第 51 卷 第 8 期 2002 年 8 月
1000-3290/2002/51(08)/1897-04

物 理 学 报
ACTA PHYSICA SINICA

Vol.51, No.8, August, 2002

静态球对称黑洞 Dirac 场的统计熵

李传安　孟庆苗　苏九清

（山东省菏泽教育学院物理系，菏泽　274016）

（2001 年 12 月 10 日收到；2002 年 2 月 5 日收到修改稿）

利用改进的 brick-wall 模型，给出了一类静态球对称黑洞 Dirac 场的熵，结果表明，在取相同的截断因子时，Dirac 场的熵均为标量场的熵的 7/2 倍.

关键词：黑洞，统计熵，brick-wall 模型，Dirac 场

PACC：9760L，0420

1. 引　　言

目前人们在黑洞熵的研究方面已经做了大量的工作[1—4]，特别是利用 't Hooft 的 brick-wall 模型[5]，已很成功地计算了各种黑洞的标量场的统计熵[6—8]，但对于矢量场，由于 Dirac 方程为一耦合方程组，对于黑洞熵的计算带来了一定的困难，文献[9，10]已对 Dirac 场的熵进行了研究，为了使结果更具有普遍性的意义，本文利用改进的 brick-wall 模型[11，12]，进一步研究一类球对称静态黑洞的熵所遵从的规律.

2. Dirac 场的经典动量

通常的非极端静态球对称黑洞，它周围的时空度规一般可表示为

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 - r^2d\theta - r^2\sin^2\theta d\varphi^2, \quad (1)$$

式中 $g_{tt} = -g^{rr} = -g_{rr}^{-1} = (g^{tt})^{-1} = f(r)(r - r_+)$，$r_+$ 为黑洞的外视界，利用 $\Gamma^{\lambda}_{\mu,\upsilon} = \frac{1}{2}g^{\lambda\alpha}(g_{\alpha\mu,\upsilon} + g_{\alpha\upsilon,\mu} - g_{\mu\upsilon,\alpha})$ 可求得(1)式度规所对应的不为零的联络

$$\Gamma^1_{00} = \frac{1}{2}f(r)[f'(r)(r - r_+) + f(r)](r - r_+),$$

$$\Gamma^1_{11} = -\Gamma^0_{10} = -\frac{1}{2}\left[\frac{f'(r)}{f(r)} + \frac{1}{r - r_+}\right],$$

$$\Gamma^1_{22} = -f(r)(r - r_+)r,$$

$$\Gamma^1_{33} = -f(r)(r - r_+)r\sin^2\theta,$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r},$$

$$\Gamma^3_{23} = \cot\theta, \quad (2)$$

式中 $f'(r) = df(r)/dr$，再利用零标架

$$l^\mu = \frac{1}{\sqrt{2}}[1, -f(r)(r - r_+), 0, 0],$$

$$n^\mu = \frac{1}{\sqrt{2}}[f^{-1}(r)(r - r_+)^{-1}, 1, 0, 0],$$

$$m^\mu = \frac{1}{\sqrt{2}r}(0, 0, -1, -i/\sin\theta),$$

$$m^\mu = \frac{1}{\sqrt{2}r}(0, 0, -1, i/\sin\theta) \quad (3)$$

和

$$\varepsilon = \frac{1}{2}(l_{\mu;\upsilon}n^\mu l^\upsilon - m_{\mu;\upsilon}\bar{m}^\mu l^\upsilon),$$

$$\alpha = \frac{1}{2}(l_{\mu;\upsilon}n^\mu \bar{m}^\upsilon - m_{\mu;\upsilon}\bar{m}^\mu \bar{m}^\upsilon),$$

$$\gamma = \frac{1}{2}(l_{\mu;\upsilon}n^\mu n^\upsilon - m_{\mu;\upsilon}\bar{m}^\mu n^\upsilon),$$

