第 51 卷 第 8 期 2002 年 8 月
1000-3290/2002/51(08)/1666-05
物 理 学 报
ACTA PHYSICA SINICA
Vol.51,No.8,August,2002

单面约束 Birkhoff 系统对称性的摄动与绝热不变量*

张 毅

(苏州科技学院城建系,苏州 215011)

(2001 年 12 月 3 日收到;2002 年 1 月 26 日收到修改稿)

研究了具有单面约束的 Birkhoff 系统在小扰动作用下对称性的摄动与不变量之间的关系.首先,建立了系统的运动微分方程,给出了系统的精确不变量;其次,基于力学系统的高阶绝热不变量的概念,得到了系统的绝热不变量存在的条件及其形式;最后,举例说明结果的应用.

关键词:分析力学,单面约束,Birkhoff 系统,对称性,摄动,绝热不变量

PACC:0320

1.引 言

经典的绝热不变量(adiabatic invariant)是指在系统的某参量缓慢变化时,相对该参量的变化而改变更慢的某一物理量[1].绝热不变量又称缓渐不变量或浸渐不变量[2].实际上,参量缓慢变化等同于小扰动的作用.系统在小扰动作用下对称性的改变及其不变量与力学系统的可积性之间有着密切关系,因此研究系统的对称性摄动与绝热不变量具有重要意义.最近,赵跃宇、梅凤翔[1,3]研究给出了非完整非保守力学系统的精确不变量与绝热不变量;陈向炜、梅凤翔给出了 Birkhoff 系统[4]、变质量完整系统[5]、包含伺服约束的非完整系统[6]的对称性摄动与绝热不变量.但是,所有这些研究尚限于双面约束系统.

实际上,在自然界及工程技术问题中相当多的约束是属于单面的,而不是双面的.1988 年,梅凤翔[7]专题研究了单面约束系统动力学.1993 年,Журавлев 和 Фуфаев 出版了国际上第一部单面约束系统动力学的专著[8].近年来,在单面约束系统的基本理论研究方面已取得了一些重要成果[9—14].

本文进一步研究单面约束 Birkhoff 系统对称性的摄动与绝热不变量.首先,建立了系统的运动微分方程,导出了系统的精确不变量;其次,基于力学系统的高阶绝热不变量的概念,给出了系统的绝热不变量存在的条件及形式,从而揭示了系统在小扰动作用下对称性的摄动与不变量之间的关系;最后,举例说明结果的应用.

2.系统的运动微分方程

研究受有单面约束的 Birkhoff 系统.假设系统的 Birkhoff 变量 a^{μ} $(\mu=1,\cdots,2n)$ 不是彼此独立的,而受有单面约束

$$f_{\beta}(t,a)\geqslant 0 \qquad (\beta=1,\cdots,g). \tag{1}$$

式中若不等号严格成立,就说约束放松,或说系统脱离约束;当约束(1)式取等号,就说约束起作用,或处于张紧,换言之,系统处于约束上.

当系统处于约束上时,对应的广义约束反力为

$$P_{\mu}=\lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}} \qquad (\mu=1,\cdots,2n), \tag{2}$$

于是,系统的运动微分方程可写成

$$\omega_{\mu\nu}\dot{a}^{\nu}-\frac{\partial B}{\partial a^{\mu}}-\frac{\partial R_{\mu}}{\partial t}=\lambda_{\beta}\frac{\partial f_{\beta}}{\partial a^{\mu}} \qquad (\mu=1,\cdots,2n);$$
$$\lambda_{\beta}\geqslant 0, f_{\beta}\geqslant 0, \qquad \lambda_{\beta}f_{\beta}=0 \qquad (\beta=1,\cdots,g), \tag{3}$$

式中

* 国家自然科学基金(批准号:19972010)及江苏省青蓝工程基金资助的课题.

$$\omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \quad (\mu,\nu = 1,\cdots,2n) \tag{4}$$

为 Birkhoff 协变张量;λ_β 为约束乘子,根据文献[15],可在运动微分方程积分之前解出 λ_β 作为 t,a 的函数,于是,广义约束反力 $P_\mu = P_\mu(t,a)$.

