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相空间中单面完整约束力学系统的对称性与守恒量*

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在增广相空间中研究单面完整约束力学系统的对称性与守恒量.建立了系统的运动微分方程;给出了系统的 Norther 对称性,Lie 对称性和 Mei 对称性的判据;研究了三种对称性之间的关系;得到了相空间中单面完整约束力学系统的 Noether 守恒量以及两类新守恒量——Hojman 守恒量和 Mei 守恒量,研究了三种对称性和三类守恒量之间的内在关系.文中举例说明研究结果的应用.

关键词:分析力学,单面约束,对称性,守恒量,相空间
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1. 引 言

在自然界及工程实际中相当多的约束是属于单面的,如系统在有限空间中的运动、工程振动与冲击系统、机器人绕过障碍的运动、滚轮系统又滚又滑的运动等[1].单面约束系统的对称性与守恒量研究比双面约束情形要复杂.当系统处于约束上时,单面约束成为双面约束,因此,单面约束系统对称性与守恒量的理论可以适用于双面约束系统[2].近年来,单面约束系统的对称性与守恒量研究已取得了一些重要进展[2—9].本文研究相空间中单面完整约束力学系统的对称性与守恒量.给出了系统的 Norther 对称性,Lie 对称性和 Mei 对称性的判据,研究了这些对称性之间的关系,得到了相空间中单面完整约束力学系统的 Noether 守恒量以及两类新守恒量——Hojman 守恒量和 Mei 守恒量.

2. 系统的运动微分方程

设力学系统的位形由 n 个广义坐标 q_s ($s=1,\cdots,n$)来确定,其运动受 a 个单面理想完整约束

$$f_\alpha(t,\boldsymbol{q}) \geqslant 0,(\alpha=1,\cdots,a) \tag{1}$$

则系统的运动微分方程可表为[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\lambda_\alpha\frac{\partial f_\alpha}{\partial q_s}\quad(s=1,\cdots,n). \tag{2}$$

对每一个 α,有

$$\lambda_\alpha\geqslant 0, f_\alpha\geqslant 0, \lambda_\alpha f_\alpha=0\quad(\alpha=1,\cdots,a), \tag{3}$$

其中 $L=T-V$ 为系统的 Lagrange 函数,T 为系统的动能,V 为势能;Q_s 为非势广义力;λ_α 为约束乘子.注意到方程(2)并不封闭,这是由于单面约束(1)的存在,系统的速度改变是不连续的.为使方程(2)封闭,还须考虑沿约束超曲面的连接条件.例如,设约束超曲面是绝对光滑的且碰撞是完全弹性的.

引入广义动量和 Hamilton 函数

$$p_s=\frac{\partial L}{\partial \dot{q}_s}, H=p_s\dot{q}_s-L, \tag{4}$$

则方程(2)可表为正则形式

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\dot{p}_s=-\frac{\partial H}{\partial q_s}+Q_s+\lambda_\alpha\frac{\partial f_\alpha}{\partial q_s}\quad(s=1,\cdots,n). \tag{5}$$

若系统处于约束上,约束方程(1)取等号,设系统非奇异,则可解出约束乘子 λ_α 作为 $t,\boldsymbol{q},\boldsymbol{p}$ 的函数,而方程(5)成为

$$\dot{q}_s=\frac{\partial H}{\partial p_s},\dot{p}_s=-\frac{\partial H}{\partial q_s}+Q_s+\Lambda_s, \tag{6}$$

其中

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$$\Lambda_s = \Lambda_s(t,\boldsymbol{q},\boldsymbol{p}) = \lambda_\alpha(t,\boldsymbol{q},\boldsymbol{p})\frac{\partial f_\alpha}{\partial q_s}, \qquad (7)$$

展开方程(6),有

$$\dot{q}_s = A_s(t,\boldsymbol{q},\boldsymbol{p}), \dot{p}_s = B_s(t,\boldsymbol{q},\boldsymbol{p}) \quad (s = 1,\cdots,n). \qquad (8)$$

若系统脱离约束,即约束方程(1)中不等号严格成立,则方程(5)成为

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s, \qquad (9)$$

展开方程(9),有

$$\dot{q}_s = \mathscr{A}_s(t,\boldsymbol{q},\boldsymbol{p}),$$
$$\dot{p}_s = \mathscr{B}_s(t,\boldsymbol{q},\boldsymbol{p}) \quad (s = 1,\cdots,n). \qquad (10)$$

3. 系统的对称性

3.1. 变换群与生成元

在增广相空间中,取群的无限小变换为

$$t^* = t + \Delta t,$$
$$q_s^*(t^*) = q_s(t) + \Delta q_s,$$
$$p_s^*(t^*) = p_s(t) + \Delta p_s, \qquad (11)$$

或其展开式

$$t^* = t + \varepsilon\tau(t,\boldsymbol{q},\boldsymbol{p}),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t,\boldsymbol{q},\boldsymbol{p}),$$
$$p_s^*(t^*) = p_s(t) + \varepsilon\eta_s(t,\boldsymbol{q},\boldsymbol{p}), \qquad (12)$$

