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开边界六顶角模型的边界关联函数

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讨论了具有开边界六顶角模型的关联函数,计算了涉及边界自发极化的边界关联函数,得到了它们的行列式表示.

关键词:六顶角模型, 关联函数, 开边界

PACC:0370, 0380, 7510J

1. 引 言

六顶角模型是二维经典统计力学中的一个重要模型.它是作为描述二维冰晶结构而引起的[1].与 Ising 模型一样,六顶角模型也是一个精确可解的统计模型[2,3].其自由能可由 Bethe Ansatz 方法得到.六顶角模型不光在周期性边界条件下被广泛研究,而且在其他边界条件(例如畴壁边界条件[4]、反射边界条件[5])下也被证明精确可解,其配分函数可表示成由谱函数所构成的行列式[6—8],可以精确计算其边界自发极化[9—11].本文将在开边界条件下讨论六顶角模型的关联函数,计算描述其边界自发极化的边界关联函数,得到了它们的行列式表示.

2. 边界关联函数

本文讨论的二维 $N\times 2N$ 格点如图 1 所示.其上、下和右边界为畴壁边界条件,左边界为反射边界.对应 $2M-1$ 行的谱为 $-\lambda_M$,$2M$ 行的谱为 λ_M,对应 k 列的谱为 ν_k.α 行 k 列格点的权在矩阵形式下可写为

$$L_{\alpha k}(\lambda_\alpha,\nu_k) = L_{\alpha k}(\lambda_\alpha-\nu_k) = \begin{pmatrix} \dfrac{\sinh\left(\lambda_\alpha-\nu_k+\frac{1}{2}\eta\sigma_k^3\right)}{\sinh\left(\lambda_\alpha-\nu_k+\frac{1}{2}\eta\right)} & \dfrac{\sinh(\eta)\sigma_k^-}{\sinh\left(\lambda_\alpha-\nu_k+\frac{1}{2}\eta\right)} \\ \dfrac{\sinh(\eta)\sigma_k^+}{\sinh\left(\lambda_\alpha-\nu_k+\frac{1}{2}\eta\right)} & \dfrac{\sinh\left(\lambda_\alpha-\nu_k-\frac{1}{2}\eta\sigma_k^3\right)}{\sinh\left(\lambda_\alpha-\nu_k+\frac{1}{2}\eta\right)} \end{pmatrix}_{[\alpha]}, \tag{1}$$

$\sigma_k^{1,2,3}$ 为作用在第 k 列上的泡利矩阵,$\sigma_k^{\pm}=\frac{1}{2}(\sigma_k^1\pm \mathrm{i}\sigma_k^2)$.左边界上的反射矩阵为

$$\widetilde{K}_+(\lambda_\alpha) = \begin{pmatrix} 0 & \cosh\left(\lambda_\alpha+\frac{1}{2}\eta-\frac{\mathrm{i}\pi}{2}+\zeta_+\right) \\ \cosh\left(-\lambda_\alpha-\frac{1}{2}\eta+\frac{\mathrm{i}\pi}{2}+\zeta_+\right) & 0 \end{pmatrix}_{[\alpha]}.$$

图1 中 $2M-1$ 和 $2M$ 行的格点对配分函数的贡献为

$$\downarrow T^{t_M}(\lambda_M)\widetilde{K}_+(\lambda_M)T(-\lambda_M)\downarrow = \downarrow U^{t_M}(\lambda_M)\uparrow,$$

其中 $T(\lambda_\alpha)=L_{\alpha N}(\lambda_\alpha,\nu_N)\cdots L_{\alpha 1}(\lambda_\alpha,\nu_1)$,为体中的单值矩阵,$U^{t_\alpha}(\lambda_\alpha)$为代数 Bethe Ansatz 方法中所定义的带边单值矩阵[5],

$$U^{t_\alpha}(\lambda_\alpha) = T^{t_\alpha}(\lambda_\alpha)K_+(\lambda_\alpha)\sigma_\alpha^2 T(-\lambda_\alpha)\sigma_\alpha^2$$

$$= \begin{pmatrix} \mathscr{A}(\lambda_\alpha) & \mathscr{C}(\lambda_\alpha) \\ \mathscr{B}(\lambda_\alpha) & \mathscr{D}(\lambda_\alpha) \end{pmatrix}_{[\alpha]}, \tag{2}$$

$$K_+(\lambda_\alpha) = \widetilde{K}_+(\lambda_\alpha)\sigma_\alpha^2 = \begin{pmatrix} \sinh\left(\lambda_\alpha+\frac{1}{2}\eta+\zeta_+\right) & 0 \\ 0 & \sinh\left(-\lambda_\alpha-\frac{1}{2}\eta+\zeta_+\right) \end{pmatrix}_{[\alpha]}.$$

