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三层流体界面内波的二阶 Stokes 解*

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以小振幅波理论为基础,利用摄动方法研究了三层密度成层状态下的界面内波,求得了三层成层状态下各层速度势的二阶渐近解及界面内波波面位移的二阶 Stokes 解.结果表明:一阶解为正弦波解,与传统线性理论的结果相一致;二阶解描述了界面波的二阶非线性修正及两界面波之间的非线性相互作用;一阶解及二阶解都依赖于各层流体的厚度及密度.Umeyama 导出的理论结果为本文的特殊情形.

关键词:三层密度成层流体,内波,二阶 Stokes 解,小振幅波理论
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1.引 言

长期以来,人们对内波进行了广泛的研究.例如,Stokes[1] 提出了两层密度成层海洋内波理论,Lamb[2],Defant[3] 及 Benjamin[4] 分别研究了小振幅内波、内长波及内孤立波的特性.Davis 和 Acrivos[5] 在两层密度分层流体中进行了内孤立波的实验,将实验得到的波速及波剖面图与解析和数值结果作了对比.Tsuji 和 Nagata[6] 将 Stokes 展开技术应用于沿两层无限深流体界面传播的内波,得到了五阶解.Holyer[7] 利用傅里叶展开方法计算得到了沿两层无限深流体界面传播的大振幅内波的直到三十一阶的级数解.Koop 和 Buther[8] 利用实验方法研究了从浅水区域向深水区域传播的内波特性,他们发现,在浅水区域 KdV 理论与实验所测的内波波面位移数据符合得非常好,但是在深水区域 Benjamin-Ono 深水理论与实验结果偏差较大.Funakoshi 和 Okikawa[9] 讨论了沿位于刚性上边界和平底之间的两层非黏性不可混溶流体界面传播的大振幅内长波的二维无旋运动,他们检验了这些波在波长上限处的特点,并定义了对一个大振幅孤立波的临界条件.最近,Umeyama[10,11] 导出了沿两层密度不同的有限深度流体界面传播的内波的二阶和三阶 Stokes 解,且利用一个造波水槽进行了实验,并将实验结果与理论结果作了比较;Song[12] 得到了两层密度分层流体系统中内波的二阶随机波解.这些讨论都是针对两层密度成层水域内波.然而,真实海洋密度成层现象非常复杂,有时明显呈多层成层状态.这样,利用两层界面内波理论常常难以合理地描述海洋内部的波动规律.因此,开展多层密度成层流体界面内波研究是非常必要的.章守宇和杨红[13] 在线性情形下研究了三层密度成层水域内波.

多层密度成层内波运动是通过非线性波动方程来描述的,寻求其解在非线性问题中占有非常重要的地位.目前,已有许多求解非线性波动方程的方法,如摄动方法[10—12]、齐次平衡法[14—16]、Jacobi 椭圆函数展开法[17]、各种双曲函数方法[18] 及一般的投影 Riccati 方程方法[19] 等.本文以小振幅波理论为基础,利用摄动方法研究了三层密度成层状态下的界面内波,求得了三层密度成层状态下各层流体速度势的二阶解及界面内波波面位移的二阶 Stokes 解.

2.基本方程和边界条件

我们考虑水深 H 为一常数,密度呈三层成层的不可混溶流体.设流体为无黏性不可压缩,并忽略地球旋转的影响.如图 1 所示,取静止水面向右为 x 轴

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正方向,垂直向上为 z 轴正向.自水面而下,密度成层流体的厚度分别为 h_1,h_2 和 h_3,密度分别为 $\rho^{(1)}$,$\rho^{(2)}$ 和 $\rho^{(3)}$.当水面静止时,不同密度流体之间构成的各界面水深坐标分别为 z_0,z_1,z_2 和 z_3,$z_0=0$ 表示水面,z_3 表示底面.假定流体是无旋的,其速度由势函数 $\Phi^{(i)}(x,z,t)(i=1,2,3)$ 表示为

$$u^{(i)}=\frac{\partial\Phi^{(i)}}{\partial x},\quad w^{(i)}=\frac{\partial\Phi^{(i)}}{\partial z},\tag{1}$$

这里 $u^{(i)}$ 和 $w^{(i)}$ 分别是第 i 层流体水粒子在 x,z 方向的运动速度,$\eta^{(1)}$,$\eta^{(2)}$ 分别为界面 1、界面 2 处的波面位移,即分别为相对于静止时水面 $z=z_1$ 及 $z=z_2$ 算起的位移.