其中 ε 为无限小参数,τ,ξ_s,η_s 为无限小变换的生成元.引入无限小生成元向量

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s}, \qquad (13)$$

它的一次扩展为

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s} + (\dot{\eta}_s - \dot{p}_s\dot{\tau})\frac{\partial}{\partial \dot{p}_s}. \qquad (14)$$

3.2. Noether 对称性

根据 Noether 理论[10—13],有如下判据:

判据 1　对于相空间中单面完整约束系统,如果存在规范函数 $G_N = G_N(t,\boldsymbol{q},\boldsymbol{p})$,使无限小生成元 τ,ξ_s,η_s 在系统处于约束上时满足条件

$$p_s\dot{\xi}_s + \dot{q}_s\eta_s - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_s}\xi_s - \frac{\partial H}{\partial p_s}\eta_s$$
$$- H\dot{\tau} + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\tau) + \dot{G}_N = 0 \qquad (15)$$

以及限制方程

$$X^{(0)}(f_\alpha) = 0, \qquad (16)$$

而当系统脱离约束时,满足条件

$$p_s\dot{\xi}_s + \dot{q}_s\eta_s - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_s}\xi_s$$
$$- \frac{\partial H}{\partial p_s}\eta_s - H\dot{\tau} + Q_s(\xi_s - \dot{q}_s\tau) + \dot{G}_N = 0, \qquad (17)$$

则相应不变性为系统的 Noether 对称性.

称方程(15)和(17)为相空间中单面完整约束系统的广义 Noether 等式.

将方程(15)和(17)分别展开,并考虑到相空间中相点速度 $\dot{q}_s,\dot{p}_s$ 的相互独立性,当系统处于约束上时有

$$-H\frac{\partial \tau}{\partial p_s} + p_k\frac{\partial \xi_k}{\partial p_s} + \frac{\partial G_N}{\partial p_s} = 0, \quad (s = 1,\cdots,n),$$
$$-H\frac{\partial \tau}{\partial q_s} + p_k\frac{\partial \xi_k}{\partial q_s} + \eta_s - (Q_s + \Lambda_s)\tau + \frac{\partial G_N}{\partial q_s} = 0,$$
$$(s = 1,\cdots,n),$$
$$-H\frac{\partial \tau}{\partial t} - \tau\frac{\partial H}{\partial t} - \xi_k\frac{\partial H}{\partial q_k} - \eta_k\frac{\partial H}{\partial p_k}$$
$$+ p_k\frac{\partial \xi_k}{\partial t} + (Q_k + \Lambda_k)\xi_k + \frac{\partial G_N}{\partial t} = 0, \qquad (18)$$

而当系统脱离约束时有

$$-H\frac{\partial \tau}{\partial p_s} + p_k\frac{\partial \xi_k}{\partial p_s} + \frac{\partial G_N}{\partial p_s} = 0, \quad (s = 1,\cdots,n),$$
$$-H\frac{\partial \tau}{\partial q_s} + p_k\frac{\partial \xi_k}{\partial q_s} + \eta_s - Q_s\tau + \frac{\partial G_N}{\partial q_s} = 0,$$
$$(s = 1,\cdots,n),$$
$$-H\frac{\partial \tau}{\partial t} - \tau\frac{\partial H}{\partial t} - \xi_k\frac{\partial H}{\partial q_k} - \eta_k\frac{\partial H}{\partial p_k}$$
$$+ p_k\frac{\partial \xi_k}{\partial t} + Q_k\xi_k + \frac{\partial G_N}{\partial t} = 0, \qquad (19)$$

方程(18),(19)是含有$(2n+2)$个未知函数 τ,ξ_s,η_s 和 G_N 的$(2n+1)$个线性偏微分方程,可称为广义 Killing 方程.如果方程有解,则相应对称性为相空间中单面完整约束系统的 Noether 对称性.

3.3. Lie 对称性

Lie 对称性是微分方程在无限小群变换下的一种不变性[12—15].方程(8)和(10)在无限小变换(11)下的不变性归结为如下的 Lie 对称性确定方程[3]

$$\dot{\xi}_s - \dot{q}_s\dot{\tau} = X^{(0)}(A_s), \dot{\eta}_s - \dot{p}_s\dot{\tau} = X^{(0)}(B_s)$$
$$\text{当 } f_\alpha(t,\boldsymbol{q}) = 0, \qquad (20)$$
$$\dot{\xi}_s - \dot{q}_s\dot{\tau} = X^{(0)}(\mathscr{A}_s), \dot{\eta}_s - \dot{p}_s\dot{\tau} = X^{(0)}(\mathscr{B}_s)$$

$$当\ f_\alpha(t,\boldsymbol{q}) > 0, \tag{21}$$

向量场(13)对单面约束(1)的限制为

$$X^{(0)}(f_\alpha) = 0 \qquad 当\ f_\alpha(t,\boldsymbol{q}) = 0, \tag{22}$$

于是有

判据 2 对于相空间中单面完整约束系统,如果无限小生成元 τ,ξ_s,η_s 满足确定方程(20),(21)以及限制方程(22),则相应不变性为系统的 Lie 对称性.

