11}$$

其中 $K_{\pm}(u)$分别为左、右反射矩阵. $K_{\pm}(u)$满足边界杨-Baxter 方程(BYBE),

$$R_{12}(u_1 - u_2) K_-^1(u_1) R_{12}(u_1 + u_2 - \eta) K_-^2(u_2)$$

$$
= K_-^2(u_2) R_{12}(u_1+u_2-\eta) K_-^1(u_1) R_{12}(u_1-u_2), \tag{12}
$$

$$
\begin{aligned}
&R_{12}(-u_1+u_2) K_+^1(u_1)^{t_1} \\
&\times R_{12}(-u_1-u_2-\eta) K_+^2(u_2)^{t_2} \\
&= K_+^2(u_2)^{t_2} R_{12}(-u_1-u_2-\eta) \\
&\times K_+^1(u_1)^{t_1} R_{12}(-u_1+u_2).
\end{aligned} \tag{13}
$$

在可积开边界条件下,系统的单值矩阵定义为[9]

$$
\begin{aligned}
U^{t_0}(u) &= T^{t_0}(u) K_+^{t_0}(u) \sigma^2 T(-u) \sigma^2 \\
&= \begin{pmatrix} \mathscr{A}(u) & \mathscr{C}(u) \\ \mathscr{B}(u) & \mathscr{D}(u) \end{pmatrix}_{[0]}.
\end{aligned} \tag{14}
$$

相应的转移矩阵定义为

$$
\begin{aligned}
\tau(u) &= \mathrm{Tr}_0 U(u) K_-(u), \\
&= \left(u - \frac{1}{2}\eta + \xi_-\right)\mathscr{A}(u) - \left(u - \frac{1}{2}\eta - \xi_-\right)\mathscr{D}(u).
\end{aligned} \tag{15}
$$

把(4)式代入(14)式,可得 $U(u)$和 $T(u)$中矩阵元之间的关系:

$$
\begin{aligned}
\mathscr{A}(u) &= \left(u + \frac{1}{2}\eta + \xi_+\right) A(u) D(-u) \\
&\quad + \left(u + \frac{1}{2}\eta - \xi_+\right) C(u) B(-u), \\
\mathscr{B}(u) &= \left(u + \frac{1}{2}\eta + \xi_+\right) B(u) D(-u) \\
&\quad + \left(u + \frac{1}{2}\eta - \xi_+\right) D(u) B(-u), \\
\mathscr{C}(u) &= -\left(u + \frac{1}{2}\eta + \xi_+\right) A(u) C(-u) \\
&\quad - \left(u + \frac{1}{2}\eta - \xi_+\right) C(u) A(-u), \\
\mathscr{D}(u) &= -\left(u + \frac{1}{2}\eta + \xi_+\right) B(u) C(-u) \\
&\quad - \left(u + \frac{1}{2}\eta - \xi_+\right) D(u) A(-u).
\end{aligned} \tag{16}
$$

因此,$\mathscr{A}(u)$,$\mathscr{B}(u)$,$\mathscr{C}(u)$和 $\mathscr{D}(u)$在 F 基下的表示可利用$\tilde{A}(u)$,$\tilde{B}(u)$,$\tilde{C}(u)$和 $\tilde{D}(u)$的表示写出,例如

$$
\begin{aligned}
\tilde{\mathscr{B}}(u) &= \left(u + \frac{1}{2}\eta + \xi_+\right)\tilde{B}(u)\tilde{D}(-u) + \left(u + \frac{1}{2}\eta - \xi_+\right)\tilde{D}(u)\tilde{B}(-u), \\
&= \sum_{i=1}^{N} f(u,\xi_i)\sigma_i^- \bigotimes_{j\neq i} \begin{pmatrix} b\left(u - \frac{1}{2}\eta - \xi_i\right) b\left(-u - \frac{1}{2}\eta - \xi_i\right) & 0 \\ 0 & b^{-1}(\xi_j - \xi_i) \end{pmatrix}_{[j]}, \\
f(u,\xi_i) &= \frac{\eta(\eta + 2u)(\zeta_+ + \xi_i)}{\left(u + \frac{1}{2}\eta - \xi_i\right)\left(u - \frac{1}{2}\eta + \xi_i\right)}.
\end{aligned} \tag{17}
$$

在开边界条件下,系统的赝真空态依然可定义为完全铁磁态,$\tilde{\mathscr{B}}(u)$和 $\tilde{\mathscr{C}}(u)$分别为 Bethe 态的产生和湮没算子.系统的共同体征态可写为

$$
\begin{aligned}
|i_1 \cdots i_n\rangle &= \tilde{\mathscr{B}}(v_1)\cdots\tilde{\mathscr{B}}(v_n)\,|0\rangle, \\
|i_1 \cdots i_n\,| &= \langle 0|\tilde{\mathscr{C}}(u_1)\cdots\tilde{\mathscr{C}}(u_n).
\end{aligned} \tag{18}
$$