因此该二维格点等价于图 2 所示的二维格点,其配分函数可以表示为

$$Z_N(\{\lambda_\alpha\}_N, \{\nu_k\}_N) = w_N^- \prod_{\alpha=1}^{N} \mathscr{B}(\lambda_\alpha) w_N^+,$$

$w_N^\pm$ 为赝真空态,分别定义为 $w_N^+ = \prod_{i=1}^N \uparrow_i$, $w_N^- = \prod_{i=1}^N \downarrow_i$. (2)式中定义的单值矩阵满足边界杨-Baxter 关系,

$$\begin{aligned} & R_{12}(-\lambda_1+\lambda_2)U^{t_1}(\lambda_1) \\ & \times R_{12}(-\lambda_1-\lambda_2-\eta)U^{t_2}(\lambda_2) \\ = & U^{t_2}(\lambda_2)R_{12}(-\lambda_1-\lambda_2-\eta) \\ & \times U^{t_1}(\lambda_1)R_{12}(-\lambda_1+\lambda_2), \end{aligned} \tag{3}$$

其中 R_{12} 为杨-Baxter 方程的三角形式解,

$$R(\lambda) = \begin{pmatrix} \sinh(\lambda+\eta) & & & \\ & \sinh(\lambda) & \sinh(\eta) & \\ & \sinh(\eta) & \sinh(\lambda) & \\ & & & \sinh(\lambda+\eta) \end{pmatrix}.$$

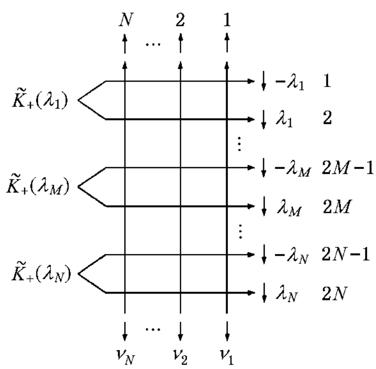

图 1　开边界的六顶角模型

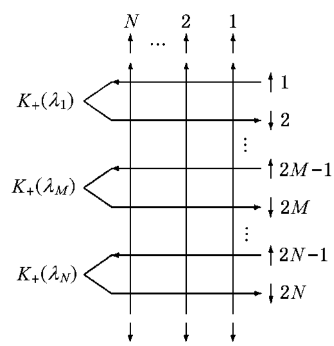

图 2　双行单值矩阵表示的六顶角模型

由(3)式可确定单值矩阵的矩阵元 $\mathscr{A}(\lambda_\alpha)$, $\mathscr{B}(\lambda_\alpha)$, $\mathscr{C}(\lambda_\alpha)$和 $\mathscr{D}(\lambda_\alpha)$之间的交换关系,其中和本文后面的计算有关的是如下三个交换关系:

$$\begin{aligned} \mathscr{A}(\lambda_\alpha)\mathscr{B}(\lambda_\beta) = & \frac{\sinh^2(\lambda_\alpha-\eta)-\sinh^2\lambda_\beta}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\beta)\mathscr{A}(\lambda_\alpha) \\ & + \frac{\sinh\eta\sinh(\lambda_\alpha+\lambda_\beta-\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\alpha)\mathscr{A}(\lambda_\beta) \\ & - \frac{\sinh(2\eta)\sinh(\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\beta)\mathscr{D}(\lambda_\alpha) \\ & + \frac{\sinh\eta\sinh(2\eta-\lambda_\alpha+\lambda_\beta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\alpha)\mathscr{D}(\lambda_\beta); \\ \mathscr{D}(\lambda_\alpha)\mathscr{B}(\lambda_\beta) = & \frac{\sinh^2(\lambda_\alpha+\eta)-\sinh^2\lambda_\beta}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\beta)\mathscr{D}(\lambda_\alpha) \\ & - \frac{\sinh\eta\sinh(\lambda_\alpha+\lambda_\beta+\eta)}{\sinh^2\lambda_\alpha-\sinh^2\lambda_\beta}\mathscr{B}(\lambda_\alpha)\mathscr{D}(\lambda_\beta) \\ & + \frac{\sinh\eta}{\sinh(\lambda_\alpha+\lambda_\beta)}\mathscr{B}(\lambda_\alpha)\mathscr{A}(\lambda_\beta); \\ \mathscr{B}(\lambda_\alpha)\mathscr{B}(\lambda_\beta) = & \mathscr{B}(\lambda_\beta)\mathscr{B}(\lambda_\alpha). \end{aligned} \tag{4}$$