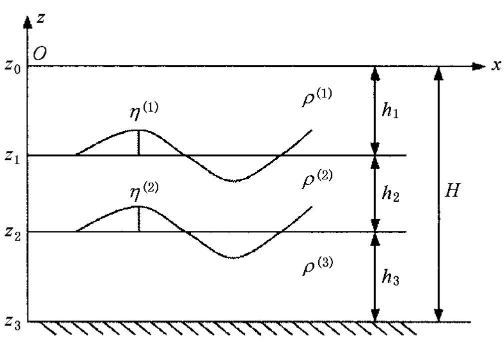

图 1 三层密度成层水域结构及内波示意图

对于第 $i(i=1,2,3)$ 层流体,有连续方程

$$\frac{\partial u^{(i)}}{\partial x}+\frac{\partial w^{(i)}}{\partial z}=0.\tag{2}$$

将(1)式代入(2)式,即得各层流体速度势满足的 Laplace 方程

$$\frac{\partial^2\Phi^{(1)}}{\partial x^2}+\frac{\partial^2\Phi^{(1)}}{\partial z^2}=0\quad(z_1+\eta^{(1)}\leqslant z\leqslant z_0),\tag{3}$$

$$\frac{\partial^2\Phi^{(2)}}{\partial x^2}+\frac{\partial^2\Phi^{(2)}}{\partial z^2}=0\quad(z_2+\eta^{(2)}\leqslant z\leqslant z_1+\eta^{(1)}),\tag{4}$$

$$\frac{\partial^2\Phi^{(3)}}{\partial x^2}+\frac{\partial^2\Phi^{(3)}}{\partial z^2}=0\quad(z_3\leqslant z\leqslant z_2+\eta^{(2)}).\tag{5}$$

上表面及下底面分别是刚性边界,即

$$\frac{\partial\Phi^{(1)}}{\partial z}=0\qquad(z=z_0=0),\tag{6}$$

$$\frac{\partial\Phi^{(3)}}{\partial z}=0\qquad(z=z_3).\tag{7}$$

各层流体界面上的运动学边界条件为

$$\frac{\partial\Phi^{(1)}}{\partial z}=\frac{\partial\eta^{(1)}}{\partial t}+\frac{\partial\eta^{(1)}}{\partial x}\frac{\partial\Phi^{(1)}}{\partial x}\qquad(z=z_1+\eta^{(1)}),\tag{8}$$

$$\frac{\partial\Phi^{(2)}}{\partial z}=\frac{\partial\eta^{(1)}}{\partial t}+\frac{\partial\eta^{(1)}}{\partial x}\frac{\partial\Phi^{(2)}}{\partial x}\qquad(z=z_1+\eta^{(1)}),\tag{9}$$

$$\frac{\partial\Phi^{(2)}}{\partial z}=\frac{\partial\eta^{(2)}}{\partial t}+\frac{\partial\eta^{(2)}}{\partial x}\frac{\partial\Phi^{(2)}}{\partial x}\qquad(z=z_2+\eta^{(2)}),\tag{10}$$

$$\frac{\partial\Phi^{(3)}}{\partial z}=\frac{\partial\eta^{(2)}}{\partial t}+\frac{\partial\eta^{(2)}}{\partial x}\frac{\partial\Phi^{(3)}}{\partial x}\qquad(z=z_2+\eta^{(2)}),\tag{11}$$