3.4. Mei 对称性

假设 $H, Q_s, \Lambda_s, f_\alpha$ 在经历变换(11)后分别成为 $H^*, Q_s^*, \Lambda_s^*, f_\alpha^*$,有

$$\begin{aligned}
H^* &= H(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)\\
&= H(t,\boldsymbol{q},\boldsymbol{p}) + \varepsilon X^{(0)}(H) + o(\varepsilon^2),\\
Q_s^* &= Q_s(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)\\
&= Q_s(t,\boldsymbol{q},\boldsymbol{p}) + \varepsilon X^{(0)}(Q_s) + o(\varepsilon^2),\\
\Lambda_s^* &= \Lambda_s(t^*, \boldsymbol{q}^*, \boldsymbol{p}^*)\\
&= \Lambda_s(t,\boldsymbol{q},\boldsymbol{p}) + \varepsilon X^{(0)}(\Lambda_s) + o(\varepsilon^2),\\
f_\alpha^* &= f_\alpha(t^*, \boldsymbol{q}^*)\\
&= f_\alpha(t,\boldsymbol{q}) + \varepsilon X^{(0)}(f_\alpha) + o(\varepsilon^2).
\end{aligned} \tag{23}$$

如果用无限小变换后的动力学函数 $H^*, Q_s^*, \Lambda_s^*, f_\alpha^*$ 代替变换前的 $H, Q_s, \Lambda_s, f_\alpha$ 时,系统的运动方程和约束方程的形式保持不变,则这种不变性称为系统的 Mei 对称性[16—22].有

$$\dot q_s = \frac{\partial H^*}{\partial p_s}, \dot p_s = -\frac{\partial H^*}{\partial q_s} + Q_s^* + \Lambda_s^*$$
$$当\ f_\alpha(t,\boldsymbol{q}) = 0, \tag{24}$$

$$\dot q_s = \frac{\partial H^*}{\partial p_s}, \dot p_s = -\frac{\partial H^*}{\partial q_s} + Q_s^*$$
$$当\ f_\alpha(t,\boldsymbol{q}) > 0, \tag{25}$$

以及

$$f_\alpha^* = f_\alpha(t^*, \boldsymbol{q}^*) = 0 \qquad 当\ f_\alpha(t,\boldsymbol{q}) = 0. \tag{26}$$

将式(23)代入方程(24)—(26),忽略 ε^2 及更高阶小项,并利用方程(6)和(9),得到

$$\frac{\partial}{\partial p_s}\{X^{(0)}(H)\} = 0,$$
$$\frac{\partial}{\partial q_s}\{X^{(0)}(H)\} = X^{(0)}(Q_s + \Lambda_s)$$
$$当\ f_\alpha(t,\boldsymbol{q}) = 0, \tag{27}$$

$$\frac{\partial}{\partial p_s}\{X^{(0)}(H)\} = 0,$$
$$\frac{\partial}{\partial q_s}\{X^{(0)}(H)\} = X^{(0)}(Q_s)$$
$$当\ f_\alpha(t,\boldsymbol{q}) > 0, \tag{28}$$

以及

$$X^{(0)}(f_\alpha) = 0 \qquad 当\ f_\alpha(t,\boldsymbol{q}) = 0, \tag{29}$$

于是有

判据 3 对于相空间中单面完整约束系统,如果无限小生成元 τ,ξ_s,η_s 满足方程(27),(28)以及限制方程(29),则相应不变性为系统的 Mei 对称性.

4. 三种对称性之间的关系

4.1. Noether 对称性与 Lie 对称性

由于,对任意函数 $F(t,\boldsymbol{q},\boldsymbol{p})$,有

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial q_s} = \frac{\partial \dot F}{\partial q_s}, \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial p_s} = \frac{\partial \dot F}{\partial p_s},$$
$$(s = 1,\cdots,n), \tag{30}$$

于是,当系统处于约束上时,利用方程(6),并考虑到上述关系式(30),可以得到

$$\begin{aligned}
&X^{(1)}\left(\dot q_s - \frac{\partial H}{\partial p_s}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left[-H\frac{\partial \tau}{\partial p_s} + p_k\frac{\partial \xi_k}{\partial p_s} + \frac{\partial G_{\mathrm N}}{\partial p_s}\right]\\
&- \frac{\partial}{\partial p_s}\Big[p_k\dot\xi_k + \dot q_k\eta_k - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_k}\xi_k - \frac{\partial H}{\partial p_k}\eta_k\\
&- H\dot\tau + (Q_k + \Lambda_k)(\xi_k - \dot q_k\tau) + \dot G_{\mathrm N}\Big]\\
&= -\frac{\partial}{\partial p_s}(Q_k + \Lambda_k)(\xi_k - \dot q_k\tau),
\end{aligned} \tag{31}$$