相应的转移矩阵的本征值为 Λ[9]

$$
\begin{aligned}
\tau(v)\prod_{\alpha=1}^{n}\tilde{\mathscr{B}}(u_\alpha)|0\rangle &= \Lambda(v,\{u_\alpha\})\prod_{\alpha=1}^{n}\tilde{\mathscr{B}}(u_\alpha)|0\rangle, \\
\Lambda(v,\{u_\alpha\}) &= \frac{\eta + 2v}{2v}\left(v - \frac{1}{2}\eta + \zeta_-\right) \\
&\quad \times \left(v - \frac{1}{2}\eta + \zeta_+\right)\delta(-v) \\
&\quad \times \prod_{\alpha=1}^{n}\frac{[(v-\eta)^2 - u_\alpha^2]}{(v^2 - u_\alpha^2)} \\
&\quad - \frac{\eta - 2v}{2v}\left(-v - \frac{1}{2}\eta + \zeta_-\right) \\
&\quad \times \left(-v - \frac{1}{2}\eta + \zeta_+\right)\delta(v) \\
&\quad \times \prod_{\alpha=1}^{n}\frac{[(v+\eta)^2 - u_\alpha^2]}{(v^2 - u_\alpha^2)}.
\end{aligned} \tag{19}
$$

作为系统的本征态,$\{u_\alpha\}$的取值受到 Bethe Ansatz 方程(BAE)的约束[9],

$$
\begin{aligned}
&\frac{\left(-u_\alpha - \frac{1}{2}\eta + \zeta_-\right)\left(-u_\alpha - \frac{1}{2}\eta + \zeta_+\right)\delta(u_\alpha)}{\left(u_\alpha - \frac{1}{2}\eta + \zeta_-\right)\left(u_\alpha - \frac{1}{2}\eta + \zeta_+\right)\delta(-u_\alpha)} \\
&= \prod_{\beta=1,\neq\alpha}^{n}\frac{[(u_\alpha - \eta)^2 - u_\beta^2]}{[(u_\alpha + \eta)^2 - u_\beta^2]}.
\end{aligned} \tag{20}
$$

事实上,不光开边界时的算子 $\tilde{\mathscr{A}}(u)$,$\tilde{\mathscr{B}}(u)$,$\tilde{\mathscr{C}}(u)$和 $\tilde{\mathscr{D}}(u)$可由周期性边界时的算子表示,甚至开边界时的本征态亦可由周期性边界时的算子表示,例如

$$\langle 0|\prod_{\alpha=1}^{n}\tilde{\mathscr{C}}(u_\alpha)=\left(\prod_{\alpha=1}^{n}\frac{2u_\alpha+\eta}{2u_\alpha}\right)\times\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_n}\left[\prod_{\beta=1}^{n}\varepsilon_\beta\left(-\varepsilon_\beta u_\beta-\frac{1}{2}\eta+\zeta_+\right)\right]\times\left(\prod_{\alpha>\beta}^{n}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta}\right)\langle 0|\prod_{\alpha=1}^{n}\tilde{C}(\varepsilon_\alpha u_\alpha),\tag{21}$$

其中 $\varepsilon_\alpha\in\{1,-1\}$,以下用递推法予以证明.

由于在 $\tilde{\mathscr{B}}$,或 $\tilde{\mathscr{C}}$ 的表达式(16)中,$\tilde{A}$,$\tilde{B}$,$\tilde{C}$ 和 $\tilde{D}$ 之间的交换满足 $\tilde{T}$ 的基本交换关系式(3),例如

$$\tilde{C}(u)\tilde{A}(v)=\frac{a(u-v)}{b(u-v)}\tilde{A}(v)\tilde{C}(u)-\frac{c(u-v)}{b(u-v)}\tilde{A}(u)\tilde{C}(v),\tag{22}$$

因此 $\tilde{\mathscr{C}}(u)$ 可写为

$$\tilde{\mathscr{C}}(u)=\frac{\eta+2u}{2u}\left[\left(-u-\frac{1}{2}\eta+\zeta_+\right)\tilde{A}(-u)\tilde{C}(u)-\left(u-\frac{1}{2}\eta+\zeta_+\right)\tilde{A}(u)\tilde{C}(-u)\right],\tag{23}$$

考虑到 $\langle 0|\tilde{A}(u_1)=\langle 0|$,可知 $n=1$ 时,(21)式成立.假设当 $n=m$ 时(21)式成立,那么

$$\langle 0|\prod_{\alpha=1}^{m+1}\tilde{\mathscr{C}}(u_\alpha)=\langle 0|\prod_{\alpha=1}^{m}\tilde{\mathscr{C}}(u_\alpha)\tilde{\mathscr{C}}(u_{m+1})=\left(\prod_{\alpha=1}^{m+1}\frac{2u_\alpha+\eta}{2u_\alpha}\right)\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_{m+1}}\times\left[\prod_{\beta=1}^{m+1}\varepsilon_\beta\left(-\varepsilon_\beta u_\beta-\frac{1}{2}\eta+\zeta_+\right)\right]\times\left(\prod_{\alpha>\beta}^{m}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta}\right)\langle 0|\prod_{\alpha=1}^{m}\times\tilde{C}(\varepsilon_\alpha u_\alpha)\tilde{A}(-\varepsilon_{m+1}u)\tilde{C}(\varepsilon_{m+1}u).\tag{24}$$

