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推广的 Tschebyshev 多项式及其在解强耦合波导方程中的应用*

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推广了 Tschebyshev 多项式，对推广的 Tschebyshev 多项式的性质作了分析，并应用于求解 $(N+1)\times(N+1)$ 强耦合环形波导耦合器方程，得出解析解. 具体计算了 5×5 环形定向耦合器的解，并对弱耦合与强耦合的关系进行了详细分析.

关键词：推广的 Tschebyshev 多项式，强耦合波导耦合器

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1. 引　言

在光波导理论中，波导耦合方程的求解是重要的，很多作者都研究过这个问题[1—7]. 波导耦合的实际应用也很多[8,9]. 文献[10]应用 Tschebyshev 多项式关系导出了 $(N+1)\times(N+1)$ 弱耦合方程的解析解. 本文在此基础上，推广了 Tschebyshev 多项式，求得了 $(N+1)\times(N+1)$ 强耦合方程的解析解，具体计算了 5×5 环形定向耦合器的解. 进一步研究了强耦合与弱耦合的一般关系，并得出强耦合情形与弱耦合情形的解可表示为同样的形式，其差别在于两者的特征值不同，因而波的传播因子不一样的结论.

2. $(N+1)\times(N+1)$ 强耦合环形耦合器

图 1 所示为 $N+1$ 个波导构成的环形耦合器. 考虑相邻及次相邻波导间的耦合作用，强耦合波导方程可写为

$$\frac{\partial \tilde{u}_i}{\partial z}=\mathrm{j}\frac{k}{2}(\tilde{u}_{i-1}+\tilde{u}_{i+1})+\mathrm{j}\frac{k'}{2}(\tilde{u}_{i-2}+\tilde{u}_{i+2})\quad(2\leqslant i\leqslant N-2),$$

$$\frac{\partial \tilde{u}_0}{\partial z}=\mathrm{j}\frac{k}{2}(\tilde{u}_N+\tilde{u}_1)+\mathrm{j}\frac{k'}{2}(\tilde{u}_{N-1}+\tilde{u}_2),$$

$$\frac{\partial \tilde{u}_1}{\partial z}=\mathrm{j}\frac{k}{2}(\tilde{u}_0+\tilde{u}_2)+\mathrm{j}\frac{k'}{2}(\tilde{u}_N+\tilde{u}_3),$$

$$\frac{\partial \tilde{u}_N}{\partial z}=\mathrm{j}\frac{k}{2}(\tilde{u}_{N-1}+\tilde{u}_0)+\mathrm{j}\frac{k'}{2}(\tilde{u}_{N-2}+\tilde{u}_1),$$

$$\frac{\partial \tilde{u}_{N-1}}{\partial z}=\mathrm{j}\frac{k}{2}(\tilde{u}_{N-2}+\tilde{u}_N)+\mathrm{j}\frac{k'}{2}(\tilde{u}_{N-3}+\tilde{u}_0),\tag{1}$$

式中 $\tilde{u}_i$ 表示第 i 通道的耦合模，$k/2$ 和 $k'/2$ 分别为相邻和次相邻波导间的耦合系数. 可将耦合系数 k 与 k' 之间的关系写为 $k'=\gamma k$.

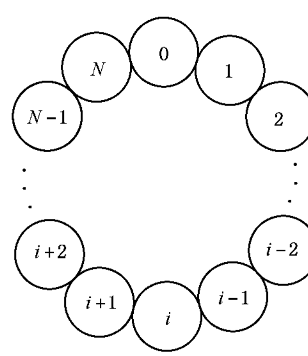

图 1　环形定向耦合器结构简图

设 $\tilde{u}_i=\tilde{\Phi}_i(\beta'/k)\mathrm{e}^{\mathrm{j}\beta'(z-ct)}$，$\beta'$ 表示模的传播常

