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相空间中非完整非保守系统的形式不变性

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研究相空间中非完整非保守系统的形式不变性.给出相空间中非完整非保守系统形式不变性的定义和判据,得到形式不变性的结构方程和守恒量形式,并举例说明结果的应用.

关键词:相空间,非完整非保守系统,形式不变性

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1. 引　　言

力学系统的对称性与守恒量之间有着密切联系,关于对称性与守恒量的研究一直是数学、物理学、力学等领域的重要课题.寻求守恒量的近代方法主要有 Noether 对称性[1] 和 Lie 对称性[2],关于这方面的研究取得一系列成果[3—10].最近,梅凤翔教授提出一种研究对称性的新方法[11],利用动力学方程的形式不变性研究系统的对称性,寻求系统的守恒量.文献[12—16]研究了位形空间中力学系统的形式不变性.本文研究相空间中非完整非保守系统的形式不变性,给出相空间中形式不变性的定义和判据,得到形式不变性的结构方程和守恒量形式,并举例说明结果的应用.

2. 运动微分方程

研究一非完整非保守力学系统.设系统的位形由 n 个广义坐标 q_s($s=1,\cdots,n$)确定,系统受有 g 个一阶非线性非完整约束

$$f_\beta = f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \qquad (\beta = 1,\cdots,g), \tag{1}$$

则在位形空间中系统的运动微分方程为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q''_s + \Lambda_s \qquad (s = 1,\cdots,n), \tag{2}$$

式中 Q''_s 为非势广义力,$\Lambda_s = \sum_{\beta=1}^{g}\lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}$ 为非完整约束力,λ_β 为约束乘子,一般说来,它们均为 $t,\boldsymbol{q},\dot{\boldsymbol{q}}$ 的函数.

在相空间中,约束方程(1)可写为

$$\bar{f}_\beta = \bar{f}_\beta(t,\boldsymbol{q},\boldsymbol{p}) = f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p})) = 0 \qquad (\beta = 1,\cdots,g), \tag{3}$$

系统的运动微分方程可写为

$$\dot{q}_s = \frac{\partial H}{\partial p_s},\ \dot{p}_s = -\frac{\partial H}{\partial p_s} + \bar{Q}_s + \bar{\Lambda}_s \qquad (s = 1,\cdots,n), \tag{4}$$

其中 $H(t,\boldsymbol{q},\boldsymbol{p})$ 为系统的 Hamilton 函数,

$$\bar{Q}_s = \bar{Q}_s(t,\boldsymbol{q},\boldsymbol{p}) = Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p})),$$

$$\bar{\Lambda}_s = \bar{\Lambda}_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}(t,\boldsymbol{q},\boldsymbol{p})).$$

3. 形式不变性

取无限小变换

$$t^* = t + \Delta t,\ q_s^*(t^*) = q_s(t) + \Delta q_s,\ p_s^*(t^*) = p_s(t) + \Delta p_s, \tag{5}$$

其展开形式为

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{q},\boldsymbol{p}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\boldsymbol{p}),$$
$$p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t,\boldsymbol{q},\boldsymbol{p}), \tag{6}$$

其中 ε 为无限小参数,ξ_0,ξ_s,η_s 为无限小单参数群变换的生成元.在无限小变换(6)下

$$H = H(t,\boldsymbol{q},\boldsymbol{p}),\ \bar{Q}_s = \bar{Q}_s(t,\boldsymbol{q},\boldsymbol{p}),$$
$$\bar{\Lambda}_s = \bar{\Lambda}_s(t,\boldsymbol{q},\boldsymbol{p}),\ \bar{f}_\beta = \bar{f}_\beta(t,\boldsymbol{q},\boldsymbol{p})$$

变为

$$H^* = H(t^*,\boldsymbol{q}^*,\boldsymbol{p}^*),\ \bar{Q}_s^* = \bar{Q}_s(t^*,\boldsymbol{q}^*,\boldsymbol{p}^*),$$

$\bar{\Lambda}_s^* = \bar{\Lambda}_s(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$, $\bar{f}_\beta^* = \bar{f}_\beta(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)$.

定义　在无限小变换(6)下,若方程(4),(3)的形式保持不变,即

$$\dot{q}_s^* = \frac{\partial H^*}{\partial p_s},\ \dot{p}_s^* = -\frac{\partial H^*}{\partial q_s} + \bar{Q}_s^* + \bar{\Lambda}_s^* \quad (s = 1,\cdots,n), \tag{7}$$

$$\bar{f}_\beta^* = \bar{f}_\beta(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*) = 0 \quad (\beta = 1,\cdots,g), \tag{8}$$

则称这种不变性为相空间中非完整非保守系统的形式不变性.