反复利用(22)式,可以把(24)式中的 $\tilde{A}(-\varepsilon_{m+1}u)$ 项移到最左端,即

$$\langle 0|\prod_{\alpha=1}^{m+1}\tilde{\mathscr{C}}(u_\alpha)=\left(\prod_{\alpha=1}^{m+1}\frac{2u_\alpha+\eta}{2u_\alpha}\right)\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_{m+1}}\times\left[\prod_{\beta=1}^{m+1}\varepsilon_\beta\left(-\varepsilon_\beta u_\beta-\frac{1}{2}\eta+\zeta_+\right)\right]\times\left(\prod_{\alpha>\beta}^{m+1}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta}\right)\langle 0|\prod_{\alpha=1}^{m+1}\times\tilde{C}(\varepsilon_\alpha u_\alpha)+\mathscr{O},\tag{25}$$

$$\mathscr{O}=\left(\prod_{\alpha=1}^{m+1}\frac{2u_\alpha+\eta}{2u_\alpha}\right)\sum_{l}\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_{l-1}}\sum_{\varepsilon_{l+1}}\cdots\sum_{\varepsilon_m}\times\left[\prod_{\beta=1,\neq l}^{m}\varepsilon_\beta\left(-\varepsilon_\beta u_\beta-\frac{1}{2}\eta+\zeta_+\right)\right]\times\left(\prod_{\alpha>\beta,\alpha,\beta\neq l}^{m}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta}\right)\times\left[\prod_{\beta=1,\neq l}^{m}\frac{(\varepsilon_\beta u_\beta+\eta)^2-u_l^2}{u_\beta^2-u_l^2}\right]\times\left\{\sum_{\varepsilon_l}\sum_{\varepsilon_{m+1}}\frac{-\eta}{\varepsilon_l u_l+\varepsilon_{m+1}u_{m+1}}\times\left[\varepsilon_l\left(-\varepsilon_l u_l-\frac{1}{2}\eta+\zeta_+\right)\right]\times\left[\varepsilon_{m+1}\left(-\varepsilon_{m+1}u_{m+1}-\frac{1}{2}\eta+\zeta_+\right)\right]\right\}\times\langle 0|\prod_{\alpha=1,\neq l}^{m}\tilde{C}(\varepsilon_\alpha u_\alpha)\tilde{C}(u_{m+1})\tilde{C}(-u_{m+1}).\tag{26}$$

通过直接计算,可知

$$\sum_{\varepsilon_l}\sum_{\varepsilon_{m+1}}\frac{-\eta\left[\varepsilon_l\left(-\varepsilon_l u_l-\frac{1}{2}\eta+\zeta_+\right)\right]\left[\varepsilon_{m+1}\left(-\varepsilon_{m+1}u_{m+1}-\frac{1}{2}\eta+\zeta_+\right)\right]}{\varepsilon_l u_l+\varepsilon_{m+1}u_{m+1}}=0,\tag{27}$$

即 $\mathscr{O}=0$.因此当 $n=m+1$ 时,(21)式也成立,证明完毕.同理可证

$$\prod_{i=1}^{n}\tilde{\mathscr{B}}(v_i)|0\rangle=\left(\prod_{i=1}^{n}\frac{2v_i+\eta}{2v_i}\right)\sum_{\sigma_1}\cdots\sum_{\sigma_n}\times\left[\prod_{k=1}^{n}(-\sigma_k)\left(-\sigma_k v_k-\frac{1}{2}\eta+\zeta_+\right)\delta(-\sigma_k v_k)\right]\times\left(\prod_{j>i}\frac{\sigma_i v_i+\sigma_j v_j-\eta}{\sigma_i v_i+\sigma_j v_j}\right)\times\prod_{i=1}^{n}B(\sigma_i v_i)|0\rangle.\tag{28}$$

4. 本征态的标积和模

这里具体计算本征态的标积和模.首先定义

$$G_c^m(\{u_\alpha\}, v_1\cdots v_m, i_{m+1}\cdots i_n)$$
$$=\langle 0|\prod_{\alpha=1}^{n}\tilde{\mathscr{C}}(u_\alpha)\prod_{i=1}^{m}\tilde{\mathscr{B}}(v_i)|i_{m+1}\cdots i_n\rangle, \quad (29)$$

其中$|i_{m+1}\cdots i_n\rangle$表示在 $i_{m+1}\cdots i_n$ 的位置自旋向下,其余位置自旋向上的态.同时,$\{u_\alpha\}_n$ 满足可积边界条件下的 BAE(20),而 $\{v_i\}_m$ 暂时取为任意参量.显然,所要计算的本征态的标积 $G_c^{(n)}$ 可以从 $G_c^{(0)}$ 通过如下递推关系得到:

$$G_c^{(m)}(\{u_\alpha\}, v_1\cdots v_m, i_{m+1}\cdots i_n)$$
$$=\sum_{i_m=1,\neq i_{m+1}\cdots i_n}^{N} G_c^{m-1}(\{u_\alpha\}, v_1\cdots v_{m-1}, i_m\cdots i_n)$$
$$\times\langle i_m\cdots i_n|\tilde{\mathscr{B}}(v_m)|i_{m+1}\cdots i_n\rangle, \quad (30)$$

其中$\langle i_m\cdots i_n|\tilde{\mathscr{B}}(v_m)|i_{m+1}\cdots i_n\rangle$可以从 $\tilde{\mathscr{B}}(v)$的表达式(16)直接得到

$$\langle i_m\cdots i_n|\tilde{\mathscr{B}}(v_m)|i_{m+1}\cdots i_n\rangle$$
$$=\delta(v_m)\delta(-v_m)\frac{\eta(2v_m+\eta)(\xi_{i_m}+\zeta_+)}{\left[v_m^2-\left(\frac{1}{2}\eta+\xi_{i_m}\right)^2\right]}$$
$$\times\left[\prod_{l=m+1}^{n}b^{-1}\left(v_m-\frac{1}{2}\eta-\xi_{i_l}\right)b^{-1}\right.$$
$$\left.\times\left(-v_m-\frac{1}{2}\eta-\xi_{i_l}\right)b^{-1}(\xi_{i_l}-\xi_{i_m})\right].$$