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数,可由本征方程求出. 令 $x' = \beta'/k$,x' 即以 k 为单位的波数,代入耦合方程(1),得

$$
\begin{aligned}
2x'\widetilde{\Phi}_i(x') &= \widetilde{\Phi}_{i-1}(x') + \widetilde{\Phi}_{i+1}(x') \\
&\quad + \gamma[\widetilde{\Phi}_{i-2}(x') + \widetilde{\Phi}_{i+2}(x')] \\
&\qquad (2 \leqslant i \leqslant N-2), \\
2x'\widetilde{\Phi}_0(x') &= \widetilde{\Phi}_N(x') + \widetilde{\Phi}_1(x') \\
&\quad + \gamma[\widetilde{\Phi}_{N-1}(x') + \widetilde{\Phi}_2(x')], \\
2x'\widetilde{\Phi}_1(x') &= \widetilde{\Phi}_0(x') + \widetilde{\Phi}_2(x') \\
&\quad + \gamma[\widetilde{\Phi}_N(x') + \widetilde{\Phi}_3(x')], \\
2x'\widetilde{\Phi}_N(x') &= \widetilde{\Phi}_{N-1}(x') + \widetilde{\Phi}_0(x') \\
&\quad + \gamma[\widetilde{\Phi}_{N-2}(x') + \widetilde{\Phi}_1(x')], \\
2x'\widetilde{\Phi}_{N-1}(x') &= \widetilde{\Phi}_{N-2}(x') + \widetilde{\Phi}_N(x') \\
&\quad + \gamma[\widetilde{\Phi}_{N-3}(x') + \widetilde{\Phi}_0(x')]. \qquad (2)
\end{aligned}
$$

(2)式为包括相邻与次邻级波导间耦合的强耦合方程,耦合系数分别用 k 与 γk 表示,若令 $\gamma=0$,便是文献[10]讨论过的弱耦合波导方程.文献[10]应用 Tschebyshev 多项式求得了它的准确解. 若 $\gamma \neq 0$,便是包括次邻级波导耦合的强耦合方程,仍用通常的 Tschebyshev 多项式是不能得到它的准确解的,但可以推广已有的 Tschebyshev 多项式定义,使之适用于求解强耦合波导方程,具体做法如下:将(2)式的解 $\widetilde{\Phi}_i(x)$ 用推广的第一类 Tschebyshev 多项式表示为

$$
\widetilde{\Phi}_i(x'_m) = \sqrt{\frac{2}{N+1}}\cos i\varphi_m \qquad (0 \leqslant i \leqslant N), \qquad (3)
$$

式中 $x'_m = \cos\varphi_m + \gamma\cos 2\varphi_m$, $\varphi_m = 2\pi\dfrac{m}{N+1}$, $m=0$, $1,2,\cdots,N$.

容易证明,(3)式是方程(2)的解. 我们注意到推广就包括在 x'_m 表示式中新增的第二项,若 $\gamma=0$,便过渡到通常的 Tschebyshev 多项式. 关于推广的 Tschebyshev 多项式的性质在下面还要研究.

3. 5×5 环形耦合器的解

3.1. 弱耦合情形

考虑 $N+1=5$,先看仅考虑相邻波导间相互作用的弱耦合情形,弱耦合方程可写为

$$
\begin{aligned}
\frac{\partial u_0}{\partial z} &= \mathrm{j}\frac{k}{2}(u_4 + u_1), \\
\frac{\partial u_1}{\partial z} &= \mathrm{j}\frac{k}{2}(u_0 + u_2), \\
\frac{\partial u_2}{\partial z} &= \mathrm{j}\frac{k}{2}(u_1 + u_3), \\
\frac{\partial u_3}{\partial z} &= \mathrm{j}\frac{k}{2}(u_2 + u_4), \\
\frac{\partial u_4}{\partial z} &= \mathrm{j}\frac{k}{2}(u_3 + u_0). \qquad (4)
\end{aligned}
$$

设 $u_i = \Phi_i(\beta/k)\mathrm{e}^{\mathrm{j}\beta(z-ct)}$,并令 $x=\beta/k$,代入耦合方程,得

$$
\begin{aligned}
2x\Phi_0(x) &= \Phi_4(x) + \Phi_1(x), \\
2x\Phi_1(x) &= \Phi_0(x) + \Phi_2(x), \\
2x\Phi_2(x) &= \Phi_1(x) + \Phi_3(x), \\
2x\Phi_3(x) &= \Phi_2(x) + \Phi_4(x), \\
2x\Phi_4(x) &= \Phi_3(x) + \Phi_0(x). \qquad (5)
\end{aligned}
$$

其解 $\Phi_i(x)$ 可用第一类 Tschebyshev 多项式表示

$$
\Phi_i(x_m) = \sqrt{\frac{2}{5}}\cos i\varphi_m \qquad (0 \leqslant i \leqslant 4), \qquad (6)
$$

