第 52 卷 第 11 期 2003 年 11 月
1000-3290/2003/52(11)/2694-06

物 理 学 报
ACTA PHYSICA SINICA

Vol.52, No.11, November, 2003
©2003 Chin. Phys. Soc.

由非简并光学参量放大系统获得压缩态光所满足的 Fokker-Planck 方程及其解*

赵超樱　谭维翰　郭奇志

(上海大学物理系，上海　200436)

(2003 年 1 月 3 日收到；2003 年 3 月 17 日收到修改稿)

通过解非简并光学参量放大的 Fokker-Planck 方程，得出压缩态光的腔内最大压缩的量子起伏为 1/16(真空起伏为 1/4)，与已知的简并光学参量放大情形腔内最大压缩为 1/8 相比，压缩度提高了一倍.

关键词：非简并光学参量放大，Fokker-Planck 方程，腔内最大压缩

PACC：0365，4250

1. 引　　言

光学参量振荡与光学参量放大是产生压缩态光较常用的也是在实验上最先实现的方法[1,2]. 对简并光学参量放大(DOPA)获得压缩态光的理论研究，最先由文献[3—6]给出腔内压缩态光在阈值附近处于最大压缩时的量子起伏为 1/8，恰为真空起伏 1/4 的 1/2 倍. 后来作者之一通过解 DOPA 与简并四波混频的 Fokker-Planck(F-P)方程也得出这一结论[7]. 这些研究均是对 DOPA 情形进行的. 如果是非简并光学参量放大(NOPA)，结果又会是怎样的呢？考虑到近年来关于对非简并光学参量的研究，包括理论与实验正在逐渐增多[8—14]，特别是产生频率简并，偏振非简并的 EPR 光束及其在量子信息方面的应用[15,16]，较受人关注. 本文在文献[7]基础上，严格求解了非简并光学参量下转换系统获得压缩态光所满足的 F-P 方程，结果表明：NOPA 情形腔内压缩光最大压缩的量子起伏为 1/16，与上面提到的 DOPA 的 1/8 相比，压缩度提高了一倍.

2. 非简并光学参量下转换系统 F-P 方程

众所周知，在描述非线性随机系统的理论中，除了 Langevin 方程外，还有 F-P 方程. 一般而言，前者要比后者容易处理，特别是产生压缩态的量子光学系统中所遇到的 F-P 方程. 扩散系数为负或零，解有可能发散. 但可以从发散的形式解出发，在求物理量的统计平均时，发散困难可以避免. 这就是文献[7]求解 DOPA 所用的方法. 将上述方法应用于求 NOPA 的 F-P 方程，发现 NOPA 系统(ε, k)可以看成是由两个 DOPA 系统 $1\left(\frac{\varepsilon}{2}, \frac{k}{\sqrt{2}}\right)$ 与 $2\left(-\frac{\varepsilon}{2}, 0\right)$ 所组成，这里 ε, k 分别为 NOPA 系统的增益与损耗. 这样就可用文献[7]方法求解系统 1 与 2，最后得到 NOPA 的解.

为确定起见，设所考虑的 NOPA 系统频率为简并而偏振为非简并的，总的哈密顿量 H 可写为

$$H = H_0 + V + W,$$

式中

$$H_0 = \hbar\omega a_1^\dagger a_1 + \hbar\omega a_2^\dagger a_2 + \sum \hbar\omega_j b_j^\dagger b_j,$$

$$V = \hbar \sum k_j b_j \frac{a_1^\dagger + a_2^\dagger}{2} + \hbar \sum k_j^* b_j^\dagger \frac{a_1 + a_2}{2},$$

$$W = -\frac{\mathrm{i}\hbar}{2}(\varepsilon a_1^\dagger a_2^\dagger - \varepsilon^* a_1 a_2), \tag{1}$$

$a_1^\dagger, a_2^\dagger$；a_1, a_2 分别为信号光和闲置光的产生和湮没算子. 由于频率简并，故有 $\hbar\omega_1 = \hbar\omega_2 = \hbar\omega$. 但信号光和闲置光的偏振为非简并的(例如分别为左、右旋圆偏振光，或 x, y 方向线偏振光). H_0 式中等号右端前两项分别对应于信号光和闲置光的哈密顿量，第三项为热库的哈密顿量，$b_j^\dagger, b_j$ 为热库的产生与湮没

* 上海市教育委员会重点学科基金资助的课题.