而 $G_c^{(0)}$ 可以利用(21)式得到.因此必须先给出$\langle 0|\prod_{\alpha=1}^{n}\tilde{C}(\varepsilon_\alpha u_\alpha)|i_1\cdots i_n\rangle$的表达式.由 $\tilde{C}$ 的表示式(9)可知

$$\langle 0|\prod_{i=1}^{n}\tilde{C}(\varepsilon_i u_i)|i_1\cdots i_n\rangle=\left[\prod_{j=1}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}\right.$$
$$\left.\times(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=1,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]\left[\prod_{\alpha=1}^{n}\right.$$
$$\left.\times\delta(\varepsilon_\alpha u_\alpha)\right]\prod_{\alpha=1}^{n}\prod_{j=1}^{n}b^{-1}\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta-\xi_{i_j}\right)$$
$$\times Z_n(\{\varepsilon_\alpha u_\alpha\}, i_1\cdots i_n),$$
$$Z_n(\{\varepsilon_\alpha u_\alpha\}, i_1\cdots i_n)$$
$$=\left[\prod_{j=1}^{n}\uparrow_{i_j}\right]\prod_{\alpha=1}^{n}\tilde{\tilde{C}}(\varepsilon_\alpha u_\alpha)\left[\prod_{j=1}^{n}\downarrow_{i_j}\right], \quad (31)$$

其中$\tilde{\tilde{C}}(\varepsilon_\alpha u_\alpha)=\tilde{C}_{i_1\cdots i_n}(\varepsilon_\alpha u_\alpha, i_1\cdots i_n)$.利用 $\tilde{C}$ 的表示式(9)可知,$Z_n(\{u_\alpha\}, i_1\cdots i_n)$满足如下的递推关系式:

$$Z_n(\{\varepsilon_\alpha u_\alpha\}, i_1\cdots i_n)$$
$$=\sum_{j=1}^{n}\frac{\eta}{\varepsilon_n u_n+\frac{1}{2}\eta-\xi_{i_j}}$$
$$\times\left[\prod_{l=1,\neq j}^{n}b\left(\varepsilon_n u_n-\frac{1}{2}\eta-\xi_{i_l}\right)\right]$$
$$\times\left[\prod_{l=1,\neq j}^{n}b^{-1}(\xi_{i_j}-\xi_{i_l})\right]$$
$$\times Z_{n-1}(\{\varepsilon_\alpha u_\alpha\}_{\alpha\neq n}, \{i_l\}_{l\neq j}). \quad (32)$$

由此可知,$Z_n(\{\varepsilon_\alpha u_\alpha\}, i_1\cdots i_n)$可以表示成如下 $n\times n$ 行列式:

$$Z_n(\{\varepsilon_\alpha u_\alpha\}, i_1\cdots i_n)=\frac{\prod_{\alpha=1}^{n}\prod_{k=1}^{n}\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta-\xi_{i_k}\right)\left(-\varepsilon_\alpha u_\alpha+\frac{1}{2}\eta-\xi_{i_k}\right)}{\prod_{\alpha>\beta}(\varepsilon_\alpha u_\alpha-\varepsilon_\beta u_\beta)(\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta-\eta)\prod_{j>l}(\xi_{i_l}^2-\xi_{i_j}^2)}\det\mathscr{N}_{\alpha j},$$
$$\mathscr{N}_{\alpha j}=\frac{-\eta}{\left[u_\alpha^2-\left(\frac{1}{2}\eta-\xi_{i_j}\right)^2\right]\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta-\xi_{i_j}\right)}. \quad (33)$$

对上式的验证需要先把行列式 $\det\mathscr{N}_{\alpha j}$的第 $\beta(\beta\neq n)$行乘以仅仅依赖参数 β 的系数 f_β,然后加到第 n 行,即

$$\mathscr{N}_{nj}\rightarrow\mathscr{N}'_{nj}=\mathscr{N}_{nj}+\sum_{\beta=1}^{n-1}f_\beta\mathscr{N}_{\beta j},$$
$$f_\beta=-\frac{\varepsilon_n u_n+\varepsilon_\beta u_\beta-\eta}{2\varepsilon_\beta u_\beta-\eta}$$
$$\times\prod_{l=1}^{n}\frac{\left[u_\beta^2-\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]}{\left[u_n^2-\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]}$$
$$\times\prod_{\alpha=1,\neq\beta}^{n-1}\frac{(\varepsilon_n u_n-\varepsilon_\alpha u_\alpha)(\varepsilon_n u_n+\varepsilon_\alpha u_\alpha-\eta)}{(\varepsilon_\beta u_\beta-\varepsilon_\alpha u_\alpha)(\varepsilon_\beta u_\beta+\varepsilon_\alpha u_\alpha-\eta)}. \quad (34)$$

