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改进的 tanh 函数方法与广义变系数 KdV 和 MKdV 方程新的精确解*

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利用改进的 tanh 函数方法将广义变系数 KdV 方程和 MKdV 方程化为一阶变系数非线性常微分方程组.通过求解这个变系数非线性常微分方程组,获得了广义变系数 KdV 方程和 MKdV 方程新的精确类孤子解、有理形式函数解和三角函数解.

关键词:改进的 tanh 函数方法,类孤子解,有理形式函数解,三角函数解

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1.引 言

自从求解 KdV 方程初值问题的反散射方法提出以来[1],在非线性演化方程的求解理论方面有了很多有效方法,如 Baclund 变换、Hirota 变换、Darboux 变换、tanh 函数方法等[1—4].但这些方法大多仅用于常系数的方程求解问题.常系数的方程只能近似地描述实际物理现象,为了准确描述实际物理现象,相应的变系数非线性方程的研究便显得非常重要.文献[5]用 Miura 方法研究了变系数 KdV 方程

$$q_t = K_0(t)(q_{xxx} + 6qq_x) + 4K_1(t)q_x - h(t)(2q + xq_x) \tag{1}$$

和变系数 MKdV 方程

$$u_t = K_0(t)(u_{xxx} - 6u^2 u_x) + 4K_1(t)u_x - h(t)(u + xu_x) \tag{2}$$

的无穷多守恒律.文献[6—9]用特殊的函数变换和截断展开法获得了广义变系数 KdV 方程和 KdV-MKdV 方程

$$u_t + 2\beta(t)u + [\alpha(t) + \beta(t)x]u_x - 3c\gamma(t)uu_x + \gamma(t)u_{xxx} = 0, \tag{3}$$

$$K_0(t)[u_{xxx} - a_1 u^2 u_x + 2a_2(u_x^2 + uu_{xx})] + a_3 h(t)K_0(t)uu_x + [K_1(t) + K_2(t)x]u_x + K_2(t)u + u_t = 0 \tag{4}$$

的多种新的精确类孤子解.式中 $a_i(i=1,2,3)$ 是任意常数,$h(t) = \exp\left[-\int_a^t K_2(s)\mathrm{d}s\right]$,$K_i(i=1,2,3)$ 是 t 的任意函数.

方程(3)是一个受数学家和物理学家感兴趣的可化为描述诸多重要物理现象的方程,如它可化为变系数的非均匀谱 KdV 方程(1)[5,10,11]、柱 KdV 方程[12]

$$u_t + \frac{1}{2t}u + 6uu_x + u_{xxx} = 0 \tag{5}$$

和具有弛豫效应非均匀介质的 KdV 方程[12]

$$u_t + \gamma(t)u + [(c_0 + \gamma(t)x)u]_x + 6uu_x + u_{xxx} = 0 \tag{6}$$

等.

近年来,在计算机的符号计算软件的支持下,出现了以 Li 等人建立的机械化求解方法的软件包[13].其中一个关键步骤是使用了 tanh 函数方法.对 tanh 函数方法 Fan[14] 和 Yan[15] 又进一步进行了有效的改进,广泛应用于常系数非线性演化方程的精确解的求解问题.这些方法的主要思想是把常系数非线性演化方程转化为一个常系数的非线性代数方程组,然后利用吴消元法求解此方程组,从而获得常系数非线性演化方程的精确解.本文进一步改进上述方

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法,将变系数非线性演化方程转化为一个变系数的非线性一阶常微分方程组,求解此方程组从而获得变系数非线性演化方程的多种精确解.以方程(2)和(3)为例说明方法的有效性.方程(1)的解可利用文献[5]给出的 Miura 变换(10)由方程(2)的解得到,而方程(5)和(6)是方程(3)的特例.

2. 改进的 tanh 函数方法和广义变系数 KdV 和 MKdV 方程新的精确解

现将文献[14]中 Fan 的方法改为如下形式.

考虑一给定的具有两个自变量 x,t 的变系数非线性偏微分方程

$$H(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{7}$$

设方程具有如下形式的解:

$$u(x,t) = \sum_{i=0}^{n} a_i(t)\phi^i[p(t)x + q(t)], \tag{8}$$

式中 $a_i(i = 1,2,\cdots,n)$, p, q 为待定函数,ϕ 满足方程

$$\phi' = \delta + \phi^2, \tag{9}$$

式中“′”表示对 $\omega = p(t)x + q(t)$ 求导,δ 为任意常数.