3. 无限小变换和精确不变量

取时间和广义坐标的无限小变换

$$t^* = t + \Delta t,$$
$$a^{\mu^*} = a^\mu + \Delta a^\mu \quad (\mu = 1,\cdots,2n) \tag{5}$$

或其展开式

$$t^* = t + \varepsilon\tau^0(t,a),$$
$$a^{\mu^*} = a^\mu + \varepsilon\xi^0_\mu(t,a) \quad (\mu = 1,\cdots,2n), \tag{6}$$

式中 ε 为无限小参数,τ^0,ξ^0_μ 为无限小生成元.取无限小生成元向量

$$X_0^{(0)} = \tau^0\frac{\partial}{\partial t} + \xi^0_\mu\frac{\partial}{\partial a^\mu}, \tag{7}$$

其一次扩张

$$X_0^{(1)} = \tau^0\frac{\partial}{\partial t} + \xi^0_\mu\frac{\partial}{\partial a^\mu} + (\dot{\xi}^0_\mu - \dot{a}^\mu\dot{\tau}^0)\frac{\partial}{\partial \dot{a}^\mu}. \tag{8}$$

定理 1　对于无限小生成元 τ^0,ξ^0_μ,如果存在规范函数 $G^0 = G^0(t,a)$ 满足结构方程

$$(R_\mu\dot{a}^\mu - B)\dot{\tau}^0 + X_0^{(1)}(R_\mu\dot{a}^\mu - B) - P_\mu(\xi^0_\mu - \dot{a}^\mu\tau^0) + \dot{G}^0 = 0 \quad 当\ f_\beta = 0, \tag{9}$$

$$(R_\mu\dot{a}^\mu - B)\dot{\tau}^0 + X_0^{(1)}(R_\mu\dot{a}^\mu - B) + \dot{G}^0 = 0 \quad 当\ f_\beta > 0, \tag{10}$$

则系统存在如下精确不变量:

$$I_0 = R_\mu\xi^0_\mu - B\tau^0 + G^0 = \mathrm{const}. \tag{11}$$

证明　由于

$$\frac{\mathrm{d}I_0}{\mathrm{d}t} = (R_\mu\dot{a}^\mu - B)\dot{\tau}^0 + X_0^{(1)}(R_\mu\dot{a}^\mu - B) + \dot{G}^0 + \left[\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right](\xi^0_\mu - \dot{a}^\mu\tau^0) - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\dot{a}^\mu\tau^0. \tag{12}$$

显然

$$\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\dot{a}^\mu = 0. \tag{13}$$

若系统处于约束上,则由方程(3)及结构方程(9),并考虑到(13)式,有

$$\frac{\mathrm{d}I_0}{\mathrm{d}t} = (R_\mu\dot{a}^\mu - B)\dot{\tau}^0 + X_0^{(1)}(R_\mu\dot{a}^\mu - B) - P_\mu(\xi^0_\mu - \dot{a}^\mu\tau^0) + \dot{G}^0 + \left[P_\mu + \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right](\xi^0_\mu - \dot{a}^\mu\tau^0) = 0;$$

若系统脱离约束,则 $\lambda_\beta = 0$,由结构方程(10),并考虑到(13)式,有

$$\frac{\mathrm{d}I_0}{\mathrm{d}t} = (R_\mu\dot{a}^\mu - B)\dot{\tau}^0 + X_0^{(1)}(R_\mu\dot{a}^\mu - B) + \dot{G}^0 + \left[\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right](\xi^0_\mu - \dot{a}^\mu\tau^0) = 0.$$

故系统存在精确不变量(11)式.证毕.

4. 对称性的摄动与绝热不变量

定义 1[1]　若 $I_z(t,a,\varepsilon)$ 是力学系统的一个含有小参量 ε 的最高次幂为 z 的物理量,其对时间 t 的一阶导数正比于 ε^{z+1},则称 I_z 为力学系统的 z 阶绝热不变量.