这样做并不会改变行列式的值.然后再把该行列式按第 n 行展开,即可证明按第 n 行展开的行列式与递推关系式相同.计算过程中要利用下列等式:

$$\prod_{j=1,\neq l}^{n}(\xi_{i_j}-\xi_{i_l}-\eta)$$

$$=\prod_{\alpha=1}^{n-1}\frac{\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta-\xi_{i_l}\right)\left(-\varepsilon_\alpha u_\alpha+\frac{1}{2}\eta-\xi_{i_l}\right)}{(\varepsilon_n u_n-\varepsilon_\alpha u_\alpha)(\varepsilon_n u_n+\varepsilon_\alpha u_\alpha-\eta)}$$

$$\times\prod_{j=1,\neq l}^{n}\frac{\left[-u_n^2+\left(\frac{1}{2}\eta-\xi_{i_j}\right)^2\right]}{\xi_{i_j}+\xi_{i_l}}$$

$$\times\left\{1+\sum_{\beta=1}^{n-1}f_\beta\frac{\left(\varepsilon_n u_n-\frac{1}{2}\eta-\xi_{i_l}\right)\left[u_n^2-\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]}{\left(\varepsilon_\beta u_\beta-\frac{1}{2}\eta-\xi_{i_l}\right)\left[u_\beta^2-\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]}\right\}. \tag{35}$$

(35)式为 ξ_{i_l} 的 $n-1$ 次多项式,它的证明可以通过验证在 n 个点 $\xi_{i_l}=\varepsilon_\beta u_\beta-\frac{1}{2}\eta,\beta=1\cdots n$ 处该式左右两端相等.因此

$$\langle 0|\prod_{\alpha=1}^{n}\tilde{\mathscr{C}}(u_\alpha)|i_1\cdots i_n\rangle=\left(\prod_{\alpha=1}^{n}\frac{2u_\alpha+\eta}{2u_\alpha}\right)$$

$$\times\left[\prod_{j=1}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=1,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]$$

$$\times\frac{\prod_{\alpha,k=1}^{n}\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\xi_{i_k}\right)^2\right]}{\prod_{\alpha>\beta}(u_\alpha^2-u_\beta^2)\prod_{j>l}(\xi_{i_l}^2-\xi_{i_j}^2)}$$

$$\times\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_n}\left[\prod_{\beta=1}^{n}\varepsilon_\beta\left(-\varepsilon_\beta u_\beta-\frac{1}{2}\eta+\zeta_+\right)\delta(\varepsilon_\beta u_\beta)\right]$$

$$\times\left(\prod_{\alpha>\beta}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta-\eta}\right)\det\mathscr{N}_{\alpha j}. \tag{36}$$

应用边界条件下的 BAE(20),可以得到如下等式:

$$\left(\prod_{\alpha>\beta}\frac{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta-\eta}{\varepsilon_\alpha u_\alpha+\varepsilon_\beta u_\beta+\eta}\right)^2=\prod_{\alpha=1}^{n}\left\{\left(-\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_-\right)\right.$$

$$\times\left(-\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_+\right)\delta(\varepsilon_\alpha u_\alpha)\left[\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_-\right)\right.$$

$$\left.\times\left(\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_+\right)\delta(-\varepsilon_\alpha u_\alpha)\right]^{-1}\Bigg\}. \tag{37}$$

代入(36)式,并利用行列式的加法法则,可知

$$G_c^{(0)}(\{u_\alpha\},i_1\cdots i_n)=\prod_{\alpha=1}^{n}(2u_\alpha+\eta)$$

$$\times\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}$$

$$\times\left[\prod_{j=1}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=1,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]$$

$$\times\frac{\prod_{\alpha=1}^{n}\prod_{k=1}^{n}\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\xi_{i_k}\right)^2\right]}{\prod_{\alpha>\beta}(u_\alpha^2-u_\beta^2)\prod_{j>l}(\xi_{i_l}^2-\xi_{i_j}^2)}\det\mathscr{X}_{\alpha j}, \tag{38}$$

$$\mathscr{X}_{\alpha j}=\frac{\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\zeta_-\right)^2\right]^{1/2}}{2u_\alpha}$$

$$\times\sum_{\varepsilon_\alpha}\varepsilon_\alpha\left(\frac{\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_-}{-\varepsilon_\alpha u_\alpha-\frac{1}{2}\eta+\zeta_-}\right)^{1/2}\mathscr{N}_{\alpha j}$$

$$=\frac{-\eta(\zeta_-+\xi_{i_j})}{\left[u_\alpha^2-\left(\frac{1}{2}\eta+\xi_{i_j}\right)^2\right]\left[u_\alpha^2-\left(\frac{1}{2}\eta-\xi_{i_j}\right)^2\right]}. \tag{39}$$

利用递推公式(30),可以由 $G_c^{(0)}$ 得到 $G_c^{(1)}$,

$$G_c^{(1)}(\{u_\alpha\},i_1\cdots i_n)$$

$$=\prod_{\alpha=1}^{n}(2u_\alpha+\eta)\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}$$

$$\times\left[\prod_{j=2}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=2,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]$$

$$\times\frac{\prod_{\alpha=1}^{n}\prod_{k=2}^{n}\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\xi_{i_k}\right)^2\right]}{\prod_{j>l\geqslant 2}(\xi_{i_l}^2-\xi_{i_j}^2)}\det\mathscr{X}_{\alpha j}^{(1)}, \tag{40}$$

$$\mathscr{X}_{\alpha j}^{(1)}=\frac{1}{\left[-v_1^2+\left(\frac{1}{2}\eta+\xi_{i_j}\right)^2\right]}\mathscr{X}_{\alpha j},\quad 2\leqslant j\leqslant n,$$

$$\mathscr{X}_{\alpha_1}^{(1)}=\prod_{l=2}^{n}\left[-v_1^2+\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]\delta(v_1)\delta(-v_1)$$