矩阵元 $\mathscr{A}(\lambda_\alpha)$和 $\mathscr{D}(\lambda_\alpha)$作用在赝真空态 w_N^+ 上的本征值分别为 $\Delta_\pm^N(\lambda_\alpha)$,

$$\mathscr{A}(\lambda_\alpha)w_N^+ = \Delta_+^N(\lambda_\alpha)w_N^+,$$

$$\mathscr{D}(\lambda_\alpha)w_N^+ = \Delta_-^N(\lambda_\alpha)w_N^+,$$

$$\begin{aligned} \Delta_+^N(\lambda_\alpha) = & \frac{\sinh(2\lambda_\alpha+\eta)}{\sinh(2\lambda_\alpha)}\sinh\left(\lambda_\alpha-\frac{1}{2}\eta+\zeta_+\right) \\ & \times \delta_N(-\lambda_\alpha) + \frac{\sinh\eta}{\sinh(2\lambda_\alpha)} \\ & \times \sinh\left(\lambda_\alpha+\frac{1}{2}\eta-\zeta_+\right)\delta_N(\lambda_\alpha), \end{aligned}$$

$$\Delta_-^N(\lambda_\alpha) = -\sinh\left(\lambda_\alpha+\frac{1}{2}\eta-\zeta_+\right)\delta_N(\lambda_\alpha),$$

$$\delta_N(\lambda_\alpha) = \prod_{i=1}^N \frac{\sinh\left(\lambda_\alpha-\nu_i-\frac{1}{2}\eta\right)}{\sinh\left(\lambda_\alpha-\nu_i+\frac{1}{2}\right)}.$$

定义如下关联函数:

$$\begin{aligned} f_1^M = & Z_N^{-1}w_N^-\left[\prod_{\alpha=M+1}^{N}\mathscr{B}(\lambda_\alpha)\right]\left[\sinh\left(\lambda_M\right.\right. \\ & \left.+\frac{1}{2}\eta+\zeta_+\right)q_1B(\lambda_M)p_1D(-\lambda_M) \end{aligned}$$

$$+\sinh\left(\lambda_M+\frac{1}{2}\eta-\zeta_+\right)q_1 \times D(\lambda_M)p_1B(-\lambda_M)\Big]\Big[\prod_{\alpha=1}^{M-1}\mathscr{B}(\lambda_\alpha)\Big]w_N^+;$$

$$f_2^M=Z_N^{-1}w_N^-\Big[\prod_{\alpha=M+1}^{N}\mathscr{B}(\lambda_\alpha)\Big]\Big[\sinh\left(\lambda_M+\frac{1}{2}\eta+\zeta_+\right)B(\lambda_M)q_1D(-\lambda_M)p_1 +\sinh\left(\lambda_M+\frac{1}{2}\eta-\zeta_+\right)D(\lambda_M)q_1 \times B(-\lambda_M)p_1\Big]\Big[\prod_{\alpha=1}^{M-1}\mathscr{B}(\lambda_\alpha)\Big]w_N^+;$$

$$f_3^M=Z_N^{-1}w_N^-\Big[\prod_{\alpha=M+1}^{N}\mathscr{B}(\lambda_\alpha)\Big]q_1\Big[\sum_{\alpha=1}^{M}\mathscr{B}(\lambda_\alpha)\Big]w_N^+;$$

$$f_4^M=Z_N^{-1}w_N^-\Big[\prod_{\alpha=M+1}^{N}\mathscr{B}(\lambda_\alpha)\Big]\Big[\sinh\left(\lambda_M+\frac{1}{2}\eta+\zeta_+\right)B(\lambda_M)q_1D(-\lambda_M) +\sinh\left(\lambda_M+\frac{1}{2}\eta-\zeta_+\right)D(\lambda_M)q_1 \times B(-\lambda_M)\Big]\Big[\sum_{\alpha=1}^{M-1}\mathscr{B}(\lambda_\alpha)\Big]w_N^+, \tag{5}$$

其中 $q_1=\frac{1}{2}(1-\sigma_1^3)$，$p_1=\frac{1}{2}(1+\sigma_1^3)$，$f_1^M$和 f_2^M分别描述右边界处在 $2M$ 行和 $2M-1$ 行之前自旋向下的概率，f_3^M和 f_4^M分别描述在 $2M$ 行和 $2M-1$ 行的边界自发极化概率.