算子,V 为信号光、闲置光与热库相互作用哈密顿量,W 则为 NOPA 由一个抽运光子(包含在增益因子 ε 中)的湮没就导致一对非简并的即信号光、闲置光子 $a_1^\dagger$,$a_2^\dagger$ 的产生.在简并参量情形下,$a_1^\dagger$,$a_2^\dagger$;a_1,a_2 属同一偏振模式 $a_1^\dagger = a_2^\dagger = a^\dagger$,故有 $a_1 a_2^\dagger - a_2^\dagger a_1 = aa^\dagger - a^\dagger a = 1$,是不可对易的.这时(1)式的 V 与 W 便过渡到 $V = \hbar\left(\sum k_j b_j a^\dagger + \sum k_j^* b_j^\dagger a\right)$,$W = -\frac{\mathrm{i}\hbar}{2}(\varepsilon a^{\dagger 2} - \varepsilon^* a^2)$.这正是文献[7]所讨论的 DOPA 方程.而在非简并参量情形下,$a_1^\dagger$,$a_2^\dagger$;a_1,a_2 属不同的偏振模式,故有 $a_1 a_2^\dagger - a_2^\dagger a_1 = 0$,$a_2 a_1^\dagger - a_1^\dagger a_2 = 0$,是可对易的.若令

$$a_1 = \frac{b_1}{\sqrt{2}} + \frac{b_2}{\sqrt{2}},\quad a_2 = \frac{b_1}{\sqrt{2}} - \frac{b_2}{\sqrt{2}},$$

$$b_1 = \frac{a_1}{\sqrt{2}} + \frac{a_2}{\sqrt{2}},\quad b_2 = \frac{a_1}{\sqrt{2}} - \frac{a_2}{\sqrt{2}},\tag{2}$$

故有

$$b_1 b_1^\dagger - b_1^\dagger b_1 = \left(\frac{a_1 + a_2}{\sqrt{2}}\right)\left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}\right) - \left(\frac{a_1^\dagger + a_2^\dagger}{\sqrt{2}}\right)\left(\frac{a_1 + a_2}{\sqrt{2}}\right) = 1.$$

同样可证:

$$b_2 b_2^\dagger - b_2^\dagger b_2 = 1,$$
$$b_1 b_2^\dagger - b_2^\dagger b_1 = b_2 b_1^\dagger - b_1^\dagger b_2 = 0,\tag{3}$$

$$V = \hbar\sum k_j b_j\left(\frac{a_1^\dagger + a_2^\dagger}{2}\right) + \hbar\sum k_j^* b_j^\dagger\left(\frac{a_1 + a_2}{2}\right) = \hbar\left(\sum\frac{k_j}{\sqrt{2}}b_j b_1^\dagger + \sum\frac{k_j^*}{\sqrt{2}}b_j^\dagger b_1\right) = V_1.$$

将(2)式代入(1)式,则 W 可以写为

$$W = \frac{\mathrm{i}\hbar}{4}(\varepsilon b_1^{\dagger 2} - \varepsilon^* b_1^2) - \frac{\mathrm{i}\hbar}{4}(\varepsilon b_2^{\dagger 2} - \varepsilon^* b_2^2) = W_1 + W_2.\tag{4}$$

比较(1)与(3),(4)式,便得出非简并参量下转换系统(ε,k)可以看成是由两个简并参量下转换系统 $1\left(\frac{\varepsilon}{2},\frac{k}{\sqrt{2}}\right)$ 与 $2\left(-\frac{\varepsilon}{2},0\right)$ 所组成.故 NOPA 系统的密度矩阵 ρ 可通过 DOPA 系统 1 与 2 的密度矩阵 ρ_1,ρ_2 的乘积来表示.相应地在相干态 P 表示中,准概率 P 可通过准概率 P_1,P_2 的乘积来表示.而 P_1,P_2 可按 DOPA 系统写出它的 F-P 方程(即 $\rho = \rho_1\rho_2 \leftrightarrow P = P_1 P_2$)

$$\frac{\partial P_1}{\partial t} = \frac{k}{\sqrt{2}}\left(\frac{\partial}{\partial\alpha_1}\alpha_1 + \frac{\partial}{\partial\alpha_1^*}\alpha_1^*\right)P_1 - \left(\frac{\varepsilon}{2}\alpha_1^*\frac{\partial}{\partial\alpha_1} + \frac{\varepsilon^*}{2}\alpha_1\frac{\partial}{\partial\alpha_1^*}\right)P_1 + \frac{1}{2}\left(\frac{\varepsilon}{2}\frac{\partial}{\partial\alpha_1}\frac{\partial}{\partial\alpha_1} + \frac{\varepsilon^*}{2}\frac{\partial}{\partial\alpha_1^*}\frac{\partial}{\partial\alpha_1^*}\right)P_1 + \sqrt{2}k\bar{n}\frac{\partial^2}{\partial\alpha_1\partial\alpha_1^*}P_1,\tag{5}$$