式中 $x_m = \cos\varphi_m$, $\varphi_m = \dfrac{2m\pi}{5}$ $(m=0,1,2,3,4)$.即本征值 $x_0=1$, $x_1=\cos\dfrac{2\pi}{5}$, $x_2=\cos\dfrac{4\pi}{5}$, $x_3=\cos\dfrac{6\pi}{5}=\cos\dfrac{4\pi}{5}=x_2$, $x_4=\cos\dfrac{8\pi}{5}=\cos\dfrac{2\pi}{5}=x_1$ 是简并的,因而可将解设为

$$
\begin{aligned}
u_0 &= \alpha_{00}\mathrm{e}^{\mathrm{j}\beta_0 z} + (\alpha_{01} + \alpha_{04}z)\mathrm{e}^{\mathrm{j}\beta_1 z} \\
&\quad + (\alpha_{02} + \alpha_{03}z)\mathrm{e}^{\mathrm{j}\beta_2 z}, \\
u_1 &= \alpha_{10}\mathrm{e}^{\mathrm{j}\beta_0 z} + (\alpha_{11} + \alpha_{14}z)\mathrm{e}^{\mathrm{j}\beta_1 z} \\
&\quad + (\alpha_{12} + \alpha_{13}z)\mathrm{e}^{\mathrm{j}\beta_2 z}, \\
u_2 &= \alpha_{20}\mathrm{e}^{\mathrm{j}\beta_0 z} + (\alpha_{21} + \alpha_{24}z)\mathrm{e}^{\mathrm{j}\beta_1 z} \\
&\quad + (\alpha_{22} + \alpha_{23}z)\mathrm{e}^{\mathrm{j}\beta_2 z}, \\
u_3 &= \alpha_{30}\mathrm{e}^{\mathrm{j}\beta_0 z} + (\alpha_{31} + \alpha_{34}z)\mathrm{e}^{\mathrm{j}\beta_1 z} \\
&\quad + (\alpha_{32} + \alpha_{33}z)\mathrm{e}^{\mathrm{j}\beta_2 z}, \\
u_4 &= \alpha_{40}\mathrm{e}^{\mathrm{j}\beta_0 z} + (\alpha_{41} + \alpha_{44}z)\mathrm{e}^{\mathrm{j}\beta_1 z} \\
&\quad + (\alpha_{42} + \alpha_{43}z)\mathrm{e}^{\mathrm{j}\beta_2 z}, \qquad (7)
\end{aligned}
$$

式中 $\beta_0 = x_0 k$, $\beta_1 = x_1 k$, $\beta_2 = x_2 k$.

将解代入方程(4),并比较系数,得

$$
\begin{aligned}
& 2x_i\alpha_{0i} - (\alpha_{4i} + \alpha_{1i}) = 0, \\
& 2x_i\alpha_{1i} - (\alpha_{0i} + \alpha_{2i}) = 0, \\
& 2x_i\alpha_{2i} - (\alpha_{1i} + \alpha_{3i}) = 0, \\
& 2x_i\alpha_{3i} - (\alpha_{2i} + \alpha_{4i}) = 0, \\
& 2x_i\alpha_{4i} - (\alpha_{3i} + \alpha_{0i}) = 0, \\
& \alpha_{i3} = \alpha_{i4} = 0. \qquad (i = 0,1,2,3,4). \qquad (8)
\end{aligned}
$$

$i=0$ 时,方程组有一个独立参量,令 $\alpha_{00} = \alpha_{10} = \alpha_{20} = \alpha_{30} = \alpha_{40} = b_0$;$i=1$ 时,方程组有两个独立参量,令 $\alpha_{01} = b_1$, $\alpha_{11} = b_3$, $\alpha_{21} = 2x_1 b_3 - b_1$, $\alpha_{31} = -2x_1(b_1$

$+b_3$),$\alpha_{41}=2x_1b_1-b_3$;$i=2$ 时,方程组有两个独立参量,令 $\alpha_{02}=b_2$,$\alpha_{12}=b_4$,$\alpha_{22}=2x_2b_4-b_2$,$\alpha_{32}=-2x_2(b_2+b_4)$,$\alpha_{42}=2x_2b_2-b_4$.