$$\begin{aligned}
&X^{(1)}\left(\dot p_s + \frac{\partial H}{\partial q_s} - Q_s - \Lambda_s\right) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[-H\frac{\partial \tau}{\partial q_s}\\
&+ p_k\frac{\partial \xi_k}{\partial q_s} + \eta_s - (Q_s + \Lambda_s)\tau + \frac{\partial G_{\mathrm N}}{\partial q_s}\Big]\\
&+ \frac{\partial}{\partial q_s}\Big[p_k\dot\xi_k + \dot q_k\eta_k - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_k}\xi_k - \frac{\partial H}{\partial p_k}\eta_k\\
&- H\dot\tau + (Q_k + \Lambda_k)(\xi_k - \dot q_k\tau) + \dot G_{\mathrm N}\Big]\\
&= \left[\frac{\partial}{\partial q_s}(Q_k + \Lambda_k) - \frac{\partial}{\partial q_k}(Q_s + \Lambda_s)\right](\xi_k - \dot q_k\tau)\\
&- \frac{\partial}{\partial p_k}(Q_s + \Lambda_s)(\eta_k - \dot p_k\tau).
\end{aligned} \tag{32}$$

当系统脱离约束时,利用方程(9),并考虑到关系(30),有

$$\begin{aligned}
&X^{(1)}\left(\dot q_s - \frac{\partial H}{\partial p_s}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left[-H\frac{\partial \tau}{\partial p_s} + p_k\frac{\partial \xi_k}{\partial p_s} + \frac{\partial G_{\mathrm N}}{\partial p_s}\right]\\
&- \frac{\partial}{\partial p_s}\Big[p_k\dot\xi_k + \dot q_k\eta_k - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_k}\xi_k - \frac{\partial H}{\partial p_k}\eta_k
\end{aligned}$$

$$- H\dot{\tau} + Q_k(\xi_k - \dot{q}_k\tau) + \dot{G}_N\Big]$$

$$= -\frac{\partial Q_k}{\partial p_s}(\xi_k - \dot{q}_k\tau), \tag{33}$$

$$X^{(1)}\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - Q_s\right) - \frac{\mathrm{d}}{\mathrm{d}t}\Big[-H\frac{\partial \tau}{\partial q_s} + p_k\frac{\partial \xi_k}{\partial q_s} + \eta_s - Q_s\tau + \frac{\partial G_N}{\partial q_s}\Big]$$

$$+ \frac{\partial}{\partial q_s}\Big[p_k\dot{\xi}_k + \dot{q}_k\eta_k - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q_k}\xi_k - \frac{\partial H}{\partial p_k}\eta_k - H\dot{\tau} + Q_k(\xi_k - \dot{q}_k\tau) + \dot{G}_N\Big]$$

$$= \Big[\frac{\partial Q_k}{\partial q_s} - \frac{\partial Q_s}{\partial q_k}\Big](\xi_k - \dot{q}_k\tau) - \frac{\partial Q_s}{\partial p_k}(\eta_k - \dot{p}_k\tau), \tag{34}$$

于是有如下结果

命题 1　对于相空间中单面完整约束系统，如果其 Noether 对称性生成元 τ,ξ_s,η_s 满足条件

$$\Big[\frac{\partial}{\partial q_s}(Q_k + \Lambda_k) - \frac{\partial}{\partial q_k}(Q_s + \Lambda_s)\Big](\xi_k - \dot{q}_k\tau) - \frac{\partial}{\partial p_k}(Q_s + \Lambda_s)(\eta_k - \dot{p}_k\tau) = 0,$$

$$\frac{\partial}{\partial p_s}(Q_k + \Lambda_k)(\xi_k - \dot{q}_k\tau) = 0,$$

$$\text{当 } f_\alpha(t,\boldsymbol{q}) = 0, \tag{35}$$

$$\left(\frac{\partial Q_k}{\partial q_s} - \frac{\partial Q_s}{\partial q_k}\right)(\xi_k - \dot{q}_k\tau) - \frac{\partial Q_s}{\partial p_k}(\eta_k - \dot{p}_k\tau) = 0,$$

$$\frac{\partial Q_k}{\partial p_s}(\xi_k - \dot{q}_k\tau) = 0,$$

$$\text{当 } f_\alpha(t,\boldsymbol{q}) > 0, \tag{36}$$

则 Noether 对称性是 Lie 对称性.

上面的命题给出了 Noether 对称性为 Lie 对称性的条件，下面的命题给出 Lie 对称性为 Noether 对称性的条件.

