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非线性波方程求解的新方法*

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从 Legendre 椭圆积分和 Jacobi 椭圆函数的定义出发,得到了新的变换,并把它用于非线性演化方程的求解. 用三个具体的例子,如非线性 Klein-Gordon 方程、Boussinesq 方程和耦合的 mKdV 方程组,说明了具体的求解步骤. 比较方便地得到非线性演化方程或方程组的新解析解,如周期解、孤子解等.

关键词: Jacobi 椭圆函数, 非线性方程, 周期解, 孤子解

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1. 引 言

在非线性波动方程的求解中, Jacobi 椭圆函数起到了十分重要的作用[1—4], 我们把 Jacobi 椭圆函数展开方法[1—4]应用到非线性演化方程的求解, 得到了新的解析解. 下面, 从新的角度认识 Legendre 椭圆积分和 Jacobi 椭圆函数在非线性方程求解的重要作用.

我们知道, 第一类 Legendre 椭圆积分定义为

$$\theta(\phi) = \int_0^{\phi} \frac{1}{\sqrt{1 - m^2 \sin^2 \varphi}} \mathrm{d}\varphi, \tag{1}$$

其中 $0 < m < 1$ 称为模数.

若令 $y = \sin\varphi$, $\tau = \sin\phi$ 则得到 Jacobi 椭圆正弦函数的定义

$$\theta(\tau) = \int_0^{\tau = \sin\phi} \frac{1}{\sqrt{(1 - y^2)(1 - m^2 y^2)}} \mathrm{d}y \equiv \mathrm{sn}^{-1} \tau, \tag{2}$$

即

$$\tau = \mathrm{sn}\theta = \sin\phi. \tag{3}$$

若令 $y = \cos\varphi$, $\tau = \cos\phi$, 则由(1)式得到 Jacobi 椭圆余弦函数的定义

$$\theta(\tau) = \int_{\tau = \cos\phi}^{1} \frac{1}{\sqrt{(1 - y^2)(m'^2 + m^2 y^2)}} \mathrm{d}y \equiv \mathrm{cn}^{-1} \tau, \tag{4}$$

即

$$\tau = \mathrm{cn}\theta = \cos\phi, \tag{5}$$

其中 m' 称作余模数, 而且有 $m'^2 + m^2 = 1$.

事实上, 由(1)式得到

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \sqrt{1 - m^2 \sin^2 \phi}, \tag{6}$$

其基本解是

$$\sin\phi = \mathrm{sn}\theta, \cos\phi = \mathrm{cn}\theta. \tag{7}$$

同样, 第三类 Legendre 椭圆积分定义为

$$\theta(\phi) = \int_0^{\phi} \frac{1}{\sqrt{m^2 - \sin^2 \varphi}} \mathrm{d}\varphi. \tag{8}$$

若令 $y = \cos\varphi$, $\tau = \cos\phi$, 则由(8)式得到第三类 Jacobi 椭圆函数的定义

$$\theta(\tau) = \int_{\tau = \cos\phi}^{1} \frac{1}{\sqrt{(1 - y^2)(y^2 - m'^2)}} \mathrm{d}y \equiv \mathrm{dn}^{-1} \tau, \tag{9}$$

即

$$\tau = \mathrm{dn}\theta = \cos\phi. \tag{10}$$

事实上, 由(8)式得到

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \sqrt{m^2 - \sin^2 \phi}, \tag{11}$$

其基本解是

$$\cos\phi = \mathrm{dn}\theta, \sin\phi = m\,\mathrm{sn}\theta. \tag{12}$$

(6)和(11)式可以看作两个基本变换, 这两个变换可以用于非线性方程的求解中, 简化求解过程. 下面, 用三个具体的例子进行说明, 分别是非线性 Klein-Gordon 方程、Boussinesq 方程和耦合的 mKdV 方程组.