3. 行列式表示

由于 q_1 和 p_1 是作用在第一列上的算子，因此在计算这些边界关联函数时可以把(5)式中的所有算子分解为第一列上的算子和其余 $N-1$ 列上的算子.利用 $L_{\alpha1}(\lambda_\alpha-\nu_1)$的具体表达式(1)，$f_1^M$可以写为

$$f_1^M=\frac{-\sinh\eta}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_M\right)} \times\prod_{\alpha=1}^{M}\frac{\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}{\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]} \times Z_N^{-1}w_{N-1}^-\Big[\prod_{\gamma=M+1}^{N}\hat{\mathscr{B}}(\lambda_\gamma)\Big] \times\hat{\mathscr{D}}(\lambda_M)\Big[\prod_{\gamma=1}^{M-1}\hat{\mathscr{B}}(\lambda_\gamma)\Big]w_{N-1}^+, \tag{6}$$

其中 $\hat{\mathscr{B}}(\lambda_\gamma)$和 $\hat{\mathscr{D}}(\lambda_\gamma)$为作用在从第二列到第 N 列的算子，由下式确定：

$$\hat{U}^{t_\alpha}(\lambda_\alpha)=\hat{T}^{t_\alpha}(\lambda_\alpha)K_+(\lambda_\alpha)\sigma_\alpha^2\hat{T}(-\lambda_\alpha)\sigma_\alpha^2 =\begin{pmatrix}\hat{\mathscr{A}}(\lambda_\alpha) & \hat{\mathscr{C}}(\lambda_\alpha)\\ \hat{\mathscr{B}}(\lambda_\alpha) & \hat{\mathscr{D}}(\lambda_\alpha)\end{pmatrix}_{[\alpha]},$$

$$\hat{T}_\alpha(\lambda_\alpha)=L_{\alpha N}(\lambda_\alpha,\nu_N)\cdots L_{\alpha2}(\lambda_\alpha,\nu_1).$$

$\hat{\mathscr{A}}$，$\hat{\mathscr{B}}$，$\hat{\mathscr{C}}$和 $\hat{\mathscr{D}}$之间的交换关系也与(4)式中所定义的关系式一样.应用(4)式中的关系式，可知

$$\hat{\mathscr{A}}(\lambda_\alpha)\Big[\prod_{\beta=1}^{\alpha-1}\hat{\mathscr{B}}(\lambda_\beta)\Big]w_{N-1}^+ =\sum_{\beta=1}^{\alpha}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta\right]}{\prod_{\gamma=1,\neq\beta}^{\alpha}(\sinh^2\lambda_\beta-\sinh^2\lambda_\gamma)} \times a_\beta^{(\alpha)}\Big[\prod_{\gamma=1,\neq\beta}^{\alpha}\hat{\mathscr{B}}(\lambda_\gamma)\Big]w_{N-1}^+,$$

$$\hat{\mathscr{D}}(\lambda_\alpha)\Big[\prod_{\beta=1}^{\alpha-1}\hat{\mathscr{B}}(\lambda_\beta)\Big]w_{N-1}^+ =\sum_{\beta=1}^{\alpha}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta\right]}{\prod_{\gamma=1,\neq\beta}^{\alpha}(\sinh^2\lambda_\beta-\sinh^2\lambda_\gamma)} \times d_\beta^{(\alpha)}\Big[\prod_{\gamma=1,\neq\beta}^{\alpha}\hat{\mathscr{B}}(\lambda_\gamma)\Big]w_{N-1}^+, \tag{7}$$