取无限小变换的生成元向量

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{s=1}^n \xi_s \frac{\partial}{\partial q_s} + \sum_{s=1}^n \eta_s \frac{\partial}{\partial p_s}, \tag{9}$$

其扩展形式为

$$X^{(1)} = X^{(0)} + \sum_{s=1}^n (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s} + \sum_{s=1}^n (\dot{\eta}_s - \dot{p}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{p}_s}. \tag{10}$$

将 H^*, $\bar{Q}_s^*$, $\bar{\Lambda}_s^*$, $\bar{f}_\beta^*$ 在无限小变换(6)下展开得

$$H^* = H + \varepsilon(X^{(1)}(H)) + o(\varepsilon^2) = H + \varepsilon(X^{(0)}(H)) + o(\varepsilon^2), \tag{11}$$

$$\bar{Q}_s^* = \bar{Q}_s + \varepsilon(X^{(1)}(\bar{Q}_s)) + o(\varepsilon^2) = \bar{Q}_s + \varepsilon(X^{(0)}(\bar{Q}_s)) + o(\varepsilon^2), \tag{12}$$

$$\bar{\Lambda}_s^* = \bar{\Lambda}_s + \varepsilon(X^{(1)}(\bar{\Lambda}_s)) + o(\varepsilon^2) = \bar{\Lambda}_s + \varepsilon(X^{(0)}(\bar{\Lambda}_s)) + o(\varepsilon^2), \tag{13}$$

$$\bar{f}_s^* = \bar{f}_s + \varepsilon(X^{(1)}(\bar{f}_s)) + o(\varepsilon^2) = \bar{f}_s + \varepsilon(X^{(0)}(\bar{f}_s)) + o(\varepsilon^2). \tag{14}$$

判据　对相空间中非完整非保守系统(4),(3),若无限小变换的生成元 ξ_0, ξ_s, η_s 满足

$$\dot{\xi}_s = \frac{\partial}{\partial p_s}(X^{(0)}(H)), \tag{15}$$

$$\dot{\eta}_s = -\frac{\partial}{\partial q_s}(X^{(0)}(H)) + X^{(0)}(\bar{Q}_s + \bar{\Lambda}_s), \tag{16}$$

$$X^{(0)}(\bar{f}_\beta(t, \boldsymbol{q}, \boldsymbol{p})) = 0, \tag{17}$$

则相应的不变性是形式不变性.

证明　对 q_s^* 求导,得 $\dot{q}_s^* = \dot{q}_s + \varepsilon \dot{\xi}_s$,将此式和(11)式代入(7)式中第一式,去掉 ε^2 及其更高阶小项得

$$\dot{q}_s + \varepsilon\dot{\xi}_s = \frac{\partial H}{\partial p_s} + \varepsilon \frac{\partial}{\partial p_s}(X^{(0)}(H)),$$

将 $\dot{q}_s = \dfrac{\partial H}{\partial p_s}$代入上式,则(15)式得证.

同理可证(16)式.而将(14)式代入(8)式,并利用(3)式便可证得(17)式.

4. 结构方程与守恒量

相空间中非完整非保守系统的形式不变性导致的守恒量由下述定理确定.

定理　在无限小变换(6)下,若相空间中非完整非保守系统(4),(3)是形式不变的,且存在规范函数 $G_F = G_F(t, \boldsymbol{q}, \boldsymbol{p})$满足结构方程

$$L_p \dot{\xi}_0 + X^{(1)}(L_p) + \sum_{s=1}^n (\xi_s - \dot{q}_s \xi_0) \times (\bar{Q}_s + \bar{\Lambda}_s) + \dot{G}_F = 0, \tag{18}$$

则系统存在如下守恒量:

$$I = \sum_{s=1}^n p_s \xi_s - H\xi_0 + G_F = \text{const}, \tag{19}$$

其中

$$L_p = L_p(t, \boldsymbol{q}, \boldsymbol{p}) = \sum_{s=1}^n p_s \dot{q}_s(t, \boldsymbol{q}, \boldsymbol{p}) - H.$$

证明　对(19)式求导,有

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \sum_{s=1}^n (\dot{p}_s \xi_s + p_s \dot{\xi}_s) - \left(\frac{\partial H}{\partial t} + \sum_{s=1}^n \frac{\partial H}{\partial q_s}\dot{q}_s + \sum_{s=1}^n \frac{\partial H}{\partial p_s}\dot{p}_s\right)\xi_0 - H\dot{\xi}_0 + \dot{G}_F, \tag{20}$$

由(18)式有

$$\left(\sum_{s=1}^n p_s \dot{q}_s - H\right)\dot{\xi}_0 - \frac{\partial H}{\partial t}\xi_0 - \sum_{s=1}^n \frac{\partial H}{\partial q_s}\xi_s + \sum_{s=1}^n \left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right)\eta_s + \sum_{s=1}^n (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) p_s + \sum_{s=1}^n (\bar{Q}_s + \bar{\Lambda}_s)(\xi_s - \dot{q}_s \xi_0) + \dot{G}_F = 0, \tag{21}$$

将(21)式代入(20)式,整理得$\dfrac{\mathrm{d}I}{\mathrm{d}t} = 0$,证毕.