$$\frac{\partial P_2}{\partial t} = \left(\frac{\varepsilon}{2}\alpha_2^*\frac{\partial}{\partial\alpha_2} + \frac{\varepsilon^*}{2}\alpha_2\frac{\partial}{\partial\alpha_2^*}\right)P_2 - \frac{1}{2}\left(\frac{\varepsilon}{2}\frac{\partial}{\partial\alpha_2}\frac{\partial}{\partial\alpha_2} + \frac{\varepsilon^*}{2}\frac{\partial}{\partial\alpha_2^*}\frac{\partial}{\partial\alpha_2^*}\right)P_2.\tag{6}$$

(5)和(6)式就是将非简并参量下转换系统(ε,k)表示为两个简并参量下转换系统 $1\left(\frac{\varepsilon}{2},\frac{k}{\sqrt{2}}\right)$ 与 $2\left(-\frac{\varepsilon}{2},0\right)$ 时的 F-P 方程.

3. 简并光学参量下转换系统 F-P 方程的求解

(5)和(6)式的求解相当于求解 DOPA 的 F-P 方程,即

$$\frac{\partial P}{\partial t} = \left[k\left(\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^*\right) - \left(\varepsilon\alpha^*\frac{\partial}{\partial\alpha} + \varepsilon^*\alpha\frac{\partial}{\partial\alpha^*}\right) + \left(\frac{\varepsilon}{2}\frac{\partial^2}{\partial\alpha^2} + \frac{\varepsilon^*}{2}\frac{\partial^2}{\partial\alpha^{*2}}\right) + 2k\bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]P.\tag{7}$$

将(7)式的 F-P 方程用实变量来表示可换系数,

$$\beta = \frac{\alpha + \alpha^*}{\sqrt{2}},\quad \tilde{\beta} = \frac{\alpha - \alpha^*}{\sqrt{2}\mathrm{i}},\tag{8}$$

故有

$$\frac{\partial}{\partial\alpha} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\beta} - \mathrm{i}\frac{\partial}{\partial\tilde{\beta}}\right),$$
$$\frac{\partial}{\partial\alpha^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\beta} + \mathrm{i}\frac{\partial}{\partial\tilde{\beta}}\right).\tag{9}$$

在此基础上求得

$$\frac{\partial}{\partial\alpha}\alpha + \frac{\partial}{\partial\alpha^*}\alpha^* = \frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\tilde{\beta}}\tilde{\beta},$$
$$\alpha^*\frac{\partial}{\partial\alpha} + \alpha\frac{\partial}{\partial\alpha^*} = \frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\tilde{\beta}}\tilde{\beta},$$
$$\frac{1}{2}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\alpha^{*2}}\right) = \frac{1}{2}\left(\frac{\partial^2}{\partial\beta^2} - \frac{\partial^2}{\partial\tilde{\beta}^2}\right),$$
$$\frac{\partial^2}{\partial\alpha\partial\alpha^*} = \frac{1}{2}\left(\frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\tilde{\beta}^2}\right).\tag{10}$$

将上面的结果代入(7)式,得

$$\frac{\partial P}{\partial t} = \left[(k - \varepsilon)\frac{\partial}{\partial\beta}\beta + (k + \varepsilon)\frac{\partial}{\partial\tilde{\beta}}\tilde{\beta}\right.$$

$$+\left(\frac{\varepsilon}{2}+k\bar{n}\right)\frac{\partial^2}{\partial\beta^2}-\left(\frac{\varepsilon}{2}-k\bar{n}\right)\frac{\partial^2}{\partial\tilde{\beta}^2}\Big]P. \tag{11}$$