$$\beta = \frac{1}{2}(l_{\mu;\upsilon}n^\mu m^\upsilon - m_{\mu;\upsilon}\bar{m}^\mu m^\upsilon),$$

$$\rho = l_{\mu;\upsilon}m^\mu \bar{m}^\upsilon,$$

$$\pi = -n_{\mu;\upsilon}\bar{m}^\mu l^\upsilon,$$

$$\mu = -n_{\mu;\upsilon}\bar{m}^\mu m^\upsilon,$$

$$\tau = l_{\mu;\upsilon}m^\mu n^\upsilon, \quad (4)$$

将(2)和(3)式代入(4)式，可求得不为零的旋系数

$$\varepsilon = -\frac{1}{2\sqrt{2}}[f'(r)(r - r_+) + f(r)],$$

$$\rho = f(r)(r - r_+)/\sqrt{2}r,$$

$$\alpha = -\beta = \frac{1}{2\sqrt{2}r}\cot\theta,$$

$$\mu = 1/\sqrt{2}r, \quad (5)$$

及微分算子

$$D=l^{\mu}\partial\mu=\frac{1}{\sqrt{2}}\frac{\partial}{\partial t}-\frac{1}{\sqrt{2}}f(r)(r-r_{+})\frac{\partial}{\partial r},$$

$$\bar{D}=n^{\mu}\partial\mu=\frac{1}{\sqrt{2}}f^{-1}(r)(r-r_{+})^{-1}\frac{\partial}{\partial t}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial r},$$

$$\delta=m^{\mu}\partial\mu=\frac{-1}{\sqrt{2}r}\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi},$$

$$\bar{\delta}=\bar{m}^{\mu}\partial\mu=\frac{-1}{\sqrt{2}r}\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sqrt{2}r\sin\theta}\frac{\partial}{\partial\varphi}.\tag{6}$$

将(5)和(6)式代入旋量粒子四分量场方程

$$(D+\varepsilon-\rho)\psi_{11}+(\bar{\delta}+\pi-\alpha)\psi_{12}-\frac{\mathrm{i}\mu_0}{\sqrt{2}}\psi_{21}=0,$$

$$(\bar{D}+\mu-\gamma)\psi_{12}+(\delta+\beta-\tau)\psi_{11}-\frac{\mathrm{i}\mu_0}{\sqrt{2}}\psi_{22}=0,$$

$$(D+\varepsilon^{*}-\rho^{*})\psi_{22}-(\delta+\pi^{*}-\alpha^{*})\psi_{21}-\frac{\mathrm{i}\mu_0}{\sqrt{2}}\psi_{12}=0,$$

$$(\bar{D}+\mu^{*}-\gamma^{*})\psi_{21}-(\bar{\delta}+\beta^{*}-\tau^{*})\psi_{22}-\frac{\mathrm{i}\mu_0}{\sqrt{2}}\psi_{11}=0.\tag{7}$$

采取小质量近似,整理后可得到

$$\left[r\frac{\partial}{\partial t}-rf(r)(r-r_{+})\frac{\partial}{\partial r}-\frac{r\Delta}{2}\right]\psi_{11}$$
$$-\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)\psi_{12}=0,$$

$$\left[rf^{-1}(r)(r-r_{+})^{-1}\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}+1\right]\psi_{12}$$
$$-\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)\psi_{11}=0,$$

$$\left[r\frac{\partial}{\partial t}-rf(r)(r-r_{+})\frac{\partial}{\partial r}-\frac{r}{2}\Delta\right]\psi_{22}$$
$$+\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)\psi_{21}=0,$$

$$\left[rf^{-1}(r)(r-r_{+})^{-1}\frac{\partial}{\partial t}+r\frac{\partial}{\partial r}+1\right]\psi_{21}$$
$$+\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)\psi_{22}=0,\tag{8}$$