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2. 非线性 Klein-Gordon 方程

非线性 Klein-Gordon 方程为

$$u_{tt} - c_0^2 u_{xx} + \alpha u - \beta u^3 = 0, \tag{13}$$

我们寻求它的行波解为

$$u = u(\xi), \xi = k(x - ct), \tag{14}$$

其中 k 和 c 分别为波数和波的传播速度.从而有

$$k^2(c^2 - c_0^2)\frac{d^2 u}{d\xi^2} + \alpha u - \beta u^3 = 0. \tag{15}$$

将 $u(\xi)$ 展开为下列三角函数的级数：

$$u(\xi) = \sum_{j=1}^{n} \sin^{j-1}\omega(a_j \sin\omega + b_j \cos\omega) + a_0, \tag{16}$$

其中 ω 满足(6)式：

$$\frac{d\omega}{d\xi} = \sqrt{1 - m^2 \sin^2\omega}, \tag{17}$$

从而有

$$\frac{d^2\omega}{d\xi^2} = - m^2 \sin\omega\cos\omega. \tag{18}$$

这样,在(16)式中,选择适当的 n 使得非线性波动方程(13)中的非线性项和最高阶导数项平衡,即可得到方程的形式解,具体的求解步骤可参看文献[1—4].这里,可以得到 $n = 1$,因此,方程(13)的形式解为

$$u = a_0 + a_1 \sin\omega + b_1 \cos\omega. \tag{19}$$

注意到

$$\begin{aligned}\frac{d^2 u}{d\xi^2} &= (a_1 \cos\omega - b_1 \sin\omega)\frac{d^2\omega}{d\xi^2} \\ &\quad - (a_1 \sin\omega + b_1 \cos\omega)\left(\frac{d\omega}{d\xi}\right)^2 \\ &= - (1 + m^2) a_1 \sin\omega - b_1 \cos\omega \\ &\quad + 2m^2 b_1 \cos\omega \sin^2\omega + 2m^2 a_1 \sin^3\omega.\end{aligned} \tag{20}$$

$$\begin{aligned}u^3 &= (a_0^3 + 3a_0 b_1^2) + 3(a_0^2 + b_1^2) a_1 \sin\omega \\ &\quad + (3a_0^2 + b_1^2) b_1 \cos\omega + 6a_0 a_1 b_1 \cos\omega\sin\omega \\ &\quad + 3a_0(a_1^2 - b_1^2)\sin^2\omega \\ &\quad + (3a_1^2 - b_1^2) b_1 \cos\omega \sin^2\omega \\ &\quad + (a_1^2 - 3b_1^2) a_1 \sin^3\omega.\end{aligned} \tag{21}$$

这样,(19)式代入(15)式得到

$$\begin{aligned}&[\alpha a_0 - \beta(a_0^3 + 3a_0 b_1^2)] \\ &+ [- k^2(c^2 - c_0^2)(1 + m^2) a_1 \\ &+ \alpha a_1 - 3\beta(a_0^2 + b_1^2) a_1]\sin\omega \\ &+ [- k^2(c^2 - c_0^2) b_1 + \alpha b_1 - \beta(3a_0^2 + b_1^2) b_1]\cos\omega \\ &+ [- 6\beta a_0 a_1 b_1]\cos\omega\sin\omega \\ &+ [- 3\beta a_0(a_1^2 - b_1^2)]\sin^2\omega + [2k^2(c^2 - c_0^2) m^2 b_1 \\ &- \beta(3a_1^2 - b_1^2) b_1]\cos\omega\sin^2\omega \\ &+ [2k^2(c^2 - c_0^2) m^2 a_1 - \beta(a_1^2 - 3b_1^2) a_1]\sin^3\omega = 0.\end{aligned} \tag{22}$$