设 $P=P(\beta)\tilde{P}(\tilde{\beta})$，则(11)式为

$$\frac{\partial P(\beta)}{\partial t}=\left[(k-\varepsilon)\frac{\partial}{\partial\beta}\beta+\left(\frac{\varepsilon}{2}+k\bar{n}\right)\frac{\partial^2}{\partial\beta^2}\right]P(\beta), \tag{12}$$

$$\frac{\partial\tilde{P}(\tilde{\beta})}{\partial t}=\left[(k+\varepsilon)\frac{\partial}{\partial\tilde{\beta}}\tilde{\beta}-\left(\frac{\varepsilon}{2}-k\bar{n}\right)\frac{\partial^2}{\partial\tilde{\beta}^2}\right]\tilde{P}(\tilde{\beta}). \tag{13}$$

(12)和(13)式为 Ornstein-Uhlenbeck 方程，参照文献[17,18]，可得(12)和(13)式的归一化形式解.

$$P(\beta)\mathrm{d}\beta=\exp\left[-\frac{c(\beta-\beta_0\exp[-(k-\varepsilon)t])^2}{(1-\exp[-2(k-\varepsilon)t])}\right]\times\frac{\mathrm{d}\beta}{\int\exp[\,]\mathrm{d}\beta},\qquad c=\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}, \tag{14}$$

$$\tilde{P}(\tilde{\beta})\mathrm{d}\tilde{\beta}=\exp\left[\frac{\tilde{c}\,(\tilde{\beta}-\tilde{\beta}_0\exp[-(k+\varepsilon)t])^2}{(1-\exp[-2(k+\varepsilon)t])}\right]\times\frac{\mathrm{d}\tilde{\beta}}{\int\exp[\,]\mathrm{d}\tilde{\beta}},\qquad \tilde{c}=\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}. \tag{15}$$

这里用的是 Ornstein-Uhlenbeck 方程的解法[17,18]，与文献[7]用格林函数解 F-P 方程[7]得出的结果一致.

在阈值以下，$k-\varepsilon>0$，且热噪音背景甚少，$\varepsilon-2k\bar{n}>0$ 情况下，$c,\tilde{c}>0$.(14)式含 β 的因子 $\exp[\,]$ 当 $\beta\to\infty$ 是收敛的，归一化因子为 $\frac{1}{\int\exp[\,]\mathrm{d}\beta}$. 但(15)式含 $\tilde{\beta}$ 的因子 $\exp[\,]$ 当 $\tilde{\beta}\to\infty$ 则是发散的. 虽然也引进归一化因子 $\frac{1}{\int\exp[\,]\mathrm{d}\tilde{\beta}}$，解(15)式只能看作形式解. 虽然如此，正如在文献[7]中所做的，并不妨碍求 $(\tilde{\beta}-\tilde{\beta}_0\exp[-(k+\varepsilon)t])^2$ 的统计平均值，亦即应用(14)和(15)式，可得出如下结果：

$$\langle(\beta-\beta_0\exp[-(k-\varepsilon)t])^2\rangle=\frac{1}{2c}(1-\exp[-2(k-\varepsilon)t]), \tag{16}$$

$$\langle(\tilde{\beta}-\tilde{\beta}_0\exp[-(k+\varepsilon)t])^2\rangle=-\frac{1}{2\tilde{c}}(1-\exp[-2(k+\varepsilon)t]). \tag{17}$$

将(16)和(17)式中参数(ε,k)用(15)式中参数 $\left(\frac{\varepsilon}{2},\frac{k}{\sqrt{2}}\right)$ 代替，便得出 DOPA 系统 1 的解，故有

$$\left\langle\left(\beta_1-\beta_{10}\exp\left[-\left(\frac{k}{\sqrt{2}}-\frac{\varepsilon}{2}\right)t\right]\right)^2\right\rangle=\frac{1}{2c_1}\left(1-\exp\left[-2\left(\frac{k}{\sqrt{2}}-\frac{\varepsilon}{2}\right)t\right]\right),$$
$$\left\langle\left(\tilde{\beta}_1-\tilde{\beta}_{10}\exp\left[-\left(\frac{k}{\sqrt{2}}+\frac{\varepsilon}{2}\right)t\right]\right)^2\right\rangle=-\frac{1}{2\tilde{c}_1}\left(1-\exp\left[-2\left(\frac{k}{\sqrt{2}}+\frac{\varepsilon}{2}\right)t\right]\right), \tag{18}$$

$$c_1=\frac{\frac{k}{\sqrt{2}}-\frac{\varepsilon}{2}}{\frac{\varepsilon}{2}+\sqrt{2}k\bar{n}},\qquad \tilde{c}_1=\frac{\frac{k}{\sqrt{2}}+\frac{\varepsilon}{2}}{\frac{\varepsilon}{2}-\sqrt{2}k\bar{n}}. \tag{19}$$