式中 $\Delta=r[f'(r)(r-r_{+})+f(r)]+2f(r)(r-r_{+})$,分离变量,令

$$\psi_{11}=\mathrm{e}^{-\mathrm{i}Et}R_{+}(r)\mathrm{Y}_{+}(\theta,\varphi),$$

$$\psi_{12}=\mathrm{e}^{-\mathrm{i}Et}\frac{1}{r}R_{-}(r)\mathrm{Y}_{-}(\theta,\varphi),$$

$$\psi_{21}=\mathrm{e}^{-\mathrm{i}Et}\frac{1}{r}R_{-}(r)\mathrm{Y}_{+}(\theta,\varphi),$$

$$\psi_{22}=\mathrm{e}^{-\mathrm{i}Et}R_{+}(r)\mathrm{Y}_{-}(\theta,\varphi),\tag{9}$$

代入(8)式整理后可得到四个独立方程,其中角向方程

$$\left[\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin^2\theta}\left(\frac{1}{4}\cos^2\theta\mp\mathrm{i}\cos\theta\frac{\partial}{\partial\varphi}\right.\right.$$
$$\left.\left.+\frac{\partial^2}{\partial\varphi^2}-\frac{1}{2}\right)+\lambda^2\right]\mathrm{Y}_{\pm}=0.\tag{10}$$

由文献[13]知,方程的解为权是 1/2 的球谐函数,分离变量常数为 $\lambda=l+1/2$,其中 $l>1/2$.径向方程

$$\left[\frac{\partial}{\partial r}-\mathrm{i}Ef^{-1}(r)(r-r_{+})^{-1}\right]R_{-}+\lambda R_{+}=0,$$

$$\left[r^2f(r)(r-r_{+})\frac{\partial}{\partial r}+\mathrm{i}Er^2+\frac{r}{2}\Delta\right]R_{+}+\lambda R_{-}=0.\tag{11}$$

整理(11)式可获得关于 R_{-} 的方程

$$r^2f(r)(r-r_{+})\frac{\partial^2R_{-}}{\partial r^2}+\frac{r}{2}\Delta\frac{\partial R_{-}}{\partial r}$$
$$+[E^2r^2f^{-1}(r)(r-r_{+})^{-1}-\lambda^2]R_{-}$$
$$-\mathrm{i}Erf^{-1}(r)(r-r_{+})^{-1}\left[\frac{\Delta}{2}-rf'(r)\right.$$
$$\left.\times(r-r_{+})-rf(r)\right]R_{-}=0.\tag{12}$$

采取 Wengel-Kramers-Brillouin(缩写为 WKB)近似,令 $R_{-}(r)=\mathrm{e}^{\mathrm{i}S_{-}(r)}$ 并将分离变量常数 $\lambda=l+1/2$ 代入,可得到旋量场一个分量所对应的经典动量

$$p=\frac{\partial S_{-}}{\partial r}=f(r)^{-1}(r-r_{+})^{-1}\left[E^2-f(r)(r\right.$$
$$\left.-r_{+})(l+\frac{1}{2})^2/r^2\right]^{1/2}.\tag{13}$$

3.Dirac 场的自由能和熵

根据正则系综理论,系统的自由能可表示为

$$\beta F=\int_0^{\infty}\mathrm{d}\Gamma(E)\ln(1+\mathrm{e}^{-\beta E})=-\beta\int_0^{\infty}\frac{\Gamma(E)\mathrm{d}E}{\mathrm{e}^{\beta E}+1},\tag{14}$$

式中 $\Gamma(E)$ 为系统的能量 $\leqslant E$ 的微观态数,由半经典的索末菲量子理论 $\oint p\mathrm{d}l=2\pi n$,并利用改进的 brick-wall 模型可求得

$$\Gamma(E)=\sum_l(2l+1)n(E)$$
$$=\frac{1}{\pi}\int_l(2l+1)\mathrm{d}l\int_{r_{+}+\varepsilon}^{r_{+}+\varepsilon+h}p\mathrm{d}r.\tag{15}$$