动力学边界条件为

$$\rho^{(1)}\left\{g\eta^{(1)}+\frac{\partial\Phi^{(1)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(1)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(1)}}{\partial z}\right)^2\right]\right\}=\rho^{(2)}\left\{g\eta^{(1)}+\frac{\partial\Phi^{(2)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(2)}}{\partial z}\right)^2\right]\right\}\quad(z=z_1+\eta^{(1)}),\tag{12}$$

$$\rho^{(2)}\left\{g\eta^{(2)}+\frac{\partial\Phi^{(2)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(2)}}{\partial z}\right)^2\right]\right\}=\rho^{(3)}\left\{g\eta^{(2)}+\frac{\partial\Phi^{(3)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(3)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(3)}}{\partial z}\right)^2\right]\right\}\quad(z=z_2+\eta^{(2)}).\tag{13}$$

这里 g 为重力加速度.

3.界面内波二阶 Stokes 解

类似于 Umeyama[10,11] 和 Song[12] 对两层流体系统中内波的研究,利用摄动方法来求解上述基本方程和边界条件(3)—(13)式.设 ε 为一个小参数,将 $\Phi^{(i)}(i=1,2,3)$,$\eta^{(i)}(i=1,2)$ 按小参数 ε 展开为

$$\Phi^{(1)}=\varepsilon\Phi_{\mathrm{I}}^{(1)}+\varepsilon^2\Phi_{\mathrm{II}}^{(1)}+O(\varepsilon^3),\tag{14}$$

$$\Phi^{(2)}=\varepsilon\Phi_{\mathrm{I}}^{(2)}+\varepsilon^2\Phi_{\mathrm{II}}^{(2)}+O(\varepsilon^3),\tag{15}$$

$$\Phi^{(3)}=\varepsilon\Phi_{\mathrm{I}}^{(3)}+\varepsilon^2\Phi_{\mathrm{II}}^{(3)}+O(\varepsilon^3),\tag{16}$$

$$\eta^{(1)}=\varepsilon\eta_{\mathrm{I}}^{(1)}+\varepsilon^2\eta_{\mathrm{II}}^{(1)}+O(\varepsilon^3),\tag{17}$$

$$\eta^{(2)}=\varepsilon\eta_{\mathrm{I}}^{(2)}+\varepsilon^2\eta_{\mathrm{II}}^{(2)}+O(\varepsilon^3),\tag{18}$$

这里,O 是阶符号,下标 Ⅰ 和 Ⅱ 分别表示一阶和二阶近似.在推导基本方程组时,为方便处理,将界面 $z=z_i+\eta^{(i)}(i=1,2)$ 处的边界条件分别替代为 $z=z_i(i=1,2)$ 处的边界条件.为此,将 $\Phi^{(i)}(i=,1,2,3)$ 关于 z 在 $z=z_i(i=1,2)$ 分别展开为 Taylor 级数,连同(14)—(18)式一起代入(3)—(13)式,比较 ε 同次幂的系数,即可得到 $\Phi_{\mathrm{I}}^{(i)}$,$\eta_{\mathrm{I}}^{(i)}$,$\Phi_{\mathrm{II}}^{(i)}$,$\eta_{\mathrm{II}}^{(i)}(i=1,2,3)$ 满足的控制方程及边界条件.

一阶方程及边界条件为

$$\frac{\partial^2\Phi_{\mathrm{I}}^{(1)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{I}}^{(1)}}{\partial z^2}=0 \quad (z_1+\eta^{(1)}\leqslant z\leqslant z_0), \tag{19}$$

$$\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial z^2}=0 \quad (z_2+\eta^{(2)}\leqslant z\leqslant z_1+\eta^{(1)}), \tag{20}$$