其中 $w_{N-1}^+=\prod_{i=2}^{N}\uparrow_i$，$a_\beta^{(\alpha)}$ 和 $d_\beta^{(\alpha)}$ 由下式确定：

$$a_\beta^{(\alpha)}=\frac{\sinh(\eta)\sinh(2\lambda_\alpha)\sinh(\eta+2\lambda_\beta)}{\sinh(2\lambda_\beta)}\Bigg[\sinh(\eta+\lambda_\beta-\lambda_\alpha)\sinh\left(\lambda_\beta+\frac{1}{2}\eta-\zeta_+\right)\frac{\prod_{\gamma=1}^{\alpha-1}[\sinh^2(\lambda_\beta+\eta)-\sinh^2\lambda_\gamma]}{\prod_{i=2}^{N}\left[\sinh^2\nu_i-\sinh^2\left(\frac{1}{2}\eta+\lambda_\beta\right)\right]} -\sinh(\eta-\lambda_\beta-\lambda_\alpha)\sinh\left(-\lambda_\beta+\frac{1}{2}\eta-\zeta_+\right)\frac{\prod_{\gamma=1}^{\alpha-1}[\sinh^2(\lambda_\beta-\eta)-\sinh^2\lambda_\gamma]}{\prod_{i=2}^{N}\left[\sinh^2\nu_i-\sinh^2\left(\frac{1}{2}\eta-\lambda_\beta\right)\right]}\Bigg],$$

$$d_\beta^{(\alpha)}=-\frac{\sinh(\eta)\sinh(\eta+2\lambda_\beta)}{\sinh(2\lambda_\beta)}\Bigg[\sinh(\lambda_\alpha+\lambda_\beta)\sinh\left(\lambda_\beta+\frac{1}{2}\eta-\zeta_+\right)\frac{\prod_{\gamma=1}^{\alpha-1}[\sinh^2(\lambda_\beta+\eta)-\sinh^2\lambda_\gamma]}{\prod_{i=2}^{N}\left[\sinh^2\nu_i-\sinh^2\left(\frac{1}{2}\eta+\lambda_\beta\right)\right]}$$

$$- \sinh(\lambda_\alpha - \lambda_\beta)\sinh\left(-\lambda_\beta + \frac{1}{2}\eta - \zeta_+\right)\frac{\prod_{\gamma=1}^{\alpha-1}[\sinh^2(\lambda_\beta - \eta) - \sinh^2\lambda_\gamma]}{\prod_{i=2}^{N}\left[\sinh^2\nu_i - \sinh^2\left(\frac{1}{2}\eta - \lambda_\beta\right)\right]}\Bigg].$$

把(7)式代入(6)式,可得

$$f_1^M = \frac{-\sinh\eta}{\sinh\left(\nu_1 + \frac{1}{2}\eta - \lambda_M\right)}\prod_{\gamma=1}^{M}\frac{\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]}{\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]}$$
$$\times \sum_{\alpha=1}^{M}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i + \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]}{\prod_{\gamma=1,\neq\alpha}^{M}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma)}$$
$$\times a_\alpha^{(M)}\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq\alpha}, \{\nu_k\}_{k\neq 1})}{Z_N(\{\lambda_\beta\}_N, \{\nu_k\}_N)}. \qquad (8)$$

$Z_{Z-1}(\{\lambda_\beta\}_{\beta\neq\alpha}, \{\nu_k\}_{k\neq 1})$ 为图 1 中去掉第 1 例,第 $2\alpha-1$ 和 2α 行的 $(N-1)\times 2(N-1)$ 格点体系的配分函数.对应于开边界条件的配分函数 $Z_N(\{\lambda_\beta\}_N, \{\nu_k\}_N)$ 可以表示成由 $\{\lambda_\beta\}_N$ 和 $\{\nu_k\}_N$ 的函数构成的 $N\times N$ 行列式[8,12]

$$Z_N(\{\lambda_\alpha\}_N, \{\nu_k\}_N) = \frac{\prod_{\alpha=1}^{N}\prod_{i=1}^{N}\left[\sinh^2\left(\nu_i + \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]}{\prod\limits_{1\leqslant i<j\leqslant N}(\sinh^2\nu_j - \sinh^2\nu_i)\prod\limits_{1\leqslant\alpha<\beta\leqslant N}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\beta)}\det{}_N\mathscr{X}(\{\lambda_\alpha\}_N, \{\nu_k\}_N), \qquad (9)$$

$$\mathscr{X}_{\alpha i} = \frac{-\sinh\eta\sinh(2\lambda_\alpha + \eta)\sinh(\nu_i + \zeta_+)}{\left[\sinh^2\left(\nu_i + \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]\left[\sinh^2\left(\nu_i - \frac{1}{2}\eta\right) - \sinh^2\lambda_\alpha\right]}.$$