对于系统 $2\left(-\frac{\varepsilon}{2},0\right)$，在作这代换之前，先应注意到(14)和(15)式可分别看成正能系统 ε 与负能系统 $-\varepsilon$ 的准概率分布解. 因为将(14)式中 ε 换成 $-\varepsilon$，便得

$$c=\frac{k-\varepsilon}{\varepsilon+2k\bar{n}}\Rightarrow-\frac{k+\varepsilon}{\varepsilon-2k\bar{n}}=-\tilde{c},$$
$$P(\beta)\mathrm{d}\beta=\exp\left[-\frac{c(\beta-\beta_0\exp[-(k-\varepsilon)t])^2}{(1-\exp[-2(k-\varepsilon)t])}\right]\times\frac{\mathrm{d}\beta}{\int\exp[\,]\mathrm{d}\beta}$$
$$\Rightarrow\exp\left[\frac{\tilde{c}\,(\beta-\beta_0\exp[-(k+\varepsilon)t])^2}{(1-\exp[-2(k+\varepsilon)])}\right]\times\frac{\mathrm{d}\beta}{\int\exp[\,]\mathrm{d}\beta}. \tag{20}$$

将(20)式等号右端与(15)式相比，除了 β 与 $\tilde{\beta}$ 记号外，与 $\tilde{P}(\tilde{\beta})\mathrm{d}\tilde{\beta}$ 的一致，亦即(20)式等号左端 $P(\beta)\times\mathrm{d}\beta$为正能系统的准概率分布，而右端则为经置换 $\varepsilon\to-\varepsilon$后，即负能系统的准概率分布. 现在回到系统 $2\left(-\frac{\varepsilon}{2},0\right)$，用这个参量代入(14)和(15)式，并注意：

$$c_2=\frac{0-\left(\frac{-\varepsilon}{2}\right)}{-\frac{\varepsilon}{2}+0}=-1,\qquad \tilde{c}_2=\frac{0+\left(\frac{-\varepsilon}{2}\right)}{-\frac{\varepsilon}{2}+0}=1,$$

便得

$$P(\beta)\mathrm{d}\beta=\exp\left[\frac{\left(\beta_2-\beta_{20}\exp\left(-\frac{\varepsilon}{2}t\right)\right)^2}{1-\exp(-\varepsilon t)}\right]\frac{\mathrm{d}\beta}{\int\exp[\,]\mathrm{d}\beta}, \tag{21}$$

$$\tilde{P}(\tilde{\beta})\mathrm{d}\tilde{\beta} = \exp\left[\frac{\left(\tilde{\beta}_2 - \tilde{\beta}_{20}\exp\frac{\varepsilon}{2}t\right)^2}{1-\exp\varepsilon t}\right]\frac{\mathrm{d}\tilde{\beta}}{\int\exp[\,]\,\mathrm{d}\tilde{\beta}}. \tag{22}$$

由(21)和(22)式求得

$$\left\langle\left(\beta_2 - \beta_{20}\exp\left(-\frac{\varepsilon}{2}t\right)\right)^2\right\rangle = -\frac{1}{2}(1-\exp(-\varepsilon t)), \tag{23}$$

$$\left\langle\left(\tilde{\beta}_2 - \tilde{\beta}_{20}\exp\frac{\varepsilon}{2}t\right)^2\right\rangle = -\frac{1}{2}(1-\exp\varepsilon t). \tag{24}$$

现在要强调的是,系统初始能量为负的,亦即(21)和(23)式分别代表能量为 $-\frac{\varepsilon}{2}$ 的系统的准概率与均方起伏.而(22)和(24)式则为代表能量为 $\frac{\varepsilon}{2}$ 的系统的准概率与均方起伏,故在求总的包括系统 1 与 2 的量子起伏时,应将系统 1 的正能部分起伏 $\langle(\Delta\beta_1)^2\rangle$ 与系统 2 的正能部分 $\langle(\Delta\tilde{\beta}_2)^2\rangle$ 相加并乘以权重1/2,同样系统 2 的负能部分量子起伏由 $\langle(\Delta\tilde{\beta}_1)^2\rangle$ 与 $\langle(\Delta\beta_2)^2\rangle$ 相加并乘以权重 1/2,即得

$$\langle(\Delta\beta)^2\rangle = \frac{1}{2}[\langle(\Delta\beta_1)^2\rangle + \langle(\Delta\tilde{\beta}_2)^2\rangle],$$

$$\langle(\Delta\tilde{\beta})^2\rangle = \frac{1}{2}[\langle(\Delta\tilde{\beta}_1)^2\rangle + \langle(\Delta\beta_2)^2\rangle]. \tag{25}$$