$$\frac{\partial^2\Phi_{\mathrm{I}}^{(3)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{I}}^{(3)}}{\partial z^2}=0 \quad (z_3\leqslant z\leqslant z_2+\eta^{(2)}), \tag{21}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial z}=0 \quad (z=z_0=0), \tag{22}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial z}=\frac{\partial\eta_{\mathrm{I}}^{(1)}}{\partial t} \quad (z=z_1), \tag{23}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial z}=\frac{\partial\eta_{\mathrm{I}}^{(1)}}{\partial t} \quad (z=z_1), \tag{24}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial z}=\frac{\partial\eta_{\mathrm{I}}^{(2)}}{\partial t} \quad (z=z_2), \tag{25}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial z}=\frac{\partial\eta_{\mathrm{I}}^{(2)}}{\partial t} \quad (z=z_2), \tag{26}$$

$$\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial z}=0 \quad (z=z_3), \tag{27}$$

$$\rho^{(1)}\left\{g\eta_{\mathrm{I}}^{(1)}+\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial t}\right\}=\rho^{(2)}\left\{g\eta_{\mathrm{I}}^{(1)}+\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial t}\right\} \quad (z=z_1), \tag{28}$$

$$\rho^{(2)}\left\{g\eta_{\mathrm{I}}^{(2)}+\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial t}\right\}=\rho^{(3)}\left\{g\eta_{\mathrm{I}}^{(2)}+\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial t}\right\} \quad (z=z_2). \tag{29}$$

二阶方程及边界条件为

$$\frac{\partial^2\Phi_{\mathrm{II}}^{(1)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{II}}^{(1)}}{\partial z^2}=0 \quad (z_1+\eta^{(1)}\leqslant z\leqslant z_0), \tag{30}$$

$$\frac{\partial^2\Phi_{\mathrm{II}}^{(2)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{II}}^{(2)}}{\partial z^2}=0 \quad (z_2+\eta^{(2)}\leqslant z\leqslant z_1+\eta^{(1)}), \tag{31}$$

$$\frac{\partial^2\Phi_{\mathrm{II}}^{(3)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{II}}^{(3)}}{\partial z^2}=0 \quad (z_3\leqslant z\leqslant z_2+\eta^{(2)}), \tag{32}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(1)}}{\partial z}=0 \quad (z=z_0=0), \tag{33}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(1)}}{\partial z}+\frac{\partial^2\Phi_{\mathrm{I}}^{(1)}}{\partial z^2}\eta_{\mathrm{I}}^{(1)}=\frac{\partial\eta_{\mathrm{II}}^{(1)}}{\partial t}+\frac{\partial\eta_{\mathrm{I}}^{(1)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial x} \quad (z=z_1), \tag{34}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(2)}}{\partial z}+\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial z^2}\eta_{\mathrm{I}}^{(1)}=\frac{\partial\eta_{\mathrm{II}}^{(1)}}{\partial t}+\frac{\partial\eta_{\mathrm{I}}^{(1)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial x} \quad (z=z_1), \tag{35}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(2)}}{\partial z}+\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial z^2}\eta_{\mathrm{I}}^{(2)}=\frac{\partial\eta_{\mathrm{II}}^{(2)}}{\partial t}+\frac{\partial\eta_{\mathrm{I}}^{(2)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial x} \quad (z=z_2), \tag{36}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(3)}}{\partial z}+\frac{\partial^2\Phi_{\mathrm{I}}^{(3)}}{\partial z^2}\eta_{\mathrm{I}}^{(2)}=\frac{\partial\eta_{\mathrm{II}}^{(2)}}{\partial t}+\frac{\partial\eta_{\mathrm{I}}^{(2)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial x} \quad (z=z_2), \tag{37}$$

$$\frac{\partial\Phi_{\mathrm{II}}^{(3)}}{\partial z}=0 \quad (z=z_3), \tag{38}$$

$$\begin{aligned}&\rho^{(1)}\left\{g\eta_{\mathrm{II}}^{(1)}+\frac{\partial\Phi_{\mathrm{II}}^{(1)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(1)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(1)}+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial z}\right)^2\right]\right\}\\&=\rho^{(2)}\left\{g\eta_{\mathrm{II}}^{(1)}+\frac{\partial\Phi_{\mathrm{II}}^{(2)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(1)}+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial z}\right)^2\right]\right\} \quad (z=z_1),\end{aligned} \tag{39}$$