$$\times(2v_1+\eta)f^{(1)}, \tag{41}$$

$$f^{(1)}=\sum_{i_1=1,\neq i_2\cdots i_n}^{N}\frac{\eta^2(\zeta_-+\xi_{i_1})(\zeta_++\xi_{i_1})}{\left[-v_1^2+\left(\frac{1}{2}\eta+\xi_{i_1}\right)^2\right]\left[-u_\alpha^2+\left(\frac{1}{2}\eta+\xi_{i_1}\right)^2\right]}$$

$$\times\left(\prod_{k=1,\neq i_1}^{N}\frac{\xi_{i_1}-\xi_k+\eta}{\xi_{i_1}-\xi_k}\right)\frac{\prod_{\beta=1,\neq\alpha}^{n}\left[-u_\beta^2+\left(\frac{1}{2}\eta-\xi_{i_1}\right)^2\right]}{\prod_{l=2}^{n}(\xi_{i_1}+\xi_{i_l})(\xi_{i_1}-\xi_{i_l}+\eta)}, \tag{42}$$

其中 $f^{(1)}$ 为 v_1 的有理函数，在 $v_1=\pm\left(\frac{1}{2}\eta+\xi_{i_l}\right)(i_1=1\cdots N,\neq i_2\cdots i_n)$ 处有一阶极点，并且当 $v_1\to\infty$ 时，$f\to0$，因此 $f^{(1)}$ 可以写为如下形式：

$$f^{(1)}=\prod_{l=2}^{n}\left[-v_1^2+\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]^{-1}\times\frac{1}{2v_1}\delta^{-1}(v_1)\delta^{-1}(-v_1)\mathscr{H}_{\alpha_1}-\sum_{b=2}^{n}\frac{g_b^{(1)}}{\left(v_1-\frac{1}{2}\eta+\xi_{i_b}\right)}\mathscr{X}_{\alpha b}-\sum_{b=2}^{n}\frac{g_b^{(1)}}{\left(v_1+\frac{1}{2}\eta-\xi_{i_b}\right)}\mathscr{X}_{\alpha b},\tag{43}$$

$$\mathscr{H}_{\alpha j}=\frac{\eta}{(u_\alpha^2-v_j^2)}\left\{\left(v_j-\frac{1}{2}\eta+\zeta_+\right)\left(v_j-\frac{1}{2}\eta+\zeta_-\right)\times\delta(-v_j)\prod_{\beta=1,\neq\alpha}^{n}[u_\beta^2-(v_j-\eta)^2]-\left(-v_j-\frac{1}{2}\eta+\zeta_+\right)\left(-v_j-\frac{1}{2}\eta+\zeta_-\right)\times\delta(v_j)\prod_{\beta=1,\neq\alpha}^{n}[u_\beta^2-(v_j+\eta)^2]\right\},\tag{44}$$

$g_b^{(j)}$ 为依赖参数 b，而不依赖参数 α 的系数，

$$g_b^{(j)}=\frac{\eta(\zeta_+-\xi_{i_b})(\zeta_--\xi_{i_b})\delta^{-1}\left(\frac{1}{2}\eta-\xi_{i_b}\right)\prod_{\beta=1}^{n}\left[-u_\beta^2+\left(\frac{1}{2}\eta+\xi_{i_b}\right)^2\right]}{(-\eta+2\xi_{i_b})\left[\prod_{l=j+1}^{n}(\eta-\xi_{i_b}-\xi_{i_l})\right]\left[\prod_{l=j+1,\neq b}^{n}(\xi_{i_l}-\xi_{i_b})\right]}.\tag{45}$$

事实上，(43)式中后两项对行列式 $\det\mathscr{X}_{\alpha j}^{(1)}$ 的值没有贡献，只有第一项有意义，因此在保持 $\det\mathscr{X}_{\alpha j}^{(1)}$ 不变的意义上，$\mathscr{X}_{\alpha_1}^{(1)}$ 可写为

$$\mathscr{X}_{\alpha_1}^{(1)}=\frac{2v_1+\eta}{2v_1}\mathscr{H}_{\alpha_1}.\tag{46}$$

假设 $G_c^{(m-1)}(\{u_\alpha\},v_1\cdots v_{m-1},i_m\cdots i_n)$ 有如下形式：

$$\begin{aligned}&G_c^{(m-1)}(\{u_\alpha\},v_1\cdots v_{m-1},i_m\cdots i_n)\\&=\prod_{\alpha=1}^{n}(2u_\alpha+\eta)\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}\\&\times\left[\prod_{j=m}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=m,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]\\&\times\frac{\prod_{\alpha=1}^{n}\prod_{k=m}^{n}\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\xi_{i_k}\right)^2\right]}{\prod_{j>l\geqslant m}(\xi_{i_l}^2-\xi_{i_j}^2)}\det\mathscr{X}_{\alpha j}^{(m-1)},\end{aligned}\tag{47}$$

$$\mathscr{X}_{\alpha j}^{m-1}=\frac{2v_j+\eta}{2v_j}\mathscr{H}_{\alpha j},\quad 1\leqslant j\leqslant m-1,$$
$$\mathscr{X}_{\alpha j}^{m-1}=\prod_{l=1}^{m-1}\left[-v_l^2+\left(\frac{1}{2}\eta+\xi_{i_j}\right)^2\right]^{-1}\mathscr{X}_{\alpha j},\quad m\leqslant j\leqslant n,\tag{48}$$