假设单面约束 Birkhoff 系统(3)式受到小扰动 εQ_μ 的作用,则系统的运动微分方程变为

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} + \varepsilon Q_\mu = P_\mu \quad 当\ f_\beta = 0; \tag{14}$$

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} + \varepsilon Q_\mu = 0 \quad 当\ f_\beta > 0. \tag{15}$$

在小扰动 εQ_μ 的作用下,系统原有的对称性与不变

量相应地会发生改变.假设扰动后的时间与空间对应的无限小生成元 τ,ξ_μ 是在系统无扰动的对称性变换生成元基础上发生的小摄动,有

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots,\quad \xi_\mu = \xi_\mu^0 + \varepsilon\xi_\mu^1 + \varepsilon^2\xi_\mu^2 + \cdots, \tag{16}$$

相应的规范函数 G 同时也发生了小摄动,即

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \tag{17}$$

无限小生成元向量及其一次扩张为

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu}, \tag{18}$$

$$X^{(1)} = X^{(0)} + (\dot{\xi}_\mu - \dot{a}^\mu\dot{\tau})\frac{\partial}{\partial\dot{a}^\mu} = \tau\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu} + (\dot{\xi}_\mu - \dot{a}^\mu\dot{\tau})\frac{\partial}{\partial\dot{a}^\mu}. \tag{19}$$

将(16)式代入(19)式,有

$$X^{(1)} = \varepsilon^k X_k^{(1)} \quad (k = 0,1,2,\cdots,), \tag{20}$$

式中

$$X_k^{(1)} = \tau^k\frac{\partial}{\partial t} + \xi_\mu^k\frac{\partial}{\partial a^\mu} + (\dot{\xi}_\mu^k - \dot{a}^\mu\dot{\tau}^k)\frac{\partial}{\partial\dot{a}^\mu} \quad (k = 0,1,2,\cdots,). \tag{21}$$

定理 2 对于受到小扰动 εQ_μ 作用的单面约束 Birkhoff 系统,如果存在规范函数 $G^k(t,a)$,使无限小变换的生成元 $\tau^k(t,a)$,$\xi_\mu^k(t,a)$满足结构方程

$$(R_\mu\dot{a}^\mu - B)\dot{\tau}^k + X_k^{(1)}(R_\mu\dot{a}^\mu - B) - P_\mu(\xi_\mu^k - \dot{a}^\mu\tau^k) + Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1}) + \dot{G}^k = 0 \quad 当 f_\beta = 0; \tag{22}$$

$$(R_\mu\dot{a}^\mu - B)\dot{\tau}^k + X_k^{(1)}(R_\mu\dot{a}^\mu - B) + Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1}) + \dot{G}^k = 0 \quad 当 f_\beta > 0, \tag{23}$$

式中 $k=0$ 时,约定 $\xi_\mu^{-1} = \tau^{-1} = 0$,则

$$I_z = \sum_{k=0}^{z}\varepsilon^k[R_\mu\xi_\mu^k - B\tau^k + G^k] \tag{24}$$

是该系统的一个 z 阶绝热不变量.

证明

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \sum_{k=0}^{z}\varepsilon^k\Bigg\{(R_\mu\dot{a}^\mu - B)\dot{\tau}^k + X_0^{(1)}(R_\mu\dot{a}^\mu - B) + \dot{G}^k + \left[\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right] \times (\xi_\mu^k - \dot{a}^\mu\tau^k) - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\dot{a}^\mu\tau^k\Bigg\},$$

若系统处于约束上,则由结构方程(22)和方程(14),并考虑到关系(13)式,有

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \sum_{k=0}^{z}\varepsilon^k\Bigg\{\left[P_\mu + \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right] \times (\xi_\mu^k - \dot{a}^\mu\tau^k) - Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1})\Bigg\}$$

$$= \sum_{k=0}^{z}\varepsilon^k[\varepsilon Q_\mu(\xi_\mu^k - \dot{a}^\mu\tau^k) - Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1})] = \varepsilon^{z+1}Q_\mu(\xi_\mu^z - \dot{a}^\mu\tau^z);$$

若系统脱离约束,则由结构方程(23)和方程(15),并考虑到关系(13)式,有

$$\frac{\mathrm{d}I_z}{\mathrm{d}t} = \sum_{k=0}^{z}\varepsilon^k\Bigg\{\left[\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right] \times (\xi_\mu^k - \dot{a}^\mu\tau^k) - Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1})\Bigg\}$$

$$= \sum_{k=0}^{z}\varepsilon^k[\varepsilon Q_\mu(\xi_\mu^k - \dot{a}^\mu\tau^k) - Q_\mu(\xi_\mu^{k-1} - \dot{a}^\mu\tau^{k-1})] = \varepsilon^{z+1}Q_\mu(\xi_\mu^z - \dot{a}^\mu\tau^z).$$

因此,I_z 为单面约束 Birkhoff 系统的一个 z 阶绝热不变量.证毕.