$$\begin{aligned}&\rho^{(2)}\left\{g\eta_{\mathrm{II}}^{(2)}+\frac{\partial\Phi_{\mathrm{II}}^{(2)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(2)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(2)}+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(2)}}{\partial z}\right)^2\right]\right\}\\&=\rho^{(3)}\left\{g\eta_{\mathrm{II}}^{(2)}+\frac{\partial\Phi_{\mathrm{II}}^{(3)}}{\partial t}+\frac{\partial^2\Phi_{\mathrm{I}}^{(3)}}{\partial t\partial z}\eta_{\mathrm{I}}^{(2)}+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial z}\right)^2\right]\right\} \quad (z=z_2),\end{aligned} \tag{40}$$

3.1. 一阶方程的解

设一阶方程(19)—(29)的解形式为

$$\eta_{\mathrm{I}}^{(1)}=b^{(1)}\cos(kx-\omega t), \tag{41}$$

$$\eta_{\mathrm{I}}^{(2)}=b^{(2)}\cos(kx-\omega t),\tag{42}$$

$$\Phi_{\mathrm{I}}^{(1)}=A^{(1)}(z)\sin(kx-\omega t),\tag{43}$$

$$\Phi_{\mathrm{I}}^{(2)}=A^{(2)}(z)\sin(kx-\omega t),\tag{44}$$

$$\Phi_{\mathrm{I}}^{(3)}=A^{(3)}(z)\sin(kx-\omega t),\tag{45}$$

这里，$b^{(1)}$，$b^{(2)}$分别表示两个界面内波的振幅，ω 为角频率，k 为波数.

将(41)—(45)式代入(19)—(27)式，可得

$$\begin{aligned}
&\frac{\mathrm{d}^2A^{(1)}}{\mathrm{d}z^2}-k^2A^{(1)}=0 && (z_1+\eta^{(1)}\leqslant z\leqslant z_0=0),\\
&\frac{\mathrm{d}A^{(1)}}{\mathrm{d}z}=0 && (z=z_0=0),\\
&\frac{\mathrm{d}A^{(1)}}{\mathrm{d}z}=b^{(1)}\omega && (z=z_1);
\end{aligned}\tag{46}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2A^{(2)}}{\mathrm{d}z^2}-k^2A^{(2)}=0 && (z_2+\eta^{(2)}\leqslant z\leqslant z_1+\eta^{(1)}),\\
&\frac{\mathrm{d}A^{(2)}}{\mathrm{d}z}=b^{(1)}\omega && (z=z_1),\\
&\frac{\mathrm{d}A^{(2)}}{\mathrm{d}z}=b^{(2)}\omega && (z=z_2);
\end{aligned}\tag{47}$$

$$\begin{aligned}
&\frac{\mathrm{d}^2A^{(3)}}{\mathrm{d}z^2}-k^2A^{(3)}=0 && (z_3\leqslant z\leqslant z_2+\eta^{(2)}),\\
&\frac{\mathrm{d}A^{(3)}}{\mathrm{d}z}=b^{(2)}\omega && (z=z_2),\\
&\frac{\mathrm{d}A^{(3)}}{\mathrm{d}z}=0 && (z=z_3).
\end{aligned}\tag{48}$$