令三角函数各幂次方的系数为零,则得到关于展开系数的方程组

$$\alpha a_0 - \beta(a_0^3 + 3a_0 b_1^2) = 0, \tag{23}$$

$$\begin{aligned}&- k^2(c^2 - c_0^2)(1 + m^2) a_1 + \alpha a_1 \\ &- 3\beta(a_0^2 + b_1^2) a_1 = 0,\end{aligned} \tag{24}$$

$$- k^2(c^2 - c_0^2) b_1 + \alpha b_1 - \beta(3a_0^2 + b_1^2) b_1 = 0, \tag{25}$$

$$- 6\beta a_0 a_1 b_1 = 0, \tag{26}$$

$$- 3\beta a_0(a_1^2 - b_1^2) = 0, \tag{27}$$

$$2k^2(c^2 - c_0^2) m^2 b_1 - \beta(3a_1^2 - b_1^2) b_1 = 0, \tag{28}$$

$$2k^2(c^2 - c_0^2) m^2 a_1 - \beta(a_1^2 - 3b_1^2) a_1 = 0. \tag{29}$$

由(23)—(29)式得到解的两种情况：

1) $a_1 \neq 0, b_1 = 0$：

$$a_0 = 0, a_1 = \pm\sqrt{\frac{2m^2\alpha}{(1 + m^2)\beta}},$$

$$k^2 = \frac{\alpha}{(1 + m^2)(c^2 - c_0^2)}, \tag{30}$$

2) $b_1 \neq 0, a_1 = 0$：

$$a_0 = 0, b_1 = \pm\sqrt{\frac{2m^2\alpha}{(2m^2 - 1)\beta}},$$

$$k^2 = \frac{\alpha}{(1 - 2m^2)(c^2 - c_0^2)}. \tag{31}$$

结合(6),(7),(17)和(19)式得到方程(13)的两种解：

$$\begin{aligned}u_1 &= a_1 \sin\omega = \pm\sqrt{\frac{2m^2\alpha}{(1 + m^2)\beta}} \\ &\quad \times \mathrm{sn}\left[\pm\sqrt{\frac{\alpha}{(1 + m^2)(c^2 - c_0^2)}}(x - ct)\right],\end{aligned} \tag{32}$$

$$\begin{aligned}u_2 &= b_1 \cos\omega = \pm\sqrt{\frac{2m^2\alpha}{(2m^2 - 1)\beta}} \\ &\quad \times \mathrm{cn}\left[\pm\sqrt{\frac{\alpha}{(1 - 2m^2)(c^2 - c_0^2)}}(x - ct)\right].\end{aligned} \tag{33}$$

很明显,这是方程(13)的两类周期解 椭圆正弦波解和椭圆余弦波解.

当 $m \to 1$ 时, $\mathrm{sn}\xi \to \tanh\xi$, $\mathrm{cn}\xi \to \mathrm{sech}\xi$,因此,同时可以得到

$$u_3 = \pm\sqrt{\frac{\alpha}{\beta}}\tanh\left[\pm\sqrt{\frac{\alpha}{2(c^2 - c_0^2)}}(x - ct)\right], \tag{34}$$

$$u_4 = \pm\sqrt{\frac{2\alpha}{\beta}}\operatorname{sech}\left[\pm\sqrt{-\frac{\alpha}{(c^2-c_0^2)}}(x-ct)\right] \tag{35}$$

两类孤子解.

若(17)式中 ω 满足(11)式,即

$$\frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \sqrt{m^2-\sin^2\omega}, \tag{36}$$

从而有

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2} = -\sin\omega\cos\omega. \tag{37}$$

方程的形式解仍然为(19)式,类似,把(19)式代入(15)式,并考虑到(36)式,则可以得到方程(13)的另一类解:

3) $b_1\neq 0, a_1=0$:

$$a_0 = 0, b_1 = \pm\sqrt{\frac{2\alpha}{(2-m^2)\beta}},$$

$$k^2 = \frac{\alpha}{(m^2-2)(c^2-c_0^2)}. \tag{38}$$

结合(11),(12),(36)和(19)式得到方程(13)的解

$$u_5 = b_1\cos\omega = \pm\sqrt{\frac{2\alpha}{(2-m^2)\beta}} \times \operatorname{dn}\left[\pm\sqrt{\frac{\alpha}{(m^2-2)(c^2-c_0^2)}}(x-ct)\right]. \tag{39}$$

很明显,这是方程(13)的另一类周期解:第三类椭圆函数解.

