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变质量力学系统的一般形式的非 Noether 守恒量

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研究一般的无限小变换下变质量力学系统 Lie 对称性的非 Noether 守恒量,进一步推广 Hojman 定理.给出变质量力学系统的一般形式的非 Noether 守恒量,并举例说明结果的应用.

关键词:变质量系统,一般的无限小变换,非 Noether 守恒量
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1.引 言

力学系统守恒量的研究具有重要的理论意义和实际意义.分析力学的方法仅能找到极少数的守恒量,如广义动量守恒、广义能量守恒等.近代寻求守恒量的方法是对称性方法,主要有 Noether 对称性、Lie 对称性和 Mei 对称性[1—17].Noether 对称性总可导致守恒量,而 Lie 对称性和 Mei 对称性一般没有这种性质.由 Lie 对称性和 Mei 对称性寻求守恒量往往要通过 Noether 对称性来找到 Noether 型的守恒量.1992 年 Hojman 给出了由 Lie 对称性找守恒量的一种直接方法,得到了一类新的守恒量[18],被称为 Hojman 定理[19].Pillay 和 Leach 证明了利用 Hojman 守恒律导出的守恒量对所有 Noether 对称性都是平凡的[19].因此,要用 Hojman 方法找有意义的非平凡守恒量,需要的对称性不是 Noether 的,这种守恒量称为非 Noether 守恒量.对 Hojman 定理已有了一些推广[20—26],然而以往的研究仅限于时间不变的特殊无限小变换.本文研究一般的无限小变换下变质量力学系统的非 Noether 守恒量,将 Hojman 定理的结果进一步推广,给出变质量力学系统的一般形式的非 Noether 守恒量.

2.运动微分方程

研究 N 个质点组成的力学系统.在时刻 t,第 i 个质点的质量为 $m_i(i=1,2,\cdots,N)$,m_i 有一般的形式 $m_i=m_i(t,\boldsymbol{q},\dot{\boldsymbol{q}})$,在时刻 $t+\mathrm{d}t$,由 i 质点分离(或并入)的微粒质量为 $\mathrm{d}m_i$.设系统受到理想的完整约束,系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$确定,则系统运动微分方程为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+P_s \quad (s=1,2,\cdots,n), \tag{1}$$

其中 L 为系统的 Lagrange 函数,Q_s 为非势广义力,P_s 为广义反推力,有

$$P_s=\dot{m}_i(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s}-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial \dot{q}_s}\right), \tag{2}$$

式中 $\boldsymbol{u}_i$ 为分离或并入 m_i 的微粒相对 m_i 的速度.假设系统非奇异,即

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0, \tag{3}$$

则由方程(1)可求出所有的广义加速度,记为

$$\ddot{q}_s=F_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \tag{4}$$

3.无限小变换与 Lie 对称性的确定方程

取一般的无限小变换

$$\begin{aligned}t^*&=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\ q_s^*(t^*)&=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\end{aligned} \tag{5}$$

其中 ε 为无限小参数,ξ_0,ξ_s 为无限小单参数群变换的生成元.在无限小变换(5)式下的 Lie 对称性的确定方程为

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_0 - 2F_s\frac{\bar{d}}{dt}\xi_0 - \frac{\partial F_s}{\partial t}\xi_0 - \frac{\partial F_s}{\partial q_k}\xi_k - \frac{\partial F_s}{\partial \dot{q}_k}\left(\frac{\bar{d}}{dt}\xi_k - \dot{q}_k\frac{\bar{d}}{dt}\xi_0\right) = 0, \tag{6}$$

其中

$$\frac{\bar{d}}{dt} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + F_s\frac{\partial}{\partial \dot{q}_s}. \tag{7}$$

如果无限小变换(5)式的生成元 ξ_0 和 ξ_s 满足方程(6)，则变换是 Lie 对称的.

4. Hojman 定理的推广

定理 对变质量力学系统(1)，如果无限小变换的生成元 ξ_0 和 ξ_s 满足确定方程(6)，且存在某函数 $\lambda = \lambda(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$，使得

$$\frac{\partial F_s}{\partial \dot{q}_s} + \frac{\bar{d}}{dt}\ln\lambda = 0, \tag{8}$$

则系统存在非 Noether 守恒量

$$I = \frac{1}{\lambda}\frac{\partial(\lambda\xi_0)}{\partial t} + \frac{1}{\lambda}\frac{\partial(\lambda\xi_s)}{\partial q_s} + \frac{1}{\lambda}\frac{\partial}{\partial \dot{q}_s}\left[\lambda\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right)\right] - \frac{\bar{d}}{dt}\xi_0 = \text{const}. \tag{9}$$

证 我们有

$$\frac{\bar{d}I}{dt} = \frac{\bar{d}}{dt}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial t}\xi_0\right) + \frac{\bar{d}}{dt}\frac{\partial\xi_0}{\partial t} + \frac{\bar{d}}{dt}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\xi_s\right) + \frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s} + \frac{\bar{d}}{dt}\left[\frac{1}{\lambda}\frac{\partial\lambda}{\partial \dot{q}_s}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right)\right] + \frac{\bar{d}}{dt}\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right) - \frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_0, \tag{10}$$