这时解可写为

$$
\begin{aligned}
u_0 &= b_0e^{j\beta_0 z}+b_1e^{j\beta_1 z}+b_2e^{j\beta_2 z},\\
u_1 &= b_0e^{j\beta_0 z}+b_3e^{j\beta_1 z}+b_4e^{j\beta_2 z},\\
u_2 &= b_0e^{j\beta_0 z}+(2x_1b_3-b_1)e^{j\beta_1 z}\\
&\quad+(2x_2b_4-b_2)e^{j\beta_2 z},\\
u_3 &= b_0e^{j\beta_0 z}-2x_1(b_1+b_3)e^{j\beta_1 z}\\
&\quad-2x_2(b_2+b_4)e^{j\beta_2 z},\\
u_4 &= b_0e^{j\beta_0 z}+(2x_1b_1-b_3)e^{j\beta_1 z}\\
&\quad+(2x_2b_2-b_4)e^{j\beta_2 z}.
\end{aligned}\tag{9}
$$

若 $z=0$ 时,u_0,u_1,u_2,u_3,u_4 的初值分别为 a_0,a_1,a_2,a_3,a_4.由此可定出

$$
\begin{aligned}
b_0 &= \frac{1}{5}(a_0+a_1+a_2+a_3+a_4),\\
b_1 &= \frac{1}{10(x_2-x_1)}\Big[(2a_0-3a_1+2a_2+2a_3-3a_4)\\
&\quad+x_2(8a_0-2a_1-2a_2-2a_3-2a_4)\Big],\\
b_2 &= -\frac{1}{10(x_2-x_1)}\Big[(2a_0-3a_1+2a_2+2a_3-3a_4)\\
&\quad+x_1(8a_0-2a_1-2a_2-2a_3-2a_4)\Big],\\
b_3 &= \frac{1}{10(x_2-x_1)}\Big[(-3a_0+2a_1-3a_2+2a_3+2a_4)\\
&\quad+x_2(-2a_0+8a_1-2a_2-2a_3-2a_4)\Big],\\
b_4 &= -\frac{1}{10(x_2-x_1)}\Big[(-3a_0+2a_1-3a_2+2a_3+2a_4)\\
&\quad+x_1(-2a_0+8a_1-2a_2-2a_3-2a_4)\Big].
\end{aligned}\tag{10}
$$

将(10)式代入(9)式,可得到方程的解,用矩阵表示为

$$
\begin{bmatrix}u_0\\u_1\\u_2\\u_3\\u_4\end{bmatrix}=\begin{pmatrix}c_1&c_2&c_3&c_3&c_2\\c_2&c_1&c_2&c_3&c_3\\c_3&c_2&c_1&c_2&c_3\\c_3&c_3&c_2&c_1&c_2\\c_2&c_3&c_3&c_2&c_1\end{pmatrix}\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\end{bmatrix},\tag{11}
$$

式中 $c_1=\frac{1}{5}\left[e^{j\beta_0 z}+2e^{j\beta_1 z}+2e^{j\beta_2 z}\right]$,$c_2=\frac{1}{10}\left[2e^{j\beta_0 z}+(\sqrt{5}-1)e^{j\beta_1 z}-(\sqrt{5}+1)e^{j\beta_2 z}\right]$,$c_3=\frac{1}{10}\left[2e^{j\beta_0 z}-(\sqrt{5}+1)e^{j\beta_1 z}+(\sqrt{5}-1)e^{j\beta_2 z}\right]$.

3.2. 强耦合情形

对于考虑相邻及次相邻波导之间相互作用的强耦合情形,其耦合方程为

$$
\begin{aligned}
\frac{\partial\tilde{u}_0}{\partial z} &= j\frac{k}{2}(\tilde{u}_4+\tilde{u}_1)+j\frac{k'}{2}(\tilde{u}_3+\tilde{u}_2),\\
\frac{\partial\tilde{u}_1}{\partial z} &= j\frac{k}{2}(\tilde{u}_0+\tilde{u}_2)+j\frac{k'}{2}(\tilde{u}_4+\tilde{u}_3),\\
\frac{\partial\tilde{u}_2}{\partial z} &= j\frac{k}{2}(\tilde{u}_1+\tilde{u}_3)+j\frac{k'}{2}(\tilde{u}_0+\tilde{u}_4),\\
\frac{\partial\tilde{u}_3}{\partial z} &= j\frac{k}{2}(\tilde{u}_2+\tilde{u}_4)+j\frac{k'}{2}(\tilde{u}_1+\tilde{u}_0),\\
\frac{\partial\tilde{u}_4}{\partial z} &= j\frac{k}{2}(\tilde{u}_3+\tilde{u}_0)+j\frac{k'}{2}(\tilde{u}_2+\tilde{u}_1).
\end{aligned}\tag{12}
$$