5. 算 例

例 4 阶 Birkhoff 系统的 Birkhoff 函数为

$$B = \frac{1}{2}[(a^3)^2 + (a^4)^2] + ga^2, \tag{25}$$

而 Birkhoff 函数组为

$$R_1 = a^3, R_2 = a^4, R_3 = 0, R_4 = 0, \tag{26}$$

系统的运动受有单面约束

$$f = a^1 - a^2 \geqslant 0, \tag{27}$$

试研究系统对称性的摄动与绝热不变量.

系统的运动微分方程可表为

$$\dot{a}^1 = a^3, \dot{a}^2 = a^4, \dot{a}^3 = -\lambda, \dot{a}^4 = -g + \lambda, \quad \lambda \geqslant 0, f \geqslant 0, \lambda f = 0. \tag{28}$$

若系统处于约束上,即

$$f = a^1 - a^2 = 0, \tag{29}$$

则由方程(28)和(29),得到

$$\lambda = \frac{1}{2}g, \tag{30}$$

于是

$$\dot{a}^1 = a^3, \dot{a}^2 = a^4, \dot{a}^3 = -\frac{1}{2}g, \dot{a}^4 = -\frac{1}{2}g; \tag{31}$$

若系统脱离约束,即

$$f = a^1 - a^2 > 0, \tag{32}$$

则方程(28)给出

$$\dot a^1 = a^3, \dot a^2 = a^4, \dot a^3 = 0, \dot a^4 = -g. \tag{33}$$

系统的结构方程(9)和(10)给出

$$-\frac{1}{2}[(a^3)^2 + (a^4)^2]\dot\tau^0 - ga^2\dot\tau^0 - \xi_2^0 g + a^3\dot\xi_1^0 + a^4\dot\xi_2^0 - \frac{1}{2}g(\xi_1^0 - \dot a^1\tau^0) + \frac{1}{2}g(\xi_2^0 - \dot a^2\tau^0) + \dot G^0 = 0 \quad \text{当} f = 0; \tag{34}$$

$$-\frac{1}{2}[(a^3)^2 + (a^4)^2]\dot\tau^0 - ga^2\dot\tau^0 - \xi_2^0 g + a^3\dot\xi_1^0 + a^4\dot\xi_2^0 + \dot G^0 = 0 \quad \text{当} f > 0. \tag{35}$$

方程(34)和(35)有解

$$\tau^0 = -1, \quad \xi_1^0 = a^4, \quad \xi_2^0 = a^3, \quad \xi_3^0 = 0, \quad \xi_4^0 = 0, \quad G^0 = a^1 g - a^3 a^4. \tag{36}$$

由定理 1,系统有如下精确不变量

$$I_0 = \frac{1}{2}(a^3 + a^4)^2 + g(a^1 + a^2) = \text{const}. \tag{37}$$

下面研究系统的绝热不变量.假设系统受到的小扰动为

$$\varepsilon Q_1 = \frac{1}{2}\varepsilon g, \varepsilon Q_2 = \frac{1}{2}\varepsilon g, \varepsilon Q_3 = 0, \varepsilon Q_4 = 0, \tag{38}$$

结构方程(22)和(23)给出

$$-\frac{1}{2}[(a^3)^2 + (a^4)^2]\dot\tau^1 - ga^2\dot\tau^1 - \xi_2^1 g + a^3\dot\xi_2^1 + a^4\dot\xi_2^1 - \frac{1}{2}g(\xi_1^1 - \dot a^1\tau^1) + \frac{1}{2}g(\xi_2^1 - \dot a^2\tau^1) + \frac{1}{2}g(\xi_1^0 - \dot a^1\tau^0) + \frac{1}{2}g(\xi_2^0 - \dot a^2\tau^0) + \dot G^1 = 0 \quad \text{当} f = 0; \tag{39}$$

$$-\frac{1}{2}[(a^3)^2 + (a^4)^2]\dot\tau^1 - ga^2\dot\tau^1 - \xi_2^1 g + a^3\dot\xi_1^1 + a^4\dot\xi_2^1 + \frac{1}{2}g(\xi_1^0 - \dot a^1\tau^0) + \frac{1}{2}g(\xi_2^0 - \dot a^2\tau^0) + \dot G^1 = 0 \quad \text{当} f > 0. \tag{40}$$