命题 2　对于相空间中单面完整约束系统，如果其 Lie 对称性生成元 τ,ξ_s,η_s 和规范函数 $G_N = G_N(t,\boldsymbol{q},\boldsymbol{p})$ 满足如下结构方程

$$- H\dot{\tau} + \dot{q}_s\eta_s + p_s\dot{\xi}_s - X^{(0)}(H) + (Q_s + \Lambda_s)(\xi_s - \dot{q}_s\tau) + \dot{G}_N = 0$$

$$\text{当 } f_\alpha(t,\boldsymbol{q}) = 0, \tag{37}$$

$$- H\dot{\tau} + \dot{q}_s\eta_s + p_s\dot{\xi}_s - X^{(0)}(H) + Q_s(\xi_s - \dot{q}_s\tau) + \dot{G}_N = 0$$

$$\text{当 } f_\alpha(t,\boldsymbol{q}) > 0, \tag{38}$$

则 Lie 对称性是 Noether 对称性.

4.2. Lie 对称性与 Mei 对称性

由判据 2 和判据 3，并利用方程(6)和(9)，容易得到下面的命题.

命题 3　对于相空间中单面完整约束系统，Lie 对称性为 Mei 对称性的充分必要条件是无限小生成元 τ,ξ_s,η_s 在系统处于约束上时满足

$$\dot{\xi}_s - \frac{\partial H}{\partial p_s}\dot{\tau} + \frac{\partial \tau}{\partial p_s}\frac{\partial H}{\partial t} + \frac{\partial \xi_k}{\partial p_s}\frac{\partial H}{\partial q_k} + \frac{\partial \eta_k}{\partial p_s}\frac{\partial H}{\partial p_k} = 0, \tag{39}$$

$$\dot{\eta}_s - \left(-\frac{\partial H}{\partial q_s} + Q_s + \Lambda_s\right)\dot{\tau} - \frac{\partial \tau}{\partial q_s}\frac{\partial H}{\partial t} - \frac{\partial \xi_k}{\partial q_s}\frac{\partial H}{\partial q_k} - \frac{\partial \eta_k}{\partial q_s}\frac{\partial H}{\partial p_k} = 0; \tag{40}$$

而当系统脱离约束时满足式(39)和下式

$$\dot{\eta}_s - \left(-\frac{\partial H}{\partial q_s} + Q_s\right)\dot{\tau} - \frac{\partial \tau}{\partial q_s}\frac{\partial H}{\partial t} - \frac{\partial \xi_k}{\partial q_s}\frac{\partial H}{\partial q_k} - \frac{\partial \eta_k}{\partial q_s}\frac{\partial H}{\partial p_k} = 0. \tag{41}$$

5. 对称性导致的守恒量

5.1. Noether 守恒量

对于相空间中单面完整约束力学系统，由 Noether 对称性可直接导出 Noether 守恒量. 有如下结果

命题 4　对于相空间中单面完整约束系统，如果无限小生成元 τ,ξ_s,η_s 相应于系统的 Noether 对称性，则系统存在 Noether 守恒量，形如

$$I_N = p_s\xi_s - H\tau + G_N = \text{const.} \tag{42}$$

Noether 理论的优势在于，有一个 Noether 对称性，就可以找到与之相应的一个守恒量[12]. 而 Lie 对称性或 Mei 对称性不一定总能导致守恒量，研究表明，由 Lie 对称性或 Mei 对称性通过 Noether 对称性可间接导出 Noether 守恒量. 有如下结果

命题 5[3]　对于相空间中单面完整约束系统，如果无限小生成元 τ,ξ_s,η_s 相应于系统的 Lie 对称性，且存在规范函数 $G_N = G_N(t,\boldsymbol{q},\boldsymbol{p})$ 满足结构方程

(37),(38),则 Lie 对称性导致 Noether 守恒量(42).

命题 6[8] 对于相空间中单面完整约束系统,如果无限小生成元 τ,ξ_s,η_s 相应于系统的 Mei 对称性,且存在规范函数 $G_N=G_N(t,\boldsymbol{q},\boldsymbol{p})$ 满足广义 Noether 等式(15),(17),则 Mei 对称性导致 Noether 守恒量(42).

5.2. Hojman 守恒量

对于相空间中单面完整约束力学系统,在一定条件下,由 Lie 对称性可直接导出一类新守恒量,称之为 Hojman 守恒量[23—29].

利用方程(8),可将 Lie 对称性的确定方程(20)写成

$$\frac{\bar{d}}{dt}\xi_s-A_s\frac{\bar{d}}{dt}\tau=X^{(0)}(A_s),$$

$$\frac{\bar{d}}{dt}\eta_s-B_s\frac{\bar{d}}{dt}\tau=X^{(0)}(B_s),$$

$$当\ f_\alpha(t,\boldsymbol{q})=0, \tag{43}$$

其中

$$\frac{\bar{d}}{dt}=\frac{\partial}{\partial t}+A_s\frac{\partial}{\partial q_s}+B_s\frac{\partial}{\partial p_s}. \tag{44}$$