设 $\tilde{u}_i=\tilde{\Phi}_i(\beta'/k)e^{j\beta'(z-ct)}$,并令 $x'=\beta'/k$ 及 $k'=\gamma k$,代入耦合方程,得

$$
\begin{aligned}
2x'\tilde{\Phi}_0(x') &= \tilde{\Phi}_4(x')+\tilde{\Phi}_1(x')\\
&\quad+\gamma(\tilde{\Phi}_3(x')+\tilde{\Phi}_2(x')),\\
2x'\tilde{\Phi}_1(x') &= \tilde{\Phi}_0(x')+\tilde{\Phi}_2(x')\\
&\quad+\gamma(\tilde{\Phi}_4(x')+\tilde{\Phi}_3(x')),\\
2x'\tilde{\Phi}_2(x') &= \tilde{\Phi}_1(x')+\tilde{\Phi}_3(x')\\
&\quad+\gamma(\tilde{\Phi}_0(x')+\tilde{\Phi}_4(x')),\\
2x'\tilde{\Phi}_3(x') &= \tilde{\Phi}_2(x')+\tilde{\Phi}_4(x')\\
&\quad+\gamma(\tilde{\Phi}_1(x')+\tilde{\Phi}_0(x')),\\
2x'\tilde{\Phi}_4(x') &= \tilde{\Phi}_3(x')+\tilde{\Phi}_0(x')\\
&\quad+\gamma(\tilde{\Phi}_2(x')+\tilde{\Phi}_1(x')).
\end{aligned}\tag{13}
$$

本征值为 $x'_0=1+\gamma$,$x'_1=\cos\frac{2\pi}{5}+\gamma\cos\frac{4\pi}{5}$,$x'_2=\cos\frac{4\pi}{5}+\gamma\cos\frac{8\pi}{5}$,$x'_3=\cos\frac{6\pi}{5}+\gamma\cos\frac{12\pi}{5}=\cos\frac{4\pi}{5}+\gamma\cos\frac{8\pi}{5}=x'_2$,$x'_4=\cos\frac{8\pi}{5}+\gamma\cos\frac{16\pi}{5}=\cos\frac{2\pi}{5}+\gamma\cos\frac{4\pi}{5}=x'_1$ 是简并的,同样可将解设为

$$
\begin{aligned}
\tilde{u}_0 &= \alpha'_{00}e^{j\beta'_0 z}+(\alpha'_{01}+\alpha'_{04}z)e^{j\beta'_1 z}\\
&\quad+(\alpha'_{02}+\alpha'_{03}z)e^{j\beta'_2 z},\\
\tilde{u}_1 &= \alpha'_{10}e^{j\beta'_0 z}+(\alpha'_{11}+\alpha'_{14}z)e^{j\beta'_1 z}\\
&\quad+(\alpha'_{12}+\alpha'_{13}z)e^{j\beta'_2 z},\\
\tilde{u}_2 &= \alpha'_{20}e^{j\beta'_0 z}+(\alpha'_{21}+\alpha'_{24}z)e^{j\beta'_1 z}\\
&\quad+(\alpha'_{22}+\alpha'_{23}z)e^{j\beta'_2 z},\\
\tilde{u}_3 &= \alpha'_{30}e^{j\beta'_0 z}+(\alpha'_{31}+\alpha'_{34}z)e^{j\beta'_1 z}\\
&\quad+(\alpha'_{32}+\alpha'_{33}z)e^{j\beta'_2 z},
\end{aligned}
$$

$$\tilde{u}_4 = \alpha'_{40}e^{j\beta'_0 z} + (\alpha'_{41} + \alpha'_{44}z)e^{j\beta'_1 z} + (\alpha'_{42} + \alpha'_{43}z)e^{j\beta'_2 z}, \tag{14}$$

式中 $\beta'_0 = x'_0 k, \beta'_1 = x'_1 k, \beta'_2 = x'_2 k$.