将(15)和(13)式,代入(14)式可得到自由能

$$F=-\frac{1}{\pi}\int_0^{\infty}\frac{\mathrm{d}E}{\mathrm{e}^{\beta E}+1}\int_l(2l+1)\mathrm{d}l\int_{r_{+}+\varepsilon}^{r_{+}+\varepsilon+h}f^{-1}(r)(r$$
$$-r_{+})^{-1}\left[E^2-f(r)(r-r_{+})(l+\frac{1}{2})^2/r^2\right]^{1/2}\mathrm{d}r$$
$$\approx-\frac{7}{180}\frac{\pi^3r_{+}{}^2}{\beta^4\varepsilon'}f^{-2}(r)\mid_{r=r_{+}},\tag{16}$$

式中 $\varepsilon' = \varepsilon(\varepsilon + h)/h$，$\varepsilon$ 为截断的紫光波长，h 为膜的厚度，积分只保留了 ε^{-1} 主导项. 于是可得到旋量场一个分量的熵为

$$S_1 = \left(\beta^2 \frac{\partial F}{\partial \beta}\right)\Big|_{\beta=\beta_H} = \frac{7\pi^3 r_+{}^2}{45\beta_H^3\varepsilon'} f^{-2}(r)\,|_{r=r_+}, \tag{17}$$

式中 β_H 为视界温度倒数，利用文献[14] $k = \frac{1}{2}\lim_{r\to r_+} g_{00,1}\sqrt{\frac{-g^{11}}{g_{00}}}$ 和 $\beta_H = 2\pi/\kappa$ 可求得 $\beta_H = 4\pi f^{-1}(r)|_{r=r_+}$，根据熵的可加性可求得一类静态球对称黑洞 Dirac 场的统计熵为

$$\begin{aligned} S_F &= S_1 + S_2 + S_3 + S_4 = 4S_1 \\ &= \frac{28\pi^3 r_+{}^2}{45\beta_H^3\varepsilon'} f^{-2}(r)\,|_{r=r_+} = \frac{7\pi^2 A_H}{45\beta_H^3\varepsilon'} f^{-2}(r)\,|_{r=r_+} \\ &= \frac{7A_H}{2880\pi\varepsilon'} f(r)\,|_{r=r_+}, \end{aligned} \tag{18}$$

式中 $A_H = \iint\left(\begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{vmatrix}\right)^{1/2} d\theta d\varphi = 4\pi r_+{}^2$ 为黑洞视界面积，可见黑洞熵是与黑洞视界面积成正比的. 利用求出的(18)式便可不难得到各种静态球对称黑洞 Dirac 场的统计熵.

4. 几种典型的球对称黑洞 Dirac 场的统计熵

4.1. Schwarzschild 黑洞

Schwarzschild 时空中的四维不变线元为

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 \\ &\quad - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \end{aligned} \tag{19}$$

由上式可得到 $f(r)|_{r=r_+} = 1/r_+$ 代入(18)式可得到

$$S_{F(Sch)} = \frac{7A_H}{2880\pi r_+\ \varepsilon'}. \tag{20}$$

当调整截断参数，使 $4r_+\varepsilon' = 1/90\pi$ 时，则

$$S_{F(Sch)} = 7A_H/8 = 7S_{K-G(Sch)}/2, \tag{21}$$

式中 S_{K-G} 为标量场的熵.

4.2. 含荷球对称黑洞

含荷球对称黑洞周围的时空性质由著名的 Ressner-Nordstrŏm 度规确定.