而利用方程(10),可将方程(21)写成

$$\frac{\bar{d}}{dt}\xi_s-\mathscr{A}_s\frac{\bar{d}}{dt}\tau=X^{(0)}(\mathscr{A}_s),$$

$$\frac{\bar{d}}{dt}\eta_s-\mathscr{B}_s\frac{\bar{d}}{dt}\tau=X^{(0)}(\mathscr{B}_s),$$

$$当\ f_\alpha(t,\boldsymbol{q})>0, \tag{45}$$

其中

$$\frac{\bar{d}}{dt}=\frac{\partial}{\partial t}+\mathscr{A}_s\frac{\partial}{\partial q_s}+\mathscr{B}_s\frac{\partial}{\partial p_s}, \tag{46}$$

有如下结果

命题 7 对于相空间中单面完整约束力学系统,如果无限小生成元 τ,ξ_s,η_s 相应于系统的 Lie 对称性,且存在规范函数 $G_H=G_H(t,\boldsymbol{q},\boldsymbol{p})$,当系统处于约束上时,满足

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\tau=0,\frac{\partial A_s}{\partial q_s}+\frac{\partial B_s}{\partial p_s}+\frac{\bar{d}}{dt}\ln G_H=0, \tag{47}$$

而当系统脱离约束时,满足

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\tau=0,\frac{\partial \mathscr{A}_s}{\partial q_s}+\frac{\partial \mathscr{B}_s}{\partial p_s}+\frac{\bar{d}}{dt}\ln G_H=0, \tag{48}$$

则 Lie 对称性直接导致 Hojman 守恒量,形如

$$I_H=\frac{1}{G_H}\frac{\partial}{\partial t}(G_H\tau)+\frac{1}{G_H}\frac{\partial}{\partial q_s}(G_H\xi_s)+\frac{1}{G_H}\frac{\partial}{\partial p_s}(G_H\eta_s)=\text{const}. \tag{49}$$

证 若系统处于约束上,容易验证关系式

$$\frac{\bar{d}}{dt}\frac{\partial\tau}{\partial t}=\frac{\partial}{\partial t}\frac{\bar{d}}{dt}\tau-\frac{\partial\tau}{\partial q_k}\frac{\partial A_k}{\partial t}-\frac{\partial\tau}{\partial p_k}\frac{\partial B_k}{\partial t},$$

$$\frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s}=\frac{\partial}{\partial q_s}\frac{\bar{d}}{dt}\xi_s-\frac{\partial\xi_s}{\partial q_k}\frac{\partial A_k}{\partial q_s}-\frac{\partial\xi_s}{\partial p_k}\frac{\partial B_k}{\partial q_s},$$

$$\frac{\bar{d}}{dt}\frac{\partial\eta_s}{\partial p_s}=\frac{\partial}{\partial p_s}\frac{\bar{d}}{dt}\eta_s-\frac{\partial\eta_s}{\partial q_k}\frac{\partial A_k}{\partial p_s}-\frac{\partial\eta_s}{\partial p_k}\frac{\partial B_k}{\partial p_s}, \tag{50}$$

利用方程(43)和条件(47),并考虑关系式(50),容易得到

$$\begin{aligned}\frac{\bar{d}}{dt}I_H&=\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial t}\tau\right)+\frac{\bar{d}}{dt}\frac{\partial\tau}{\partial t}+\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial q_s}\xi_s\right)\\&\quad+\frac{\bar{d}}{dt}\frac{\partial\xi_s}{\partial q_s}+\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial p_s}\eta_s\right)+\frac{\bar{d}}{dt}\frac{\partial\eta_s}{\partial p_s}\\&=\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial t}\tau\right)+\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial q_s}\xi_s\right)\\&\quad+\frac{\bar{d}}{dt}\left(\frac{1}{G_H}\frac{\partial G_H}{\partial p_s}\eta_s\right)-\frac{\bar{d}}{dt}\ln G_H\frac{\bar{d}}{dt}\tau-\tau\frac{\partial}{\partial t}\frac{\bar{d}}{dt}\ln G_H\\&\quad-\xi_k\frac{\partial}{\partial q_k}\frac{\bar{d}}{dt}\ln G_H-\eta_k\frac{\partial}{\partial p_k}\frac{\bar{d}}{dt}\ln G_H\\&=0,\end{aligned}$$

当系统脱离约束时,同理可得

$$\frac{\bar{d}}{dt}I_H=0,$$

故系统有守恒量(49).证毕.

由系统的 Noether 对称性或 Mei 对称性,通过 Lie 对称性可间接导致 Hojman 守恒量.有如下结果

命题 8 对于相空间中单面完整约束系统,如果其 Noether 对称性生成元 τ,ξ_s,η_s 满足条件(35),(36),且存在规范函数 $G_H=G_H(t,\boldsymbol{q},\boldsymbol{p})$ 满足条件(47),(48),则系统的 Noether 对称性导致 Hojman 守恒量(49).

命题 9 对于相空间中单面完整约束系统,如果其 Mei 对称性生成元 τ,ξ_s,η_s 满足条件(39)—(41),且存在规范函数 $G_H=G_H(t,\boldsymbol{q},\boldsymbol{p})$ 满足条件(47),(48),则系统的 Mei 对称性导致 Hojman 守恒量(49).