将(14)式代入耦合方程(12),并利用初始条件 $z=0$ 时,$\tilde{u}_0 = a_0, \tilde{u}_1 = a_1, \tilde{u}_2 = a_2, \tilde{u}_3 = a_3, \tilde{u}_4 = a_4$,可得出强耦合方程的解为

$$\begin{bmatrix} \tilde{u}_0 \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{bmatrix} = \begin{bmatrix} c'_1 & c'_2 & c'_3 & c'_3 & c'_2 \\ c'_2 & c'_1 & c'_2 & c'_3 & c'_3 \\ c'_3 & c'_2 & c'_1 & c'_2 & c'_3 \\ c'_3 & c'_3 & c'_2 & c'_1 & c'_2 \\ c'_2 & c'_3 & c'_3 & c'_2 & c'_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \tag{15}$$

式中 $c'_1 = \frac{1}{5}\left[e^{j\beta'_0 z} + 2e^{j\beta'_1 z} + 2e^{j\beta'_2 z}\right]$, $c'_2 = \frac{1}{10}\left[2e^{j\beta'_0 z} + (\sqrt{5}-1)e^{j\beta'_1 z} - (\sqrt{5}+1)e^{j\beta'_2 z}\right]$, $c'_3 = \frac{1}{10}\left[2e^{j\beta'_0 z} - (\sqrt{5}+1)e^{j\beta'_1 z} + (\sqrt{5}-1)e^{j\beta'_2 z}\right]$.

4. $(N+1)\times(N+1)$ 强耦合方程与弱耦合方程解的分析

比较 5×5 强耦合方程与弱耦合方程的解,可以看出,考虑相邻及次相邻波导之间相互作用的强耦合方程的解与仅考虑相邻波导间相互作用的弱耦合方程的解从形式上看是一样的. 但强耦合情形特征值为 $x'_m = \cos\varphi_m + \gamma\cos2\varphi_m$,其中包含了耦合系数 γ,而弱耦合情形特征值为 $x_m = \cos\varphi_m$,两者是不一样的,这样解的传播因子 $e^{jx'_m k(z-ct)}$ 也就不一样. 事实上,强耦合方程(13)是弱耦合方程(5)的推广,如将强耦合方程的特征值 $x'_m = \cos\varphi_m + \gamma\cos2\varphi_m$,用弱耦合方程特征值 $x_m = \cos\varphi_m$ 表示,即:$x'_m = x_m + \gamma(2x_m^2 - 1)$,并注意到强耦合与弱耦合特征函数有如下关系:$\tilde{\Phi}_i(x'_m) = \Phi_i(x_m) = \sqrt{\frac{2}{N+1}}\cos i\varphi_m$(其中 $\varphi_m = 2\pi\frac{m}{N+1}$). 一般地,以上关系对去掉下标后的 x', x 及 $\tilde{\Phi}_i(x'), \Phi_i(x)$ 也成立,故可用这些关系将强耦合方程(2)用弱耦合特征函数 $\Phi_i(x)$ 及坐标 x 表示为

$$\begin{aligned} &2[x + \gamma(2x^2 - 1)]\Phi_i(x) \\ &= \Phi_{i-1}(x) + \Phi_{i+1}(x) + \gamma[\Phi_{i-2}(x) + \Phi_{i+2}(x)]. \end{aligned} \tag{16}$$

上式成立,只要求 $\Phi_i(x)$ 满足弱耦合关系,即上式中不含 γ 的项相等,含 γ 的项为

$$\begin{aligned} &2\gamma[2x^2\Phi_i(x) - \Phi_i(x)] \\ &= 2\gamma\left\{x[\Phi_{i-1}(x) + \Phi_{i+1}(x)] - \Phi_i(x)\right\} \\ &= \gamma[\Phi_{i-2}(x) + \Phi_{i+2}(x)]. \end{aligned} \tag{17}$$

故只要弱耦合关系成立,(16)式就成立,亦即强耦合关系成立. 以上强耦合与弱耦合关系可推广到包括更高次邻级波导间耦合的强耦合方程及其解,若包含 n 级次邻级波导耦合的强耦合方程

$$\begin{aligned} 2x^{(n)}\Phi_i(x^{(n)}) &= \Phi_{i-1}(x^{(n)}) + \Phi_{i+1}(x^{(n)}) \\ &+ \gamma_1[\Phi_{i-2}(x^{(n)}) \\ &+ \Phi_{i+2}(x^{(n)})] + \cdots \\ &+ \gamma_{n-1}[\Phi_{i-n}(x^{(n)}) \\ &+ \Phi_{i+n}(x^{(n)})] \end{aligned} \tag{18}$$