那么由递推关系(30)，可得

$$\begin{aligned}&G_c^{(m)}(\{u_\alpha\},v_1\cdots v_m,i_{m+1}\cdots i_n)\\&=\prod_{\alpha=1}^{n}(2u_\alpha+\eta)\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}\\&\times\left[\prod_{j=m+1}^{n}\prod_{k=1,\neq i_j}^{N}b^{-1}(\xi_{i_j}-\xi_k)\right]\left[\prod_{j,k=m+1,j\neq k}^{n}b(\xi_{i_j}-\xi_{i_k})\right]\\&\times\frac{\prod_{\alpha=1}^{n}\prod_{k=m+1}^{n}\left[-u_\alpha^2+\left(\frac{1}{2}\eta-\xi_{i_k}\right)^2\right]}{\prod_{j>l\geqslant m+1}(\xi_{i_l}^2-\xi_{i_j}^2)}\det\mathscr{X}_{\alpha j}^{(m)},\end{aligned}\tag{49}$$

$$\mathscr{X}_{\alpha j}^{(m)}=\frac{2v_j+\eta}{2v_j}\mathscr{H}_{\alpha_j},\quad 1\leqslant j\leqslant m-1,$$
$$\mathscr{X}_{\alpha j}^{(m)}=\prod_{l=1}^{m}\left[-v_l^2+\left(\frac{1}{2}\eta+\xi_{i_j}\right)^2\right]^{-1}\mathscr{X}_{\alpha j},\quad m+1\leqslant j\leqslant n,$$
$$\mathscr{X}_{\alpha m}^{(m)}=\prod_{l=m+1}^{m}\left[-v_m^2+\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]\times\delta(v_m)\delta(-v_m)(\eta+2v_m)f^{(m)},\tag{50}$$

其中

$$\begin{aligned}f^{(m)}&=\sum_{i_m=1,\neq i_{m+1}\cdots i_n}^{N}\frac{\eta^2(\zeta_-+\xi_{i_m})(\zeta_++\xi_{i_m})}{\left[-v_m^2+\left(\frac{1}{2}\eta+\xi_{i_m}\right)^2\right]\left[-u_\alpha^2+\left(\frac{1}{2}\eta+\xi_{i_m}\right)^2\right]}\\&\times\left(\prod_{k=1,\neq i_m}^{N}\frac{\xi_{i_m}-\xi_k+\eta}{\xi_{i_m}-\xi_k}\right)\\&\times\frac{\prod_{\beta=1,\neq\alpha}^{n}\left[-u_\beta^2+\left(\frac{1}{2}\eta-\xi_{i_m}\right)^2\right]}{\prod_{l=m+1}^{n}(\xi_{i_m}+\xi_{i_l})(\xi_{i_m}-\xi_{i_l}+\eta)}\end{aligned}$$

$$\times\prod_{j=1}^{m-1}\left[-v_j^2+\left(\frac{1}{2}\eta+\xi_{i_m}\right)^2\right]^{-1}. \tag{51}$$

类似于 $f^{(1)}$ 的讨论，$f^{(m)}$ 可以写为

$$\begin{aligned}f^{(m)} &= \prod_{l=m+1}^{n}\left[-v_m^2+\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]^{-1}\\ &\times\prod_{j=1}^{m-1}(v_j^2-v_m^2)^{-1}\frac{1}{2v_m}\mathscr{H}_{\alpha m}\\ &-\sum_{b=m+1}^{n}\frac{g_b^{(m)}}{\left(v_m-\frac{1}{2}\eta+\xi_{i_b}\right)}\mathscr{X}_{\alpha b}\\ &-\sum_{b=m+1}^{n}\frac{g_b^{(m)}}{\left(v_m+\frac{1}{2}\eta+\xi_{i_b}\right)}\mathscr{X}_{\alpha b}\\ &-\sum_{k=1}^{m-1}\frac{h_k^{(m)}}{(v_m+v_k)}\mathscr{X}_{\alpha k}\\ &-\sum_{k=1}^{m-1}\frac{h_k^{(m)}}{(v_m-v_k)}\mathscr{X}_{\alpha k},\end{aligned} \tag{52}$$

其中 $h_k^{(m)}$ 与 $g_b^{(m)}$ 类似，为依赖参数 b，而不依赖参数 α 的系数，

$$\begin{aligned}h_k^{(m)} &= \frac{1}{2v_k}\prod_{l=m+1}^{n}\left[v_k^2-\left(\frac{1}{2}\eta-\xi_{i_l}\right)^2\right]^{-1}\\ &\times\prod_{j=1,\neq k}^{m-1}(v_j^2-v_k^2)^{-1}.\end{aligned} \tag{53}$$

(52)式中后面四项对行列式 $\det\mathscr{X}_{\alpha j}^{(m)}$ 的贡献为零，只有第一项有贡献，因此 $\mathscr{X}_{\alpha m}^{(m)}$ 可以写为

$$\mathscr{X}_{\alpha m}^{(m)} = \frac{1}{\prod_{j=1}^{m-1}(v_j^2-v_m^2)}\frac{2v_m+\eta}{2v_m}\mathscr{H}_{\alpha m}. \tag{54}$$

依次递推，最后可以得到本征态的标积为

$$\begin{aligned}G_c^{(n)}(\{u_\alpha\},\{v_j\}) &= \left\{\prod_{\alpha=1}^{n}(2u_\alpha+\eta)\right.\\ &\times\left.\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}\right\}\\ &\times\left[\prod_{j=1}^{n}\frac{(2v_j+\eta)}{2v_j}\right]\frac{1}{\prod_{\alpha>\beta}(u_\alpha^2-u_\beta^2)\prod_{i<j}(v_i^2-v_j^2)}\det\mathscr{H}_{\alpha i}.\end{aligned} \tag{55}$$