这关系也可以通过将(4)式写成如下形式予以证明:

$$W = \frac{\mathrm{i}\hbar}{2}\left[\varepsilon\left(\frac{b_1^{\dagger 2}}{2} + \frac{(\mathrm{i}b_2^\dagger)^2}{2}\right) - \varepsilon^*\left(\frac{b_1^2}{2} + \frac{(-\mathrm{i}b_2)^2}{2}\right)\right]$$
$$= \frac{\mathrm{i}\hbar}{2}(\varepsilon b^{\dagger 2} - \varepsilon^* b^2), \tag{26}$$

式中 b, b_1, b_2 分别为总体系与分体系 1,2 的光子数算符.注意到 $\langle\Delta(b_1^\dagger b_2^\dagger)\rangle = \langle b_1^\dagger \Delta b_2^\dagger + (\Delta b_1^\dagger) b_2^\dagger\rangle = 0$,同样 $\langle\Delta(b_1 b_2)\rangle = 0$,便得

$$\langle\Delta b^{\dagger 2}\rangle = \left\langle\frac{\Delta(b_1^\dagger)^2}{2} + \frac{\Delta(\mathrm{i}b_2^\dagger)^2}{2}\right\rangle$$
$$= \left\langle\Delta\left(\frac{b_1^\dagger + \mathrm{i}b_2^\dagger}{\sqrt{2}}\right)^2\right\rangle,$$
$$\langle\Delta b^2\rangle = \left\langle\frac{\Delta b_1^2}{2} + \frac{\Delta(-\mathrm{i}b_2)^2}{2}\right\rangle$$
$$= \left\langle\Delta\left(\frac{b_1 - \mathrm{i}b_2}{\sqrt{2}}\right)^2\right\rangle. \tag{27}$$

在相干态 P 表示中 $b, b_1, \mathrm{i}b_2$ 可表示为

$$b = \frac{\beta + \mathrm{i}\tilde{\beta}}{\sqrt{2}},\quad b_1 = \frac{\beta_1 + \mathrm{i}\tilde{\beta}_1}{\sqrt{2}},$$
$$-\mathrm{i}b_2 = -\mathrm{i}\frac{\beta_2 + \mathrm{i}\tilde{\beta}_2}{\sqrt{2}} = \frac{\tilde{\beta}_2 - \mathrm{i}\beta_2}{\sqrt{2}},$$
$$b^* = \frac{\beta - \mathrm{i}\tilde{\beta}}{\sqrt{2}},\quad b_1^* = \frac{\beta_1 - \mathrm{i}\tilde{\beta}_1}{\sqrt{2}},$$
$$\mathrm{i}b_2^* = \mathrm{i}\frac{\beta_2 - \mathrm{i}\tilde{\beta}_2}{\sqrt{2}} = \frac{\tilde{\beta}_2 + \mathrm{i}\beta_2}{\sqrt{2}}. \tag{28}$$

由(27)和(28)式便得

$$\beta = \frac{\beta_1 + \tilde{\beta}_2}{\sqrt{2}},\quad \tilde{\beta} = \frac{\tilde{\beta}_1 - \beta_2}{\sqrt{2}}, \tag{29}$$

故 b 的实部 $x = \frac{\beta}{\sqrt{2}}$,虚部 $y = \frac{\tilde{\beta}}{\sqrt{2}}$ 的量子起伏可写为

$$\langle(\Delta x)^2\rangle = \frac{1}{4} + \langle:(\Delta x)^2:\rangle = \frac{1}{4} + \left\langle\frac{(\Delta\beta)^2}{2}\right\rangle$$
$$= \frac{1}{4} + \frac{1}{4}[\langle(\Delta\beta_1)^2\rangle + \langle(\Delta\tilde{\beta}_2)^2\rangle]$$
$$= \frac{1}{4} + \frac{1}{4}\left[\frac{1}{2}\frac{\frac{\varepsilon}{2}+\sqrt{2}k\bar{n}}{\frac{k}{\sqrt{2}}-\frac{\varepsilon}{2}}\times\left(1-\exp\left[-2\left(\frac{k}{\sqrt{2}}-\frac{\varepsilon}{2}\right)t\right]\right) - \frac{1}{2}(1-\exp\varepsilon t)\right],$$