成立,相应的本征值为 $x^{(n)} = \cos\varphi + \gamma_1\cos2\varphi + \cdots + \gamma_{n-1}\cos n\varphi$,则包含 $n+1$ 级次邻级波导间耦合的强耦合方程

$$\begin{aligned} 2x^{(n+1)}\tilde{\Phi}_i(x^{(n+1)}) &= \tilde{\Phi}_{i-1}(x^{(n+1)}) + \tilde{\Phi}_{i+1}(x^{(n+1)}) \\ &+ \gamma_1[\tilde{\Phi}_{i-2}(x^{(n+1)}) \\ &+ \tilde{\Phi}_{i+2}(x^{(n+1)})] + \cdots \\ &+ \gamma_{n-1}[\tilde{\Phi}_{i-n}(x^{(n+1)}) \\ &+ \tilde{\Phi}_{i+n}(x^{(n+1)})] \\ &+ \gamma_n[\tilde{\Phi}_{i-(n+1)}(x^{(n+1)}) \\ &+ \tilde{\Phi}_{i+(n+1)}(x^{(n+1)})] \end{aligned} \tag{19}$$

亦成立,相应的本征值为 $x^{(n+1)} = \cos\varphi + \gamma_1\cos2\varphi + \cdots + \gamma_{n-1}\cos n\varphi + \gamma_n\cos(n+1)\varphi$.

由于 $x^{(n+1)} = x^{(n)} + \gamma_n\cos(n+1)\varphi$ 及 $\Phi_i(x^{(n)}) = \tilde{\Phi}_i(x^{(n+1)})$,则方程(19)可化为

$$\begin{aligned} &2[x^{(n)} + \gamma_n\cos(n+1)\varphi]\Phi_i(x^{(n)}) \\ &= \Phi_{i-1}(x^{(n)}) + \Phi_{i+1}(x^{(n)}) \\ &\quad + \gamma_1[\Phi_{i-2}(x^{(n)}) + \Phi_{i+2}(x^{(n)})] + \cdots \\ &\quad + \gamma_{n-1}[\Phi_{i-n}(x^{(n)}) + \Phi_{i+n}(x^{(n)})] \\ &\quad + \gamma_n[\Phi_{i-(n+1)}(x^{(n)}) + \Phi_{i+(n+1)}(x^{(n)})]. \end{aligned} \tag{20}$$

(20)式不含 γ_n 的项满足方程(18),含 γ_n 的项消去 γ_n 后,得

$$\begin{aligned} 2\cos(n+1)\varphi\Phi_i(x^{(n)}) &= \Phi_{i-(n+1)}(x^{(n)}) \\ &+ \Phi_{i+(n+1)}(x^{(n)}). \end{aligned} \tag{21}$$

利用关系 $\Phi_i(x^{(n)}) = \cos i\varphi$,上式等号左端为

$$2\cos(n+1)\varphi\cos i\varphi = \cos[i + (n+1)]\varphi$$

$$+\cos[i-(n+1)]\varphi$$
$$=\Phi_{i+(n+1)}(x^{(n)})+\Phi_{i-(n+1)}(x^{(n)}).$$

因而,(21)式成立,即包含 $n+1$ 级次邻级波导间耦合的强耦合关系成立.

5. 推广的 Tschebyshev 多项式的性质

由于强耦合方程与弱耦合方程的本征值不同,则强耦合解与弱耦合解对 x_m(下面记为 x)的依赖关系不同. 对弱耦合情形,有

$$\Phi_n(x)=\cos(n\arccos x). \tag{22}$$

而对强耦合情形,由 $x=\cos\varphi'+\gamma\cos2\varphi'$,即 $\cos^2\varphi'+\frac{1}{2\gamma}\cos\varphi'-\frac{\gamma+x}{2\gamma}=0$,可解出

$$\cos\varphi'=-\frac{1}{4\gamma}+\sqrt{\left(\frac{1}{4\gamma}\right)^2+\frac{\gamma+x}{2\gamma}}. \tag{23}$$