以及

$$\frac{\bar{d}}{dt}\frac{\partial\xi_0}{\partial t} = \frac{\partial}{\partial t}\frac{\bar{d}}{dt}\xi_0 - \frac{\partial F_k}{\partial t}\frac{\partial\xi_0}{\partial \dot{q}_k}, \tag{11}$$

$$\frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{\bar{d}}{dt}\xi_s - \frac{\partial F_k}{\partial q_s}\frac{\partial\xi_s}{\partial \dot{q}_k}, \tag{12}$$

$$\frac{\bar{d}}{dt}\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right) = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{d}}{dt}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right) - \frac{\partial}{\partial q_s}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right) - \frac{\partial F_k}{\partial \dot{q}_s}\frac{\partial}{\partial \dot{q}_k}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right). \tag{13}$$

将(11)—(13)式代入(10)式，并利用(6)式，得

$$\frac{\bar{d}I}{dt} = \frac{\bar{d}}{dt}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial t}\xi_0\right) + \frac{\bar{d}}{dt}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial q_s}\xi_s\right) + \frac{\bar{d}}{dt}\left[\frac{1}{\lambda}\frac{\partial\lambda}{\partial \dot{q}_s}\left(\frac{\bar{d}}{dt}\xi_s - \dot{q}_s\frac{\bar{d}}{dt}\xi_0\right)\right] + \frac{\partial F_s}{\partial \dot{q}_s}\frac{\bar{d}}{dt}\xi_0 + \frac{\partial^2 F_s}{\partial \dot{q}_s\partial t}\xi_0 + \frac{\partial^2 F_s}{\partial \dot{q}_s\partial q_k}\xi_k + \frac{\partial^2 F_s}{\partial \dot{q}_s\partial \dot{q}_k}\left(\frac{\bar{d}}{dt}\xi_k - \dot{q}_k\frac{\bar{d}}{dt}\xi_0\right). \tag{14}$$

将(8)式分别对 t, q_s 和 $\dot{q}_s$ 求偏导数，并将其代入(14)式，利用(6)式，可得

$$\frac{\bar{d}I}{dt} = 0. \tag{15}$$

如果 $\xi_0 = 0, m_i = m_i(t, \boldsymbol{q})$，则本文的结果化为文献[23]的结果；如果 $\xi_0 = 0, \lambda = \lambda(\boldsymbol{q})$，则本文的结果化为 Hojmen 的结果[18]，当然，Hojmen 没有指明方程(4)的具体意义.

5. 算 例

系统的 Lagrange 函数为

$$L = \frac{1}{2}m(t)(\dot{q}_1^2 + \dot{q}_2^2), \tag{16}$$

质量的变化规律为

$$m = m_0 e^{-\gamma t} \quad (m_0, \gamma \text{ 为常数}), \tag{17}$$

非势广义力为

$$Q_1 = 0, \quad Q_2 = -km\dot{q}_2, \tag{18}$$

微粒分离的相对速度为

$$\boldsymbol{u} = -\dot{\boldsymbol{r}}, \tag{19}$$

试研究系统的非 Noether 守恒量.

由(2)式可知

$$P_1 = P_2 = 0, \tag{20}$$

方程(1)给出

$$\ddot{q}_1 = F_1 = \gamma\dot{q}_1, \quad \ddot{q}_2 = F_2 = (\gamma - k)\dot{q}_2, \tag{21}$$

确定方程(6)给出

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_1 - 2\gamma\dot{q}_1\frac{\bar{d}}{dt}\xi_0 - \dot{q}_1\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_0 - \left(\frac{\bar{d}}{dt}\xi_1 - \dot{q}_1\frac{\bar{d}}{dt}\xi_0\right)\gamma = 0, \tag{22}$$

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_2 - 2(\gamma - k)\dot{q}_2\frac{\bar{d}}{dt}\xi_0 - \dot{q}_2\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\xi_0 - \left(\frac{\bar{d}}{dt}\xi_2 - \dot{q}_2\frac{\bar{d}}{dt}\xi_0\right)(\gamma - k) = 0, \tag{23}$$

方程(22)，(23)有如下解：

$$\xi_0 = 1,$$
$$\xi_1 = 1, \tag{24}$$
$$\xi_2 = [\dot{q}_2 + (k - \gamma) q_2]^2,$$

方程(8)给出

$$2\gamma - k + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\lambda = 0, \tag{25}$$

解得

$$\lambda = \mathrm{e}^{(k-2\gamma)t}. \tag{26}$$

由(9)式可得到守恒量

$$I = 2\dot{q}_2 + 2(k - \gamma) q_2 - 2\gamma + k = \mathrm{const}. \tag{27}$$

利用 Noether 逆定理可知,与守恒量(27)式相应的 Noether 对称性生成元为

$$\xi_0 = 0$$
$$\xi_1 = 0, \tag{28}$$
$$\xi_2 = \frac{2}{m}.$$

因此,守恒量(27)式是一个非 Noether 守恒量.

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Non-Noether conserved quantity of a general form for mechanical systems with variable mass

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Abstract

In this paper, we study the non-Noether conserved quantity of Lie symmetry for mechanical systems with variable mass under a general infinitesimal transformation. The Hojman theorem is further generalized. The non-Noether conserved quantity of a general form for mechanical systems with variable mass is obtained. An example is given to illustrate the application of the result.

Keywords: variable mass system, general infinitesimal transformation, non-Noether conserved quantity

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