5. 算　　例

考虑三自由度系统,其 Lagrange 函数

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2), \tag{22}$$

非完整约束

$$f = \dot{q}_1 + t\dot{q}_2 - q_2 + \dot{q}_3 + t = 0, \tag{23}$$

非势广义力

$$Q_1 = t,\ Q_2 = 0,\ Q_3 = t^2, \tag{24}$$

试在相空间中研究系统的形式不变性.

由 $p_s=\frac{\partial L}{\partial \dot{q}_s}$ 得 $p_1=\dot{q}_1$，$p_2=\dot{q}_2$，$p_3=\dot{q}_3$，则系统的 Hamilton 函数为

$$H=\sum_{s=l}^{3}p_s\dot{q}_s-L=\frac{1}{2}(p_1^2+p_2^2+p_3^2),\tag{25}$$

约束方程

$$\bar{f}=p_1+tp_2-q_2+p_3+t=0,\tag{26}$$

非势广义力

$$\bar{Q}_1=t,\ \bar{Q}_2=0,\ \bar{Q}_3=t^2.\tag{27}$$

由 $\Lambda_s=\sum_{\beta=1}^{g}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}$ 知，$\bar{\Lambda}_s=\sum_{\beta=1}^{g}\bar{\lambda}_\beta\frac{\partial \bar{f}_\beta}{\partial p_s}$，将(26)式代入，得

$$\bar{\Lambda}_1=\bar{\lambda},\ \bar{\Lambda}_2=t\bar{\lambda},\ \bar{\Lambda}_3=\bar{\lambda}.\tag{28}$$

将(25)式代入系统运动微分方程式(4)，得

$$\dot{p}_1=\bar{\Lambda}_1+t,\ \dot{p}_2=\bar{\Lambda}_2,\ \dot{p}_3=\bar{\Lambda}_3+t^2.\tag{29}$$

(26)式对 t 求导，得

$$\dot{p}_1+t\dot{p}_2+\dot{p}_3+1=0.\tag{30}$$

将(28)和(29)式代入(30)式，整理得 $\bar{\lambda}=-\frac{1+t+t^2}{2+t^2}$，则广义非完整约束力为

$$\bar{\Lambda}_1=-\frac{1+t+t^2}{2+t^2},\ \bar{\Lambda}_2=-\frac{t+t^2+t^3}{2+t^2},\ \bar{\Lambda}_3=-\frac{1+t+t^2}{2+t^2}.\tag{31}$$

取

$$\xi_0=0,\ \xi_1=1,\ \xi_2=0,\ \xi_3=1,\ \eta_1=\eta_2=\eta_3=0,\tag{32}$$

则有

$$X^{(0)}(H)=\frac{\partial H}{\partial q_1}+\frac{\partial H}{\partial q_3}=0,$$
$$X^{(0)}(\bar{Q})=\frac{\partial \bar{Q}}{\partial q_1}+\frac{\partial \bar{Q}}{\partial q_3}=0,$$
$$X^{(0)}(\bar{\Lambda})=\frac{\partial \bar{\Lambda}}{\partial q_1}+\frac{\partial \bar{\Lambda}}{\partial q_3}=0,$$
$$X^{(0)}(\bar{f})=\frac{\partial \bar{f}}{\partial q_1}+\frac{\partial \bar{f}}{\partial q_3}=0.\tag{33}$$

可见(15)，(16)，(17)式满足，因此该变换是系统的形式不变性变换.

(18)式给出

$$G_F=2t-\frac{1}{2}t^2-\frac{1}{3}t^3+\ln(2+t^2)-\sqrt{2}\operatorname{arctg}\frac{\sqrt{2}}{2}t,\tag{34}$$

由(19)式可求得守恒量

$$I=p_1+p_3+2t-\frac{1}{2}t^2-\frac{1}{3}t^3+\ln(2+t^2)-\sqrt{2}\operatorname{arctg}\frac{\sqrt{2}}{2}t=\text{const}.\tag{35}$$

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Form invariance of nonconservative nonholonomic systems in the phase space

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Abstract

In this paper, the form invariance of nonconservative nonholonmic systems in the phase space is studied. The definition and criterion of the form invariance of nonholonmic nonconservative systems in the phase space is given. The structure equation and conservation law of form invariance is obtained. An example is given to illustrate the application of the result.

Keywords: phase space, nonconservative nonholonnmic, form invariance

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