令 $f(x)=-\frac{1}{4\gamma}\left[1-\sqrt{1+\left(\frac{1}{2}+\frac{x}{2\gamma}\right)16\gamma^2}\right]$,有 $\varphi'=\arccos f(x)$,则强耦合解对 x 的依赖关系为

$$\widetilde{\Phi}_n(x)=\cos[n\arccos f(x)]. \tag{24}$$

当 $\gamma\to0$ 时,$f(x)=x$,于是有 $\varphi'=\arccos x$,即通常 Tschebyshev 多项式. 在推广的 Tschebyshev 多项式(24)中,用 $f(x)$ 代替了 x,而 $f(x)$ 包含了耦合系数 γ 的信息. 图 2 给出 γ 分别取 0,0.05,0.10,0.20 时,$\widetilde{\Phi}_{10}(x)$ 随 x 的变化曲线.

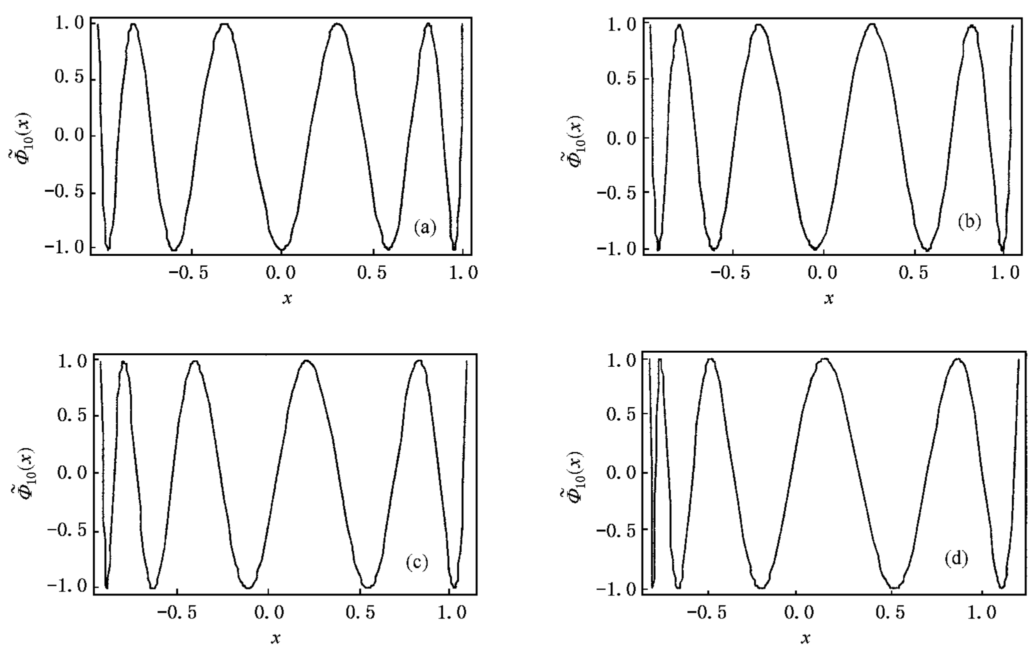

图 2 $\widetilde{\Phi}_{10}(x)$ 随 x 的变化曲线 (a),(b),(c),(d)分别对应 γ 取 0,0.05,0.10,0.20 情形

6. 结 论

本文推广了 Tschebyshev 多项式,求得了次相邻波导间耦合的 $(N+1)\times(N+1)$ 强耦合方程的解析解,具体给出了 5×5 环形定向耦合器解的表达式. 进一步研究了强耦合与弱耦合的一般关系及推广的 Tschebyshev 多项式的性质. 并得出强耦合情形与弱耦合情形的解可表示为同样的形式,其差别在于两者的特征值不同,因而波的传播因子不一样的结论.

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Generalized Tschebyshev polynomial and its application in solving the strong coupling waveguide equations*

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Abstract

In this paper, by means of generalized Tschebyshev polynomial, the analytical solution is derived for $(N+1)\times(N+1)$ waveguide couplers arranged in a ring with strong coupling. As a concrete example, the solution of 5×5 waveguide couplers is calculated, and the relationship between strong and weak couplings is analyzed. The property of generalized Tschebyshev polynomial is discussed in detail.

Keywords: generalized Tschebyshev polynomial, waveguide couplers with strong coupling
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