当 $m\to 1$ 时 $\operatorname{dn}\xi\to\operatorname{sech}\xi$,因此,同时可以得到孤子解为(35)式.

3. Boussinesq 方程的解

Boussinesq 方程形式为

$$u_{tt}-c_0^2u_{xx}-\alpha u_{xxxx}-\beta(u^2)_{xx} = 0, \tag{40}$$

求其行波解,把(14)式代入(40)式得到

$$(c^2-c_0^2)\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}-\alpha k^2\frac{\mathrm{d}^4u}{\mathrm{d}\xi^4}-\beta\frac{\mathrm{d}^2u^2}{\mathrm{d}\xi^2} = 0, \tag{41}$$

方程积分两次得到

$$(c^2-c_0^2)u-\beta u^2-\alpha k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = A, \tag{42}$$

其中 A 为积分常数.把(16)式代入(41)式得到方程(40)的形式解为

$$u = a_0+a_1\sin\omega+b_1\cos\omega+a_2\sin^2\omega+b_2\cos\omega\sin\omega. \tag{43}$$

这里,取 ω 满足变换(36)式(满足(17)式的可以类似求解).把(43)式代入(42)式得到

$$\begin{aligned}
&[(c^2-c_0^2)a_0-\beta(a_0^2+b_1^2)-2\alpha k^2m^2a_2-A]\\
&+[(c^2-c_0^2)a_1-\beta(2a_0a_1+2b_1b_2)\\
&+\alpha k^2(1+m^2)a_1]\sin\omega+[(c^2-c_0^2)b_1\\
&-2\beta a_0b_1+\alpha k^2m^2b_1]\cos\omega+[(c^2-c_0^2)b_2\\
&-\beta(2a_0b_2+2a_1b_1)+\alpha k^2(1+4m^2)b_2]\cos\omega\sin\omega\\
&+[(c^2-c_0^2)a_2-\beta(2a_0a_2+a_1^2-b_1^2+b_2^2)\\
&+4\alpha k^2(1+m^2)a_2]\sin^2\omega+[-\beta(2a_1b_2+2a_2b_1)\\
&-2\alpha k^2b_1]\cos\omega\sin^2\omega+[-\beta(2a_1a_2-2b_2b_1)\\
&-2\alpha k^2a_1]\sin^3\omega+[-2\beta a_2b_2\\
&-6\alpha k^2b_2]\cos\omega\sin^3\omega+[-\beta(a_2^2-b_2^2)\\
&-6\alpha k^2a_2]\sin^4\omega = 0.
\end{aligned} \tag{44}$$

令三角函数各幂次项的系数为零,得到展开系数的代数方程

$$(c^2-c_0^2)a_0-\beta(a_0^2+b_1^2)-2\alpha k^2m^2a_2-A = 0, \tag{45}$$

$$(c^2-c_0^2)a_1-\beta(2a_0a_1+2b_1b_2)+\alpha k^2(1+m^2)a_1 = 0, \tag{46}$$

$$(c^2-c_0^2)b_1-2\beta a_0b_1+\alpha k^2m^2b_1 = 0, \tag{47}$$

$$(c^2-c_0^2)b_2-\beta(2a_0b_2+2a_1b_1)+\alpha k^2(1+4m^2)b_2 = 0, \tag{48}$$

$$(c^2-c_0^2)a_2-\beta(2a_0a_2+a_1^2-b_1^2+b_2^2)+4\alpha k^2(1+m^2)a_2 = 0, \tag{49}$$

$$-\beta(2a_1b_2+2a_2b_1)-2\alpha k^2b_1 = 0, \tag{50}$$

$$-\beta(2a_1a_2-2b_2b_1)-2\alpha k^2a_1 = 0, \tag{51}$$

$$-2\beta a_2b_2-6\alpha k^2b_2 = 0, \tag{52}$$

$$-\beta(a_2^2-b_2^2)-6\alpha k^2a_2 = 0. \tag{53}$$

由(45)—(53)式得到方程(40)的两种解:

1) $a_1=b_1=b_2=0$:

$$a_0 = \frac{c^2-c_0^2}{2\beta}+\frac{2\alpha k^2(1+m^2)}{\beta},$$

$$\alpha_2 = -\frac{6\alpha k^2}{\beta}. \tag{54}$$

2) $a_1=b_1=0, b_2\neq 0$:

$$a_0 = \frac{c^2-c_0^2}{2\beta}+\frac{\alpha k^2(1+4m^2)}{2\beta}$$

$$a_2 = -\frac{3\alpha k^2}{\beta}, b_2 = \pm\mathrm{i}\frac{3\alpha k^2}{\beta},$$

$$\mathrm{i} = \sqrt{-1}. \tag{55}$$

结合(11),(12),(36)和(43)式得到方程(40)的解

$$u_1 = a_0+a_2\sin^2\omega = \frac{c^2-c_0^2}{2\beta}$$

$$+\frac{2\alpha k^2(1+m^2)}{\beta}-\frac{6\alpha k^2}{\beta}m^2\mathrm{sn}^2[k(x-ct)]$$
$$=\frac{c^2-c_0^2}{2\beta}+\frac{2\alpha k^2(1-2m^2)}{\beta}$$
$$+\frac{6\alpha k^2}{\beta}m^2\mathrm{cn}^2[k(x-ct)]$$
$$=\frac{c^2-c_0^2}{2\beta}+\frac{2\alpha k^2(m^2-2)}{\beta}+\frac{6\alpha k^2}{\beta}\mathrm{dn}^2[k(x-ct)],\tag{56}$$

$$u_2=a_0+a_2\sin^2\omega+b_2\cos\omega\sin\omega$$
$$=\frac{c^2-c_0^2}{2\beta}+\frac{\alpha k^2(1+4m^2)}{2\beta}$$
$$-\frac{3\alpha k^2}{\beta}m^2\mathrm{sn}^2[k(x-ct)]$$
$$\pm\mathrm{i}\frac{3\alpha k^2}{\beta}m\,\mathrm{sn}[k(x-ct)]\mathrm{dn}[k(x-ct)].\tag{57}$$

当 $m\to1$ 时,同时可以得到孤子解

$$u_3=\frac{c^2-c_0^2}{2\beta}+\frac{4\alpha k^2}{\beta}-\frac{6\alpha k^2}{\beta}\tanh^2[k(x-ct)]$$
$$=\frac{c^2-c_0^2}{2\beta}-\frac{2\alpha k^2}{\beta}+\frac{6\alpha k^2}{\beta}\mathrm{sech}^2[k(x-ct)],\tag{58}$$

$$u_4=\frac{c^2-c_0^2}{2\beta}+\frac{5\alpha k^2}{2\beta}-\frac{3\alpha k^2}{\beta}\tanh^2[k(x-ct)]$$
$$\pm\mathrm{i}\frac{3\alpha k^2}{\beta}\tanh[k(x-ct)]\mathrm{sech}[k(x-ct)].\tag{59}$$

4. 耦合 mKdV 方程组的解

耦合 mKdV 方程形式为

$$u_t+\alpha_0v_x+\alpha u^2u_x+\beta u_{xxx}=0,$$
$$v_t+\alpha_1(uv)_x+\alpha_2vv_x=0,\tag{60}$$

求其行波解,即

$$u=u(\xi),v=v(\xi),\xi=k(x-ct).\tag{61}$$

把(61)式代入方程组(60)并积分一次

$$-cu+\alpha_0v+\frac{\alpha}{3}u^3+\beta k^2\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}=C,$$
$$-cv+\alpha_1uv+\frac{\alpha_2}{2}v^2=C_2.\tag{62}$$