$$\langle(\Delta y)^2\rangle = \frac{1}{4} + \langle:(\Delta y)^2:\rangle = \frac{1}{4} + \left\langle\frac{(\Delta\tilde{\beta})^2}{2}\right\rangle$$
$$= \frac{1}{4} + \frac{1}{4}[\langle(\Delta\tilde{\beta}_1)^2\rangle + \langle(\Delta\beta_2)^2\rangle]$$
$$= \frac{1}{4} - \frac{1}{4}\left[\frac{1}{2}\frac{\frac{\varepsilon}{2}-\sqrt{2}k\bar{n}}{\frac{k}{\sqrt{2}}+\frac{\varepsilon}{2}}\times\left(1-\exp\left[-2\left(\frac{k}{\sqrt{2}}+\frac{\varepsilon}{2}\right)t\right]\right) + \frac{1}{2}(1-\exp(-\varepsilon t))\right]. \tag{30}$$

现在就上式进行讨论,当工作于阈值以下时,$\frac{k}{\sqrt{2}} - \frac{\varepsilon}{2} \geqslant 0$,压缩分量为

$$\langle(\Delta y)^2\rangle = \frac{1}{4} - \frac{1}{4}\left[\frac{1}{2}\frac{\frac{\varepsilon}{2} - \sqrt{2}k\bar{n}}{\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}} \times \left(1 - \exp\left[-2\left(\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}\right)t\right]\right) + \frac{1}{2}(1 - \exp(-\varepsilon t))\right]$$

$$\geqslant \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\frac{\frac{\varepsilon}{2} - \sqrt{2}k\bar{n}}{\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}}\right)$$

$$\approx \frac{1}{8}\frac{\frac{k}{\sqrt{2}} + \sqrt{2}k\bar{n}}{\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}} \geqslant \frac{1}{16},$$

$$\langle(\Delta x)^2\rangle\langle(\Delta y)^2\rangle \approx \frac{1}{16}\left(1 + \frac{1}{2}\frac{\frac{\varepsilon}{2}}{\frac{k}{\sqrt{2}} - \frac{\varepsilon}{2}} - \frac{1}{2}\right) \times \left(1 - \frac{1}{2}\frac{\frac{\varepsilon}{2}}{\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}} - \frac{1}{2}\right) \geqslant \frac{1}{16}.$$

最大压缩 1/16 为真空起伏 1/4 的 1/4.若工作于阈值以上，$\frac{k}{\sqrt{2}} - \frac{\varepsilon}{2} < 0$，且$\frac{\varepsilon}{2} \gg \frac{k}{\sqrt{2}}$，$t$ 很大时，

$$\langle(\Delta x)^2\rangle = \frac{1}{8}\left[1 + \frac{\frac{\varepsilon}{2}}{\frac{k}{\sqrt{2}} - \frac{\varepsilon}{2}}\left(1 - \exp\left[-2\left(\frac{k}{\sqrt{2}} - \frac{\varepsilon}{2}\right)t\right]\right) + \exp\varepsilon t\right] \approx \frac{1}{4}\exp\varepsilon t,$$

$$\langle(\Delta y)^2\rangle = \frac{1}{8}\left[1 - \frac{\frac{\varepsilon}{2}}{\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}}\left(1 - \exp\left[-2\left(\frac{k}{\sqrt{2}} + \frac{\varepsilon}{2}\right)t\right]\right) + \exp(-\varepsilon t)\right] \approx \frac{1}{4}\exp(-\varepsilon t),$$

$\langle(\Delta x)^2\rangle$，$\langle(\Delta y)^2\rangle$仍满足测不准关系：

$$\langle(\Delta x)^2\rangle\langle(\Delta y)^2\rangle \geqslant \frac{1}{16}.$$

4. 讨论与小结

本文在文献[7]基础上，首先将非简并光学参量下转换系统(ε，k)化成两个简并参量下转换系统 $1\left(\frac{\varepsilon}{2},\frac{k}{\sqrt{2}}\right)$与 $2\left(-\frac{\varepsilon}{2},0\right)$两部分组成的系统.然后利用 Ornstein-Uhlenbeck 方程的严格解求解了简并光学参量下转换系统获得压缩态光所满足的 F-P 方程.这样就可以通过求解系统 1 与 2，最后求得 NOPA 方程的 F-P 解.结果表明：NOPA 情形腔内压缩光最大压缩的量子起伏为 1/16.与已知的 DOPA 情形腔内最大压缩为 1/8 相比，压缩度提高了一倍.从上面计算可以看到，这主要是因为第 2 负能系统的阻尼 $k=0$，因而与真空场的相互作用 $2k\bar{n}=0$，为理想的系统，对总系统的量子起伏不作出贡献.而总系统的量子起伏是由第 1 系统作出的，而第 1 系统的权重为 1/2，故总系统的量子起伏在最佳压缩状态下，也就是$\frac{1}{2}\times\frac{1}{8}=\frac{1}{16}$.