由方程(46)—(48)解得 $A^{(i)}$ $(i=1,2,3)$，并代入(43)—(45)式，即得

$$\Phi_{\mathrm{I}}^{(1)}=-\frac{\omega}{k}\frac{b^{(1)}\cosh kz}{\sinh kh_1}\sin(kx-\omega t)\quad(z_1+\eta^{(1)}\leqslant z\leqslant z_0=0),\tag{49}$$

$$\Phi_{\mathrm{I}}^{(2)}=\frac{\omega}{k}\frac{b^{(1)}\cosh k(z-z_2)-b^{(2)}\cosh k(z-z_1)}{\sinh kh_2}\times\sin(kx-\omega t)\quad(z_2+\eta^{(2)}\leqslant z\leqslant z_1+\eta^{(1)}),\tag{50}$$

$$\Phi_{\mathrm{I}}^{(3)}=\frac{\omega}{k}\frac{b^{(2)}\cosh k(z-z_3)}{\sinh kh_3}\sin(kx-\omega t)\quad(z_3\leqslant z\leqslant z_2+\eta^{(2)}).\tag{51}$$

此结果与通常的线性结果(见文献[13])是完全一致的.

将(49)—(51)式代入边界条件(28)和(29)式，化简可得

$$\begin{aligned}
&\left[g(\rho^{(1)}-\rho^{(2)})+\frac{\omega^2}{k}(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2)\right]b^{(1)}-\frac{\rho^{(2)}\omega^2}{k\sinh kh_2}b^{(2)}=0,\\
&-\frac{\rho^{(2)}\omega^2}{k\sinh kh_2}b^{(1)}+\left[g(\rho^{(2)}-\rho^{(3)})+\frac{\omega^2}{k}(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)\right]b^{(2)}=0.
\end{aligned}\tag{52}$$

上述方程组有解的必要条件为

$$\begin{vmatrix}
g(\rho^{(1)}-\rho^{(2)})+\frac{\omega^2}{k}(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2) & -\frac{\rho^{(2)}\omega^2}{k\sinh kh_2}\\
-\frac{\rho^{(2)}\omega^2}{k\sinh kh_2} & g(\rho^{(2)}-\rho^{(3)})+\frac{\omega^2}{k}(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)
\end{vmatrix}=0,$$

即

$$\begin{aligned}
&\frac{\omega^4}{k^2}\left[(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2)(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)-\frac{\rho^{(2)2}}{(\sinh kh_2)^2}\right]\\
&+\frac{\omega^2}{k}[g(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)+g(\rho^{(2)}-\rho^{(3)})(\rho^{(1)}\coth kh_2+\rho^{(2)}\coth kh_3)]\\
&+g^2(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}-\rho^{(3)})=0.
\end{aligned}\tag{53}$$

记

$$\begin{aligned}
a(k)&=(\rho^{(1)}\coth kh_1+\rho^{(2)}\coth kh_2)(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)-\frac{\rho^{(2)2}}{(\sinh kh_2)^2},\\
b(k)&=g(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}\coth kh_2+\rho^{(3)}\coth kh_3)+g(\rho^{(2)}-\rho^{(3)})(\rho^{(1)}\coth kh_2+\rho^{(2)}\coth kh_3),\\
c&=g^2(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}-\rho^{(3)}).
\end{aligned}$$

于是得

$$\omega^2=\frac{-b(k)\pm\sqrt{b^2(k)-4a(k)c}}{2a(k)}k.\tag{54}$$

上述两组频散关系表明三层密度成层流体界面内波有两个运动模态，它们分别对应于两个界面波的传播方式.

另外由(52)式可得

$$b^{(2)} = \alpha_1 b^{(1)}, \tag{55}$$

式中

$$\alpha_1 = [gk(\rho^{(1)} - \rho^{(2)}) + \omega^2(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)]\sinh kh_2 \times (\rho^{(2)}\omega^2)^{-1}. \tag{56}$$

显然,频散关系(54)式依赖于各层流体的厚度及密度.此外,由(55)式可知,若在界面 1 处有一小扰动,则在界面 2 处将会诱导相应的与界面 1 处小扰动具有相同波数和频率的运动,且两个内波波面位移振幅按(55)式相互调制.