把(8)式中的 Z_N 和 Z_{N-1} 用(9)式表示,并注意到当 $\alpha > M$ 时,$\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma) = 0$,可知 f_1^M 有如下行列式表示:

$$f_1^M = \frac{-\sinh\eta\sinh\left(\nu_1 + \frac{1}{2}\eta + \lambda_M\right)\prod_{i=2}^{N}(\sinh^2\nu_i - \sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^{N}\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]}\frac{\det{}_N\mathscr{H}_1^M(\{\lambda_\beta\}_{\beta\neq\alpha}, \{\nu_k\}_{k\neq 1})}{\det{}_N\mathscr{X}(\{\lambda_\alpha\}_N, \{\nu_k\}_N)}, \qquad (10)$$

$$\begin{cases}(\mathscr{H}_1^M)_{\alpha i} = \mathscr{X}_{\alpha i}, \quad 2\leqslant i\leqslant N; \\ (\mathscr{H}_1^M)_{\alpha 1} = a_\alpha^{(M)}\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma).\end{cases}$$

类似于 f_1^M 的计算,f_2^M 有下列表示:

$$f_2^M = \frac{-\sinh\eta\sinh\left(\nu_1 - \frac{1}{2}\eta - \lambda_M\right)\prod_{i=2}^{N}(\sinh^2\nu_i - \sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^{N}\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\gamma\right]}\frac{\det{}_N\mathscr{H}_2^M(\{\lambda_\beta\}_{\beta\neq\alpha}, \{\nu_k\}_{k\neq 1})}{\det{}_N\mathscr{X}(\{\lambda_\alpha\}_N, \{\nu_k\}_N)}, \qquad (11)$$

$$\begin{cases}(\mathscr{H}_2^M)_{\alpha i} = \mathscr{X}_{\alpha i}, \quad 2\leqslant i\leqslant N; \\ (\mathscr{H}_2^M)_{\alpha 1} = d_\alpha^{(M)}\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha - \sinh^2\lambda_\gamma).\end{cases}$$

f_3^M 和 f_4^M 的计算相对要复杂.在对第一列计算以后,f_3^M 可以写为如下的求和式:

$$f_3^M = Z_N^{-1}\sum_{\alpha=1}^{M}\frac{-\sinh\eta\prod_{\beta=1}^{\alpha-1}\left[\sinh^2\left(\nu_1 + \frac{1}{2}\eta\right) - \sinh^2\lambda_\beta\right]}{\prod_{\beta=1}^{\alpha}\left[\sinh^2\left(\nu_1 - \frac{1}{2}\eta\right) - \sinh^2\lambda_\beta\right]}w_{N-1}^-\left[\prod_{\gamma=\alpha+1}^{N}\hat{\mathscr{B}}(\lambda_\gamma)\right]$$
$$\times\left\{\left[\sinh\left(\nu_1 + \frac{1}{2}\eta + \lambda_\alpha\right)\hat{\mathscr{A}}(\lambda_\alpha) + \sinh\left(\nu_1 - \frac{1}{2}\eta - \lambda_\alpha\right)\hat{\mathscr{D}}(\lambda_\alpha)\right]\right\}\left[\prod_{\gamma=1}^{\alpha-1}\hat{\mathscr{B}}(\lambda_\gamma)\right]w_{N-1}^+.$$

首先考虑求和式中 $\alpha = M$ 项,并应用交换关系式(7),可知

$$f_3^M = -\sinh\eta\left[\prod_{\beta=1}^{M}\frac{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}{\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}\right]\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_M\right]}{\prod_{\gamma=1}^{M-1}(\sinh^2\lambda_M-\sinh^2\lambda_\gamma)}$$
$$\times\left[\frac{a_M^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_M\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_M^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_M}\right]\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq M},\{\nu_k\}_{k\neq1})}{Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)}+\cdots \tag{12}$$