设(60)式的展开解为

$$u(\xi)=\sum_{j=1}^{n_1}\sin^{j-1}\omega(a_j\sin\omega+b_j\cos\omega)+a_0,$$
$$v(\xi)=\sum_{j=1}^{n_2}\sin^{j-1}\omega(A_j\sin\omega+B_j\cos\omega)+A_0.\tag{63}$$

(63)式代入(62)式可得到 $n_1=n_2=1$,即形式解为

$$u(\xi)=a_1\sin\omega+b_1\cos\omega+a_0,$$
$$v(\xi)=A_1\sin\omega+B_1\cos\omega+A_0.\tag{64}$$

当 ω 满足(17)式时,(64)式代入(62)式得到方程(60)的两类周期解:

1) $a_1\neq0,b_1=0$:

$$a_0=0,a_1=\pm\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(1+m^2)\alpha\alpha_2}},\quad A_0=\frac{2c}{\alpha_2},$$
$$A_1=-\frac{2\alpha_1}{\alpha_2}a_1,B_1=0,k^2=-\frac{c\alpha_2+2\alpha_0\alpha_1}{(1+m^2)\beta\alpha_2}.\tag{65}$$

2) $b_1\neq0,a_1=0$:

$$a_0=0,b_1=\pm\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(2m^2-1)\alpha\alpha_2}},\quad A_0=\frac{2c}{\alpha_2},$$
$$A_1=0,B_1=-\frac{2\alpha_1}{\alpha_2}b_1,$$
$$k^2=\frac{c\alpha_2+2\alpha_0\alpha_1}{(2m^2-1)\beta\alpha_2},\tag{66}$$

即周期解为

$$u_1=a_1\sin\omega=\pm\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(1+m^2)\alpha\alpha_2}}$$
$$\times\mathrm{sn}\left[\pm\sqrt{-\frac{c\alpha_2+2\alpha_0\alpha_1}{(1+m^2)\beta\alpha_2}}(x-ct)\right],$$
$$v_1=A_0+A_1\sin\omega$$
$$=\frac{2c}{\alpha_2}\mp\frac{2\alpha_1}{\alpha_2}\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(1+m^2)\alpha\alpha_2}}$$
$$\times\mathrm{sn}\left[\pm\sqrt{-\frac{c\alpha_2+2\alpha_0\alpha_1}{(1+m^2)\beta\alpha_2}}(x-ct)\right];\tag{67}$$

$$u_2=b_1\cos\omega=\pm\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(2m^2-1)\alpha\alpha_2}}$$
$$\times\mathrm{cn}\left[\pm\sqrt{\frac{c\alpha_2+2\alpha_0\alpha_1}{(2m^2-1)\beta\alpha_2}}(x-ct)\right],$$
$$v_2=A_0+B_1\cos\omega$$
$$=\frac{2c}{\alpha_2}\mp\frac{2\alpha_1}{\alpha_2}\sqrt{\frac{6m^2(c\alpha_2+2\alpha_0\alpha_1)}{(2m^2-1)\alpha\alpha_2}}$$
$$\times\mathrm{cn}\left[\pm\sqrt{\frac{c\alpha_2+2\alpha_0\alpha_1}{(2m^2-1)\beta\alpha_2}}(x-ct)\right].\tag{68}$$