方程(9)具有如下形式的通解:

$$\phi = -\sqrt{-\delta}\tanh\sqrt{-\delta}\omega, \qquad \delta < 0, \tag{10}$$

$$\phi = -\sqrt{-\delta}\coth\sqrt{-\delta}\omega, \qquad \delta < 0, \tag{11}$$

$$\phi = -\frac{1}{\omega}, \qquad \delta = 0, \tag{12}$$

$$\phi = \sqrt{\delta}\tan\sqrt{\delta}\omega, \qquad \delta > 0, \tag{13}$$

$$\phi = -\sqrt{\delta}\cot\sqrt{\delta}\omega, \qquad \delta > 0. \tag{14}$$

将(8)和(9)式代入(7)式,并令 x,$\phi^i x^j$($i = 1,2,\cdots n$, $j = 0,1$)的系数为零(n 的确定与文献[14]相同),可得一变系数非线性常微分方程组.求解此方程组,便可得原方程的精确解.

下面以此方法求解方程(2)和(3).首先解方程(2).根据本文的方法,可设解的形式为

$$u = f(t) + g(t)\phi(\omega), \tag{15}$$

将(15)式代入方程(2),并令 x,$\phi^i x^j$($i = 1,2,\cdots 4$, $j = 0,1$)的系数为零,得到如下一阶变系数非线性常微分方程组:

$$6K_0(t)g(t)p(t)(g(t) - p(t)) \times (g(t) + p(t)) = 0, \tag{16}$$

$$\delta g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t) + h(t)p(t)\right) = 0, \tag{17}$$

$$g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t) + h(t)p(t)\right) = 0, \tag{18}$$

$$h(t)g(t) + 12K_0(t)(g(t))^2 \times p(t)f(t)\delta + \frac{\mathrm{d}}{\mathrm{d}t}g(t) = 0, \tag{19}$$

$$12K_0(t)(g(t))^2 p(t)f(t) = 0, \tag{20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) + g(t)\delta\frac{\mathrm{d}}{\mathrm{d}t}q(t) + h(t)f(t) - 4K_1(t)g(t)p(t)\delta + 6K_0(t)g(t)p(t) \times (f(t))^2\delta - 2K_0(t)g(t)(p(t))^3\delta^2 = 0, \tag{21}$$

$$g(t)\left(6K_0(t)p(t)\delta(g(t))^2 + \frac{\mathrm{d}}{\mathrm{d}t}q(t) + 6K_0(t)p(t)(f(t))^2\right) - g(t)(4K_1(t) \times p(t) + 8K_0(t)(p(t))^3\delta) = 0. \tag{22}$$

由(20)式可知

$$f(t) = 0, \tag{23}$$

代入(16)—(22)式,得到

$$6g(t)K_0(t)p(t)(g(t) - p(t))(g(t) + p(t)) = 0, \tag{24}$$

$$h(t)g(t) + \frac{\mathrm{d}}{\mathrm{d}t}g(t) = 0, \tag{25}$$

$$g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t) + h(t)p(t)\right) = 0, \tag{26}$$

$$\delta g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t) + h(t)p(t)\right) = 0, \tag{27}$$

$$g(t)\left(6K_0(t)p(t)\delta(g(t))^2 - 4K_1(t)p(t) + \frac{\mathrm{d}}{\mathrm{d}t}q(t) - 8K_0(t)(p(t))^3\delta\right) = 0, \tag{28}$$

$$\delta g(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}q(t) - 4K_1(t)p(t) - 2K_0(t)(p(t))^3\delta\right) = 0. \tag{29}$$

由(25)—(27)式可知

$$g(t) = p(t) = \exp\left(-\int_a^t h(s)\mathrm{d}s\right), \tag{30}$$

由(28)和(29)式可知

$$q(t) = \int_a^t\left[4K_1(s)\exp\left(-\int_a^s h(v)\mathrm{d}v\right) + 2\delta K_0(s) \times \exp\left(-3\int_a^s h(v)\mathrm{d}v\right)\right]\mathrm{d}s. \tag{31}$$

将(10)—(14),(23),(30)和(31)式代入(15)式,得到方程(2)如下新的精确解:

类孤子解($\delta < 0$)

$$u_1(x,t) = -\sqrt{-\delta}\exp\left(-\int_a^t h(s)\mathrm{d}s\right)\tanh\sqrt{-\delta}$$

$$\times\left\{\exp\left(-\int_a^t h(s)\mathrm{d}s\right)x+\int_a^t\left[4K_1(s)\exp\left(-\int_a^s h(v)\mathrm{d}v\right)+2\delta K_0(s)\exp\left(-3\int_a^s h(v)\mathrm{d}v\right)\right]\mathrm{d}s\right\}; \quad (32)$$

奇性类孤子解($\delta<0$)

$$u_2(x,t)=-\sqrt{-\delta}\exp\left(-\int_a^t h(s)\mathrm{d}s\right)\coth\sqrt{-\delta}\times\left\{\exp\left(-\int_a^t h(s)\mathrm{d}s\right)x+\int_a^t\left[4K_1(s)\times\exp\left(-\int_a^s h(v)\mathrm{d}v\right)+2\delta K_0(s)\times\exp\left(-3\int_a^s h(v)\mathrm{d}v\right)\right]\mathrm{d}s\right\}; \quad (33)$$

有理形式解($\delta=0$)

$$u_3(x,t)=\frac{-\exp\left(-\int_a^t h(s)\mathrm{d}s\right)}{\exp\left(-\int_a^t h(s)\mathrm{d}s\right)x+\int_a^t 4K_1(s)\exp\left(-\int_a^s h(v)\mathrm{d}v\right)\mathrm{d}s}; \quad (34)$$

三角函数解($\delta>0$)

$$u_4(x,t)=\sqrt{\delta}\exp\left(-\int_a^t h(s)\mathrm{d}s\right)\tan\sqrt{\delta}\times\left\{\exp\left(-\int_a^t h(s)\mathrm{d}s\right)x+\int_a^t\left[4K_1(s)\times\exp\left(-\int_a^s h(v)\mathrm{d}v\right)+2\delta K_0(s)\times\exp\left(-3\int_a^s h(v)\mathrm{d}v\right)\right]\mathrm{d}s\right\}; \quad (35)$$

奇性三角函数解($\delta>0$)

$$u_5(x,t)=-\sqrt{\delta}\exp\left(-\int_a^t h(s)\mathrm{d}s\right)\cot\sqrt{\delta}\times\left\{\exp\left(-\int_a^t h(s)\mathrm{d}s\right)x+\int_a^t\left[4K_1(s)\times\exp\left(-\int_a^s h(v)\mathrm{d}v\right)+2\delta K_0(s)\times\exp\left(-3\int_a^s h(v)\mathrm{d}v\right)\right]\mathrm{d}s\right\}. \quad (36)$$

下面讨论方程(3)的解,可设它的解为

$$u=f(t)+g_1(t)\phi(\omega)+g_2(t)\phi^2(\omega), \quad (37)$$

将(37)式代入方程(3),并令 $x,\phi^i x^j$ ($i=1,2,\cdots,5,j=0,1$)的系数为零,得到如下一阶变系数非线性常微分方程组:

$$-3\gamma(t)p(t)g_1(t)(-2(p(t))^2+3cg_2(t))=0, \quad (38)$$

$$-6\gamma(t)p(t)g_2(t)(cg_2(t)-4(p(t))^2)=0, \quad (39)$$

$$g_1(t)\delta\left(\beta(t)p(t)+\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right)=0, \quad (40)$$

$$2\delta g_2(t)\left(\beta(t)p(t)+\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right)=0, \quad (41)$$

$$g_1(t)\left(\beta(t)p(t)+\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right)=0, \quad (42)$$

$$2g_2(t)\left(\beta(t)p(t)+\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right)=0, \quad (43)$$

$$2\beta(t)f(t)+g_1(t)\delta\frac{\mathrm{d}}{\mathrm{d}t}q(t)+\frac{\mathrm{d}}{\mathrm{d}t}f(t)-3c\gamma(t)p(t)f(t)g_1(t)\delta+\alpha(t)p(t)g_1(t)\delta+2\gamma(t)(p(t))^3g_1(t)\delta^2=0, \quad (44)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}g_1(t)+2\beta(t)g_1(t)+16\gamma(t)(p(t))^3g_2(t)\delta^2+2g_2(t)\delta\frac{\mathrm{d}}{\mathrm{d}t}q(t)-6c\gamma(t)p(t)f(t)g_2(t)\delta-3c\gamma(t)p(t)(g_1(t))^2\delta+2\alpha(t)p(t)g_2(t)\delta=0, \quad (45)$$