利用可积边界条件下的转移矩阵的本征值 Λ 及下列等式：

$$\det\nu_{\alpha i} = \frac{\prod_{\alpha>\beta}(u_\alpha^2-u_\beta^2)\prod_{i<j}(v_i^2-v_j^2)}{\prod_{\alpha=1}^{n}\prod_{i=1}^{n}(u_\alpha^2-v_i^2)},$$

$$\nu_{\alpha i} = \frac{1}{u_\alpha^2-v_i^2}, \tag{56}$$

$G_c^{(n)}$ 还可以用(19)式表示为

$$\begin{aligned}G_c^{(n)} &= \left\{\prod_{\alpha=1}^{n}\frac{(\eta+2u_\alpha)}{2u_\alpha}\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\delta(u_\alpha)\right.\right.\\ &\times\left.\left.\delta(-u_\alpha)\right]^{1/2}\right\}\left[\prod_{j=1}^{n}\frac{2v_j}{(\eta-2v_j)}\right]\frac{\det\tau_{\alpha i}}{\det\nu_{\alpha i}},\end{aligned}$$

$$\tau_{\alpha i} = \frac{\partial\Lambda(v_i,\{u_\beta\})}{\partial u_\alpha}. \tag{57}$$

最后，令 $\{v_i\}$ 也为本征态的谱参数，即 $v_\alpha=u_\alpha$，$\alpha=1,\cdots,n$. 通过对(55)式取极限，可以得到本征态的模为

$$\begin{aligned}S_n &= G_c^{(n)}(\{u_\alpha\},\{u_\alpha\}),\\ &= \eta^n\left\{\prod_{\alpha=1}^{n}\left(\frac{\eta+2u_\alpha}{2u_\alpha}\right)^2\left(-u_\alpha-\frac{1}{2}\eta+\zeta_-\right)\right.\\ &\times\left(-u_\alpha-\frac{1}{2}\eta+\zeta_+\right)\delta(u_\alpha)\left[\frac{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_+\right)^2}{u_\alpha^2-\left(\frac{1}{2}\eta-\zeta_-\right)^2}\right.\\ &\times\left.\left.\delta(u_\alpha)\delta(-u_\alpha)\right]^{1/2}\right\}\left[\prod_{\alpha,\beta=1,\alpha\neq\beta}^{n}\frac{u_\alpha^2-(u_\beta+\eta)^2}{u_\alpha^2-u_\beta^2}\right]\\ &\times\det\Phi_{\alpha\beta},\end{aligned} \tag{58}$$

其中 $\Phi_{\alpha\beta}$ 也为一 $n\times n$ 矩阵的矩阵元，

$$\Phi_{\alpha\beta} = -\frac{\partial}{\partial u_\beta}\ln\left[\frac{\left(u_\alpha-\frac{1}{2}\eta+\zeta_+\right)\left(u_\alpha-\frac{1}{2}\eta+\zeta_-\right)\delta(-u_\alpha)}{\left(-u_\alpha-\frac{1}{2}\eta+\zeta_+\right)\left(-u_\alpha-\frac{1}{2}\eta+\zeta_-\right)\delta(u_\alpha)}\prod_{\gamma=1,\neq\alpha}^{n}\frac{u_\gamma^2-(\eta-u_\alpha)^2}{u_\gamma^2-(\eta+u_\alpha)^2}\right]. \tag{59}$$

值得注意的是上式在求导时要把 $\{u_\alpha\}$ 的函数当作一般意义上的函数,即不考虑 BAE(20).

5. 总 结

本文利用 F 基下的产生和湮没算子的表达式,直接计算了可积开边界条件下 XXX-$\frac{1}{2}$ 自旋链模型的本征态的标积和模,给出了用谱参量函数的行列式表达的开边界条件下的 Gaudin 公式,为进一步计算系统的形式因子和关联函数做好了准备. 在讨论中给出的 $Z_n(\{\epsilon_\alpha u_\alpha\}, i_1 \cdots i_n)$ 表达式并不是唯一的. 事实上,在文献[8]中,Maillet 等人也计算了 $Z_n(\{\epsilon_\alpha u_\alpha\}, i_1 \cdots i_n)$,并给出了另外一种表达式. 两式相比较,可以看出本文的表达式形式上较复杂,但是却对后面的计算有利. 另外,在周期性边界条件下,人们即可先算出 $G_c^{(0)}$,再通过递推关系式算出 $G_c^{(n)}$,也可先算出 $G_b^{(0)}$,再通过递推关系式算出 $G_b^{(n)}$ ($= G_c^{(n)}$). 但在可积开边界条件下,由 $G_b^{(0)}$ 通过递推关系式计算出 $G_b^{(n)}$ 看起来非常困难,原因在于在 F 基下 A 算子的表达式稍显复杂. 这种困难也许只是表面的,有可能存在其他的基底,使得 A 和 D 的表达式完全对称起来.

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The Gaudin formula for the case of XXX-$\frac{1}{2}$ spin chain with integrable open boundary condition

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Abstract

Utlizing the factorizing F operator in the quantum space, the scalar products of the Bethe states are directly calculated and the Gaudin formula is proved for the case of XXX-$\frac{1}{2}$ spin chain under the integrable open-boundary condition.

Keywords: integrable model, correlation function, open boundary

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