相应的孤子解为

$$u_3=\pm\sqrt{\frac{3(c\alpha_2+2\alpha_0\alpha_1)}{\alpha\alpha_2}}$$
$$\times\tanh\left[\pm\sqrt{-\frac{c\alpha_2+2\alpha_0\alpha_1}{2\beta\alpha_2}}(x-ct)\right],$$

$$v_3 = \frac{2c}{\alpha_2} \mp \frac{2\alpha_1}{\alpha_2}\sqrt{\frac{3(c\alpha_2 + 2\alpha_0\alpha_1)}{\alpha\alpha_2}} \times \tanh\left[\pm\sqrt{-\frac{c\alpha_2 + 2\alpha_0\alpha_1}{2\beta\alpha_2}}(x - ct)\right]; \quad (69)$$

$$u_4 = \pm\sqrt{\frac{6(c\alpha_2 + 2\alpha_0\alpha_1)}{\alpha\alpha_2}} \times \mathrm{sech}\left[\pm\sqrt{\frac{c\alpha_2 + 2\alpha_0\alpha_1}{\beta\alpha_2}}(x - ct)\right],$$

$$v_4 = \frac{2c}{\alpha_2} \mp \frac{2\alpha_1}{\alpha_2}\sqrt{\frac{6(c\alpha_2 + 2\alpha_0\alpha_1)}{\alpha\alpha_2}} \times \mathrm{sech}\left[\pm\sqrt{\frac{c\alpha_2 + 2\alpha_0\alpha_1}{\beta\alpha_2}}(x - ct)\right]. \quad (70)$$

当 ω 满足(36)式时,(64)式代入(62)式得到方程(60)的另一类周期解:

3) $b_1 \neq 0, a_1 = 0$:

$$a_0 = 0, b_1 = \pm\sqrt{\frac{6(c\alpha_2 + 2\alpha_0\alpha_1)}{(2 - m^2)\alpha\alpha_2}}, A_0 = \frac{2c}{\alpha_2},$$

$$A_1 = 0, B_1 = -\frac{2\alpha_1}{a_2}b_1,$$

$$k^2 = \frac{c\alpha_2 + 2\alpha_0\alpha_1}{(2 - m^2)\beta\alpha_2}, \quad (71)$$

即

$$u_5 = b_1\cos\omega = \pm\sqrt{\frac{6(c\alpha_2 + 2\alpha_0\alpha_1)}{(2 - m^2)\alpha\alpha_2}} \times \mathrm{dn}\left[\pm\sqrt{\frac{c\alpha_2 + 2\alpha_0\alpha_1}{(2 - m^2)\beta\alpha_2}}(x - ct)\right],$$

$$v_5 = A_0 + B_1\cos\omega = \frac{2c}{\alpha_2} \mp \frac{2\alpha_1}{\alpha_2}\sqrt{\frac{6(c\alpha_2 + 2\alpha_0\alpha_1)}{(2 - m^2)\alpha\alpha_2}} \times \mathrm{dn}\left[\pm\sqrt{\frac{c\alpha_2 + 2\alpha_0\alpha_1}{(2 - m^2)\beta\alpha_2}}(x - ct)\right], \quad (72)$$

其对应的孤子解为(70)式.

5. 结　　论

在非线性问题中,对非线性演化方程的求解和定性分析占有很重要的地位.已经发展了很多比较系统的求解方法和分析手段,如 Jacobi 椭圆函数展开法[1—4],齐次平衡法[5—9],双曲正切函数展开法[10—12],试探函数法[13,14],非线性变换法[15,16],和 sine-cosine 方法[17].这些方法也求得了非线性波方程的周期解、冲击波解或孤立波解[1—32].本文注意了 Jacobi 椭圆函数和三角函数的转换,既简化了求解过程,又能够得到周期解和孤子解,这样便于复杂方程的求解.

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A new method to construct solutions to nonlinear wave equations*

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Abstract

From the definition of Legendre elliptic integration and Jacobi elliptic function, new transformations are obtained and applied to construct the exact solutions of nonlinear wave equations. The nonlinear Klein – Gordon equation, Boussinesq equation and the coupled mKdV equations are taken as three examples to illustrate the detailed steps in obtaining exact solutions. There new analytical solutions such as periodic solutions and soliton solutions are derived for these nonlinear evolution equation (or equations).

Keywords: Jacobi elliptic function, nonlinear wave equation, periodic solution, soliton solution
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