$$-6c\gamma(t)p(t)(g_2(t))^2\delta+40\gamma(t)(p(t))^3g_2(t)\delta-6c\gamma(t)p(t)f(t)g_2(t)+2\alpha(t)p(t)g_2(t)-3c\gamma(t)p(t)(g_1(t))^2+2g_2(t)\frac{\mathrm{d}}{\mathrm{d}t}q(t)=0, \quad (46)$$

$$2\beta(t)g_2(t)-9c\gamma(t)p(t)g_1(t)g_2(t)\delta+\alpha(t)p(t)g_1(t)+\frac{\mathrm{d}}{\mathrm{d}t}g_2(t)+g_1(t)\frac{\mathrm{d}}{\mathrm{d}t}q(t)+8\gamma(t)(p(t))^3g_1(t)\delta-3c\gamma(t)p(t)f(t)g_1(t)=0. \quad (47)$$

由(38)和(39)式可知

$$g_1(t)=0, \quad (48)$$

将(48)式代入(38)—(47)式,得

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)+2\beta(t)f(t)=0, \quad (49)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}g_2(t)+2\beta(t)g_2(t)=0, \quad (50)$$

$$-6g_2(t)\gamma(t)p(t)(cg_2(t)-4(p(t))^2)=0, \quad (51)$$

$$2\delta g_2(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t)+\beta(t)p(t)\right)=0, \quad (52)$$

$$2g_2(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}p(t)+\beta(t)p(t)\right)=0, \quad (53)$$

$$-2g_2(t)\left\{3c\gamma(t)p(t)\delta g_2(t)-20\gamma(t)(p(t))^3\delta-\frac{\mathrm{d}}{\mathrm{d}t}q(t)+3c\gamma(t)p(t)f(t)-\alpha(t)p(t)\right\}=0, \quad (54)$$

$$-2\delta g_2(t)\Big(-8\gamma(t)(p(t))^3\delta-\alpha(t)p(t)$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}q(t)+3c\gamma(t)p(t)f(t)\Big)=0. \tag{55}$$

由(49),(52),(53)和(51)式可得

$$f(t)=\exp\left(-2\int_a^t\beta(s)\mathrm{d}s\right), \tag{56}$$

$$p(t)=\exp\left(-\int_a^t\beta(s)\mathrm{d}s\right), \tag{57}$$

$$g_2(t)=\frac{4}{c}\exp\left(-2\int_a^t\beta(s)\mathrm{d}s\right). \tag{58}$$

此时(50)式自然成立,由(54)和(55)式得

$$\begin{aligned}&-8\gamma(t)(p(t))^3\delta-\alpha(t)p(t)\\&-\frac{\mathrm{d}}{\mathrm{d}t}q(t)+3c\gamma(t)p(t)f(t)=0,\end{aligned} \tag{59}$$

由(59)式有

$$\begin{aligned}q(t)=&\int_a^t\left\{-8\delta\gamma(s)\exp\left(-3\int_a^s\beta(v)\mathrm{d}v\right)\right\}\mathrm{d}s\\&-\int_a^t\left\{\alpha(s)\exp\left(-\int_a^s\beta(v)\mathrm{d}v\right)\right.\\&\left.+3c\gamma(s)\exp\left(-3\int_a^s\beta(v)\mathrm{d}v\right)\right\}\mathrm{d}s.\end{aligned} \tag{60}$$

由(10)—(14),(37),(48),(56)—(58)和(60)式可得方程(3)的5种精确解,由于表达式复杂,在此不赘述.但由(56)和(60)式可以看出,文献[7]所得的类孤子解仅是这里的 $\delta=\frac{1}{8}$ 时的一个特例.

3. 结 论

总之,通过对扩展的 tanh 函数方法的进一步地改进,本文实现了将变系数非线性演化方程转化为一变系数非线性常微分方程组这一关键步骤,获得了已有方法所不能得到的新的精确解.这一方法显然也适合其他方程,并可进一步推广到高维的变系数方程,如变系数的 KP 方程等.有关结果将另文给出.

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Improved tanh-function method and the new exact solutions for the general variable coefficient KdV equation and MKdV equation*

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Abstract

In this paper, by using the extended tanh-function method, the general variable coefficient KdV and MKdV equations are reduced to first-order variable coefficient nonlinear ordinary differential equations, and then the new exact solutions for these equations, which include exact soliton-like, rational formal and triangle function solutions, are obtained through solving these ordinary differential equations.

Keywords: improved tanh-function method, soliton-like solutions, rational formal solutions, triangle function solutions

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