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 \\ &\quad - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \end{aligned} \tag{22}$$

由上式可得到 $f(r)|_{r=r_+} = (r_+ - r_-)/r_+^2$，代入(18)式可得到

$$S_{F(R\text{-}N)} = \frac{7A_H(r_+ - r_0)}{2880\pi r_+{}^2\varepsilon'}. \tag{23}$$

当调整截断参数，使 $4r_+^2\varepsilon'/(r_+ - r_-) = 1/90\pi$ 时，则

$$S_{F(R\text{-}N)} = 7A_H/8 = 7S_{K-G(R\text{-}N)}/2. \tag{24}$$

4.3. Schwarzchild-de Sitter 黑洞

在 Schwarzschild-de Sitter 时空中四维不变线元为

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r} + \frac{1}{3}\lambda r^2\right)dt^2 - \left(1 - \frac{2M}{r} + \frac{1}{3}\lambda r^2\right)^{-1} \\ &\quad \times dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \end{aligned} \tag{25}$$

由上式可得到 $f(r)|_{r=r_+} = 3(r_c - r_+)(r_+ - r_0)/\lambda r_+$，式中 r_c 为宇宙视界，r_0 为一实根，代入(18)式可得到

$$S_{F(S\text{-}D)} = \frac{21A_H(r_c - r_+)(r_+ - r_0)}{2880\pi\lambda r_+\ \varepsilon'}. \tag{26}$$

当调整截断参数，使 $4\lambda r_+\varepsilon'/3(r_c - r_+)(r_+ - r_0) = 1/90\pi$ 时，则

$$S_{F(S\text{-}D)} = 7A_H/8 = 7S_{K-G(S\text{-}D)}/2. \tag{27}$$

以上仅仅求出了 Schwarzschild-de Sitter 视界处的熵，同样也可以计算宇宙视界附近的熵. 两者之和即为 Schwarzschild-de Sitter 黑洞的熵.

综上所述充分表明，当取相同的截断因子时，对于球对称黑洞，Dirac 场的熵均为其标量场的熵的 7/2倍.

[1] Zhao Z and Zhu J Y 1999 *Acta Phys. Sin.* **48** 1558 (in Chinese) [赵 峥、朱建阳 1999 物理学报 **48** 1558]
[2] Lin H 2000 *Acta Phys. Sin.* **49** 1413 (in Chinese) [林 海 2000 物理学报 **49** 1413]
[3] Li C A 2001 *Acta Phys. Sin.* **50** 986 (in Chinese) [李传安 2001 物理学报 **50** 986]
[4] Zhao R and Zhang L C 2001 *Acta Phys. Sin.* **50** 593 (in Chinese) [赵 仁、张丽春 2001 物理学报 **50** 593]
[5] 't Hooft G 1985 *Nucl. Phys.* B **256** 727
[6] Li X and Zhao Z 2000 *Phys. Rev.* D **62** 2821
[7] Ho J W *et al* 1997 *Class. Quant. Grav.* **14** 2617
[8] Jing J L and Yan M L 2000 *Chin. Phys.* **9** 389

[9] Luo Z J and Zhu J Y 1999 *Acta Phys. Sin.* **48** 395 [罗智坚、朱建阳 1999 物理学报 **49** 395]
[10] Liu W B and Zhao Z 2000 *Phys. Rev.* D **61** 2812
[11] Li X and Zhao Z 2001 *Chin. Phys. Lett.* **18** 463
[12] Liu W B and Zhao Z 2001 *Chin. Phys. Lett.* **18** 310
[13] Li Z H 1999 *Int. J. Theor. Phys.* **38** 925
[14] Zhao Z 1999 *Thermal Property of Black Hole and Singularity of Space-Time* (Beijing: Beijing Normal University Pres) p148 (in Chinses) [赵 峥 1999 黑洞的热性质与时空的奇异性(北京:北京师范大学出版社)第148页]

Statistical entropy of the Dirac field of static spherically symmetric black holes

Li Chuan-An Meng Qing-Miao Su Jiu-Qing
(*Heze Institute of Education*, *Heze* 274016, *China*)

(Received 10 December 2001; revised manuscript received 5 February 2002)

Abstract

A statistical entropy of the Dirac field of static spherically symmetric black holes is gvien in this paper by using the improved brick-wall model. The result shows that all these entrpies 7/2 times as large as the entropy of a scalar field when the same cutting factors are taken.

Keywords: black hole, statistical entropy, brick-wall model, Dirac feild
PACC: 9760L, 0420