[1] Slusher R E，Hollberg L W，Yurke B，Mertz J C and Valley J F 1985 *Phys*. *Rev*. *Lett*. **55** 2409

[2] Wu L A，Kimble H J，Hall J L and Wu H F 1986 *Phys*. *Rev*. *Lett*. **57** 2520

[3] Milburn G and Walls D F 1981 *Opt*. *Commun*. **39** 401

[4] Kumar P and Shapiro J H 1984 *Phys*. *Rev*. A **30** 1568

[5] Wolinsky M and Carmichael H J 1985 *Opt*. *Commun*. **55** 138

[6] Bondurant R S，Kumar P，Shapiro J H and Maeda M 1984 *Phys*. *Rev*. A **30** 343

[7] Tan W H，Li Y F and Zhang W P 1988 *Acta Phys*. *Sin*. **37** 396 (in Chinese)［谭维翰、李宇舫、张卫平 1988 物理学报 **37** 396］
Tan W H，Li Y F and Zhang W P 1987 *Opt*. *Commun*. **64** 195

[8] Drummond P D and Reid M D 1990 *Phys*. *Rev*. A **41** 3930

[9] Reid M D 1989 *Phys*. *Rev*. A **40** 913

[10] Ou Z Y *et al* 1992 *Phys*. *Rev*. *Lett*. **68** 3663

[11] Wang D L，Gong Q H，Wang K G and Yang G J 2000 *Acta Phys*. *Sin*. **49** 1484 (in Chinese)［王丹翎、龚旗煌、汪凯戈、杨国健 2000 物理学报 **49** 1484］

[12] Gamal M 1999 *Acta Phys*. *Sin*. (Overseas Edition) **8** 754

[13] Li Y M，Wu Y R，Zhang K S and Peng K C 2003 *Acta Phys*. *Sin*. **52** 849 (in Chinese)［李永民、吴迎瑞、张宽收、彭堃墀 2003 物理学报 **52** 849］

[14] Zheng S B 2003 *Chin. Phys.* **12** 51

[15] Li X Y, Jing J T, Zhang J, Pan Q, Xie C D and Peng K C 2002 *Acta Phys. Sin.* **51** 966 (in Chinese) [李小英、荆杰泰、张 靖、潘 庆、谢常德、彭堃墀 2002 物理学报 **51** 966]

[16] Zhang Y, Wang H, Li X Y, Jing J T, Xie C D and Peng K C 2000 *Phys. Rev.* A. **62** 23813

[17] Gardiner C W 1983 *Handbook of Stochastic Methods* (New York: Springer-Verlag) p75

[18] Tan W H 2000 *Nonlinear and Quantum Optics* 2nd ed (Beijing: Science Press) p374 (in Chinese) [谭维翰 2000 非线性与量子光学(第二版)(北京 科学出版社)第 374 页]

The solution of the Fokker-Planck equation of non-degenerate parametric amplification system for generation of squeezed light *

Zhao Chao-Ying Tan Wei-Han Guo Qi-Zhi

(*Department of Physics*, *Shanghai University*, *Shanghai* 200436, *China*)

(Received 3 January 2003; revised manuscript received 17 March 2003)

Abstract

In this paper, we present the analytical solution of the Fokker-Planck equation of non-degenerate optical parametric amplification (NOPA) for generation of squeezed light. The maximum intra-cavity compression of squeezed light derived from the analytical solution is 1/16 (vacuum fluctuations 1/4). To compare it with that of the previous result 1/8 of degenerate optical parametric amplification (DOPA), it seems that the squeezing for NOPA is superior to the squeezing for DOPA.

Keywords: non-degenerate optical parametric amplification, Fokker-Planck equation, maximum intra-cavity compression

PACC: 0365, 4250

* Project supported by the Foundation of the Education Commission of Shanghai, China.