方程(39)和(40)有解

$$\tau^1 = 0, \quad \xi_1^1 = a^3 + a^4, \quad \xi_2^1 = a^3 + a^4, \quad \xi_3^1 = 0, \quad \xi_4^1 = 0, \quad G^1 = -\frac{1}{2}(a^3 + a^4)^2. \tag{41}$$

由定理 2,系统有一阶绝热不变量为

$$I_1 = \frac{1}{2}(a^3 + a^4)^2 + g(a^1 + a^2) + \varepsilon\frac{1}{2}(a^3 + a^4)^2. \tag{42}$$

进一步可求得系统的更高阶绝热不变量.

[1] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Scienca Press) p 164 (in Chinese) [赵跃宇、梅凤翔 1999 力学系统的对称性与守恒量(北京:科学出版社)第 164 页]

[2] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) p 728 (in Chinese) [梅凤翔、刘　端、罗　勇 1991 高等分析力学(北京:北京理工大学出版社)第 728 页]

[3] Zhao Y Y and Mei F X 1996 *Acta Mech. Sin.* **28** 207 (in Chinese) [赵跃宇、梅凤翔 1996 力学学报 **28** 207]

[4] Chen X W, Zhang R C and Mei F X 2000 *Acta Mech. Sin.* **16** 282

[5] Chen X W and Mei F X 2000 *Chin. Phys.* **9** 721

[6] Chen X W and Mei F X 2001 *Chinese Quarterly of Mechanics* **22** 204 (in Chinese) [陈向炜、梅凤翔 2001 力学季刊 **22** 204]

[7] Mei F X 1988 *Special Problems in Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) p 318 (in Chinese) [梅凤翔 1988 分析力学专题(北京:北京工业学院出版社)第 318 页]

[8] Журавлев В Φ, Фуфаев Н А 1993 *Механика систем с неудерживающими связями* (Москва:Наука) стр1

[9] Lacomba E A and Tulczyjew W M 1990 *J. Phys. A: Math. Gen.* **23** 2801

[10] Zhang Y and Mei F X 2000 *Chin. Sci. Bull.* **45** 1354

[11] Zhang Y and Mei F X 1999 *Appl. Mathem. Mechan.* **20** 59

[12] Zhang Y and Mei F X 2000 *Appl. Mathem. Mechan.* **21** 59

[13] Li Y C, Zhang Y and Liang J H 2001 *Chin. Phys.* **10** 376

[14] Zhang H B 2001 *Acta Phys. Sin.* **50** 1837 (in Chinese) [张宏彬 2001 物理学报 **50** 1837]

[15] Mei F X, Shi R C, Zhang Y F *et al* 1996 *Dynamics of Birkhoffian System* (Beijing: Beijing Institute of Technology Press) p 1 (in Chinese) [梅凤翔、史荣昌、张永发等 1996 Birkhoff 系统动力学(北京:北京理工大学出版社)第 1 页]

Perturbation to symmetries and adiabatic invariant of Birkhoffian systems with unilateral constraints*

Zhang Yi

(*Department of Urban Construction*, *University of Science and Technology of Suzhou*, *Suzhou* 215011, *China*)

(Received 3 December 2001; revised manuscript received 26 January 2002)

Abstract

The relation between the invariant and the perturbation to symmetries of Birkhoffian systems with unilateral constraints under the action of small disturbance is studied. First, the differential equations of motion of the systems are established, and the exact invariant of the systems is obtained. Next, based on the concept of high-order adiabatic invariant of mechanical systems, the form of the adiabatic invariant and the conditions for their existence are given; and finally, an example is given to illustrate the application of the results.

Keywords: analytical mechanics, unilateral constraint, Birkhoffian system, symmetry, perturbation, adiabatic invariant

PACC: 0320

* Project supported by the National Natural Science Foundation of China (Grant No. 19972010) and by the 'Qing Lan' Project Foundation of Jiangsu Province, China.