由于各个 $\mathscr{B}(\lambda_\alpha)$ 之间互相对易((4)式),因此由(5)式中 f_3^M 的定义可知 f_3^M 中 M 个谱参量 $\lambda_1\cdots\lambda_M$ 完全对称,(12)式中其他项应是第一项对 M 个谱参量 $\lambda_1\cdots\lambda_M$ 作对称置换的结果,

$$f_3^M = -\sinh\eta\left[\prod_{\beta=1}^{M}\frac{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}{\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\beta}\right]\sum_{\alpha=1}^{M}\frac{\prod_{i=2}^{N}\left[\sinh^2\left(\nu_i+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha\right]}{\prod_{\gamma=1,\neq\alpha}^{M}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma)}$$
$$\times\left[\frac{a_\alpha^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_\alpha\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_\alpha^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha}\right]\frac{Z_{N-1}(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{Z_N(\{\lambda_\beta\}_N,\{\nu_k\}_N)}.$$

上式类似于 f_1^M 的求和表达式(8),因此通过类似的计算也可以把计算结果写为如下行列式:

$$f_3^M = \frac{-\sinh\eta\prod_{i=2}^{N}(\sinh^2\nu_i-\sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]\prod_{\gamma=M+1}^{N}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]}\frac{\det_N\mathscr{H}_3^M(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{\det_N\mathscr{K}(\{\lambda_\alpha\}_N,\{\nu_k\}_N)}, \tag{13}$$

$$\begin{cases}(\mathscr{H}_3^M)_{\alpha i} = \mathscr{K}_{\alpha i}, \quad 2\leqslant i\leqslant N;\\ (\mathscr{H}_3^M)_{\alpha 1} = \left[\frac{a_\alpha^{(M)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_\alpha\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_\alpha^{(M)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha}\right]\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma).\end{cases}$$

因为 $1=q_1+p_1$,f_4^M 可以写为 f_2^M 和 f_3^{M-1} 的和

$$f_4^M = f_3^{M-1}+f_2^M.$$

由(11)和(13)式以及行列式的加法法则,可以写出 f_4^M 的计算结果如下:

$$f_4^M = \frac{-\sinh\eta\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)\prod_{i=2}^{N}(\sinh^2\nu_i-\sinh^2\nu_1)}{\prod_{\gamma=1}^{M}\left[\sinh^2\left(\nu_1-\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]\prod_{\gamma=M}^{N}\left[\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\gamma\right]}\frac{\det_N\mathscr{H}_4^M(\{\lambda_\beta\}_{\beta\neq\alpha},\{\nu_k\}_{k\neq1})}{\det_N\mathscr{K}(\{\lambda_\alpha\}_N,\{\nu_k\}_N)}, \tag{14}$$

$$\begin{cases}(\mathscr{H}_4^M)_{\alpha i} = \mathscr{K}_{\alpha i}, \quad 2\leqslant i\leqslant N;\\ (\mathscr{H}_4^M)_{\alpha 1} = d_\alpha^{(M)}\prod_{\gamma=M+1}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma)+\sinh\left(\nu_1-\frac{1}{2}\eta+\lambda_M\right)\\ \qquad\times\left[\frac{a_\alpha^{(M-1)}}{\sinh\left(\nu_1+\frac{1}{2}\eta-\lambda_\alpha\right)}+\frac{\sinh\left(\nu_1-\frac{1}{2}\eta-\lambda_M\right)d_\alpha^{(M-1)}}{\sinh^2\left(\nu_1+\frac{1}{2}\eta\right)-\sinh^2\lambda_\alpha}\right]\prod_{\gamma=M}^{N}(\sinh^2\lambda_\alpha-\sinh^2\lambda_\gamma).\end{cases}$$

4. 总 结

关联函数的精确计算在二维精确可解统计理论中一直是一个非常重要的问题.本文对六顶角模型在反射型开边界条件下,计算了描述其边界自发极化的边界关联函数,得到了它们的行列式表示.在周期性边界或畴壁边界条件下,这些表达式在热力学极限下,可被写为 Fredholm 积分算子的行列式,并由

此得到关联函数所应满足的微分方程及其渐进行为. 因此,对本文所得到的这些关联函数取热力学极限很有意义,我们将在后续的工作中给予讨论. 另外,由于反射型边界条件破坏了系统的平移不变性,因此本文的讨论并不适用于其他列($k \geqslant 2$)上的关联函数. 对任意格点上的局域算子的关联函数的计算将是一个富于挑战性的问题.

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Boundary correlation functions of the six-vertex model with open boundary

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Abstract

The six-vertex model with open boundary conditions is discussed. The correlation functions for the model are calculated and expressed as some determinants.

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