5.3. Mei 守恒量

对于相空间中单面完整约束力学系统,在一定条件下,由 Mei 对称性可直接导致一类新守恒量,称之为 Mei 守恒量[2,30].有如下结果

命题 10 对于相空间中单面完整约束系统,如果无限小生成元 τ,ξ_s,η_s 相应于系统的 Mei 对称性,且存在规范函数 $G_M=G_M(t,\boldsymbol{q},\boldsymbol{p})$,当系统处于

约束上时,满足结构方程

$$X^{(0)}(p_s)\frac{\bar{d}}{dt}\xi_s+\frac{\bar{d}}{dt}\{X^{(0)}(p_s)\}\xi_s$$
$$-X^{(0)}\{X^{(0)}(H)\}-X^{(0)}(H)\frac{\bar{d}}{dt}\tau$$
$$+X^{(0)}(Q_s+\Lambda_s)(\xi_s-\dot{q}_s\tau)+\frac{\bar{d}}{dt}G_M=0, \quad (51)$$

而当系统脱离约束时,满足结构方程

$$X^{(0)}(p_s)\frac{\bar{d}}{dt}\xi_s+\frac{\bar{d}}{dt}\{X^{(0)}(p_s)\}\xi_s$$
$$-X^{(0)}\{X^{(0)}(H)\}-X^{(0)}(H)\frac{\bar{d}}{dt}\tau$$
$$+X^{(0)}(Q_s)(\xi_s-\dot{q}_s\tau)+\frac{\bar{d}}{dt}G_M=0, \quad (52)$$

则 Mei 对称性直接导致 Mei 守恒量,形如

$$I_M=X^{(0)}(p_s)\xi_s-X^{(0)}(H)\tau+G_M=\mathrm{const}. \quad (53)$$

证 若系统处于约束上,有

$$\frac{\bar{d}}{dt}I_M=X^{(0)}(p_s)\frac{\bar{d}}{dt}\xi_s+\frac{\bar{d}}{dt}\{X^{(0)}(p_s)\}\xi_s$$
$$-\frac{\bar{d}}{dt}\{X^{(0)}(H)\}\tau-X^{(0)}(H)\frac{\bar{d}}{dt}\tau$$
$$+\frac{\bar{d}}{dt}G_M, \quad (54)$$

由于无限小生成元 τ,ξ_s,η_s 相应于系统的 Mei 对称性,由 Mei 对称性的判据方程(27),容易得到

$$\frac{\bar{d}}{dt}\{X^{(0)}(H)\}=\frac{\partial}{\partial t}\{X^{(0)}(H)\}+A_sX^{(0)}(Q_s+\Lambda_s), \quad (55)$$

将(55)式代入(54)式,利用结构方程(51),得到

$$\frac{\bar{d}}{dt}I_M=0. \quad (56)$$

当系统脱离约束时,同理可得式(56).故系统有守恒量(53).证毕.

由系统的 Noether 对称性或 Lie 对称性,通过 Mei 对称性可间接导致 Mei 守恒量.有如下结果

命题 11 对于相空间中单面完整约束系统,如果其 Noether 对称性生成元 τ,ξ_s,η_s 满足方程(27)(28),且存在规范函数 $G_M=G_M(t,\boldsymbol{q},\boldsymbol{p})$ 满足结构方程(51)(52),则系统的 Noether 对称性导致 Mei 守恒量(53).

命题 12 对于相空间中单面完整约束系统,如果其 Lie 对称性生成元 τ,ξ_s,η_s 满足条件(39)—(41),而且存在规范函数 $G_M=G_M(t,\boldsymbol{q},\boldsymbol{p})$ 满足结构方程(51)(52),则系统的 Lie 对称性导致 Mei 守恒量(53).

6. 算 例

例 设质量为 m 的质点在不低于光滑直线 $y=x$ 的铅垂平面中运动,试研究其在相空间中的对称性与守恒量.

选取广义坐标 $q_1=x,q_2=y$,系统的 Lagrange 函数和约束方程为

$$L=m(\dot{q}_1^2+\dot{q}_2^2)/2-mgq_2, \quad (57)$$

$$f_1=q_2-q_1\geqslant 0. \quad (58)$$

令

$$p_1=m\dot{q}_1,p_2=m\dot{q}_2,$$
$$H=(p_1^2+p_2^2)/2m+mgq_2; \quad (59)$$

则系统的运动微分方程可表为[3]

$$\dot{q}_1=p_1/m,\dot{q}_2=p_2/m,$$
$$\dot{p}_1=-mg/2,\dot{p}_2=-mg/2,$$
$$\text{当 } f_1=0; \quad (60)$$

$$\dot{q}_1=p_1/m,\dot{q}_2=p_2/m,$$
$$\dot{p}_1=0,\dot{p}_2=-mg,$$
$$\text{当 } f_1>0. \quad (61)$$

广义 Noether 等式(15)(17)给出

$$p_1\dot{\xi}_1+p_2\dot{\xi}_2+\dot{q}_1\eta_1+\dot{q}_2\eta_2-mg\xi_2$$
$$-p_1\eta_1/m-p_2\eta_2/m-H\dot{\tau}-mg(\xi_1-\dot{q}_1\tau)/2$$
$$+mg(\xi_2-\dot{q}_2\tau)/2+\dot{G}_N=0$$
$$\text{当} \quad f_1=0, \quad (62)$$

$$p_1\dot{\xi}_1+p_2\dot{\xi}_2+\dot{q}_1\eta_1+\dot{q}_2\eta_2-mg\xi_2$$
$$-p_1\eta_1/m-p_2\eta_2/m-H\dot{\tau}+\dot{G}_N=0$$
$$\text{当} \quad f_1>0, \quad (63)$$

限制方程给出

$$-\xi_1+\xi_2=0 \qquad \text{当 } f_1=0. \quad (64)$$

方程(62)(63)有解

$$\tau=0,\xi_1=1,\xi_2=1,\eta_1=0,$$
$$\eta_2=0,G_N=mgt, \quad (65)$$

$$\tau=0,\xi_1=\xi_2=(p_1+p_2)/m,$$
$$\eta_1=\eta_2=-mg,$$
$$G_N=-(p_1+p_2)^2/2m+mg(q_1+q_2). \quad (66)$$

生成元(65)(66)满足限制方程(64),因此它们都对应系统的 Noether 对称性.与生成元(65)(66)相应的

Noether 守恒量为

$$I_N = p_1 + p_2 + mgt = \text{const}. \tag{67}$$

$$I_N = (p_1 + p_2)^2/2m + mg(q_1 + q_2) = \text{const}. \tag{68}$$

系统的 Lie 对称性确定方程(43)(45)给出

$$\frac{\bar{d}}{dt}\xi_1 - \frac{p_1}{m}\frac{\bar{d}}{dt}\tau = \frac{\eta_1}{m},$$

$$\frac{\bar{d}}{dt}\xi_2 - \frac{p_2}{m}\frac{\bar{d}}{dt}\tau = \frac{\eta_2}{m},$$

$$\frac{\bar{d}}{dt}\eta_1 + \frac{1}{2}mg\frac{\bar{d}}{dt}\tau = 0,$$

$$\frac{\bar{d}}{dt}\eta_2 + \frac{1}{2}mg\frac{\bar{d}}{dt}\tau = 0,$$

$$\text{当 } f_1 = 0, \tag{69}$$

$$\frac{\bar{d}}{dt}\eta_1 = 0, \frac{\bar{d}}{dt}\eta_2 + mg\frac{\bar{d}}{dt}\tau = 0;$$

$$\text{当 } f_1 > 0, \tag{70}$$

方程(69)(70)有解

$$\tau = 1, \xi_1 = \xi_2 = (p_1 + p_2)/m,$$

$$\eta_1 = \eta_2 = -mg. \tag{71}$$

生成元(71)满足限制方程(64),它对应系统的 Lie 对称性.将生成元(71)代入方程(47)(48),并利用方程(60)(61),有

$$\frac{\bar{d}}{dt}\ln G_H = 0, \tag{72}$$

取

$$\ln G_H = (p_1 + p_2)t - m(q_1 + q_2) + mgt^2/2, \tag{73}$$

将生成元(71)和函数 G_H 代入(49)式,得到系统的一个 Hojman 守恒量

$$I_H = -p_1 - p_2 - mgt = \text{const}. \tag{74}$$

做计算,有

$$X^{(0)}(H) = mg\xi_2 + p_1\eta_1/m + p_2\eta_2/m, \tag{75}$$

取生成元为

$$\tau = 0, \xi_1 = \xi_2 = t, \eta_1 = p_2, \eta_2 = -p_1, \tag{76}$$

则有

$$X^{(0)}(H) = mgt. \tag{77}$$

由判据 3,与生成元(76)相应的不变性是系统的 Mei 对称性.结构方程(51)(52)给出

$$G_M = -m(q_1 + q_2) + mgt^2/2 + 2p_1 t. \tag{78}$$

由命题 10,系统有如下形式的 Mei 守恒量

$$I_M = (p_1 + p_2)t - m(q_1 + q_2) + mgt^2/2 = \text{const}. \tag{79}$$

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Symmetries and conserved quantities of mechanical systems with unilateral holonomic constraints in phase space*

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Abstract

The symmetries and conserved quantities of mechanical systems with unilateral holonomic constraints in extended phase space is studied. The differential equations of motion of the systems are established. The criterions of Noether symmetry, Lie symmetry and Mei symmetry are given, and the relations between the symmetries are researched. The Noether conserved quantity and two types of new conserved quantities, called the Hojman quantity and Mei quantity, for the systems are obtained, and intrinsic relations between the three symmetries and three types of conserved quantities are researched. An example is given to illustrate the application of the results.

Keywords: analytical mechanics, unilateral constraint, symmetry, conserved quantity, phase space

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