3.2. 二阶解

为求二阶解,我们先来求二阶方程的解.由于在二阶方程的边界条件中含有一阶解,所以先将一阶解代入二阶方程的边界条件中进行化简.将(41),(42)和(49)—(51)式代入(34)—(37),(39)及(40)式中,可得

$$\frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(1)}}{\partial t} + k\omega b^{(1)2}\coth kh_1 \sin 2(kx - \omega t) \quad (z = z_1), \tag{57}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(2)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(1)}}{\partial t} - k\omega b^{(1)}\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2}\sin 2(kx - \omega t) \quad (z = z_1), \tag{58}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(2)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(2)}}{\partial t} - k\omega b^{(2)}\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2}\sin 2(kx - \omega t) \quad (z = z_2), \tag{59}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(3)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(2)}}{\partial t} - k\omega b^{(2)2}\coth kh_3 \sin 2(kx - \omega t) \quad (z = z_2), \tag{60}$$

$$\begin{aligned}
&\rho^{(1)}\left\{g\eta_{\mathrm{II}}^{(1)} + \frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial t} + \frac{1}{2}\omega^2 b^{(1)2}[(\coth kh_1)^2 - 2]\cos^2(kx - \omega t) + \frac{1}{2}\omega^2 b^{(1)2}\sin^2(kx - \omega t)\right\} \\
&= \rho^{(2)}\left\{g\eta_{\mathrm{II}}^{(1)} + \frac{\partial \Phi_{\mathrm{II}}^{(2)}}{\partial t} + \frac{1}{2}\omega^2 b^{(1)2}\sin^2(kx - \omega t)\right. \\
&\left. + \frac{1}{2}\omega^2\left[\left(\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2}\right)^2 - 2b^{(1)2}\right]\cos^2(kx - \omega t)\right\} \quad (z = z_1),
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\rho^{(2)}\left\{g\eta_{\mathrm{II}}^{(2)} + \frac{\partial \Phi_{\mathrm{II}}^{(2)}}{\partial t} + \frac{1}{2}\omega^2\left[\left(\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2}\right)^2 - 2b^{(2)2}\right]\cos^2(kx - \omega t) + \frac{1}{2}\omega^2 b^{(2)2}\sin^2(kx - \omega t)\right\} \\
&= \rho^{(3)}\left\{g\eta_{\mathrm{II}}^{(2)} + \frac{\partial \Phi_{\mathrm{II}}^{(3)}}{\partial t} + \frac{1}{2}\omega^2 b^{(2)2}\sin^2(kx - \omega t)\right. \\
&\left. + \frac{1}{2}\omega^2 b^{(2)2}[(\coth kh_3)^2 - 2]\cos^2(kx - \omega t)\right\} \quad (z = z_2).
\end{aligned} \tag{62}$$

利用类似于求解一阶方程的方法可得二阶方程的解为

$$\Phi_{\mathrm{II}}^{(1)} = -\left[\frac{\omega}{k}d^{(1)} + \frac{1}{2}\omega b^{(1)2}\coth kh_1\right]\frac{\cosh 2kz}{\sinh 2kh_1}\sin 2(kx - \omega t) \quad (z_1 + \eta^{(1)} \leqslant z \leqslant z_0 = 0), \tag{63}$$

$$\begin{aligned}
\Phi_{\mathrm{II}}^{(2)} = &\left\{\frac{\left[\frac{\omega}{k}d^{(1)} - \frac{1}{2}\omega b^{(1)}\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2}\right]\cosh 2k(z - z_2)}{\sinh 2kh_2}\right. \\
&\left. - \frac{\left[\frac{\omega}{k}d^{(2)} - \frac{1}{2}\omega b^{(2)}\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2}\right]\cosh 2k(z - z_1)}{\sinh 2kh_2}\right\}\sin 2(kx - \omega t) \\
&(z_2 + \eta^{(2)} \leqslant z \leqslant z_1 + \eta^{(1)}),
\end{aligned} \tag{64}$$

$$\Phi_{\mathrm{II}}^{(3)} = \left[\frac{\omega}{k}d^{(2)} - \frac{1}{2}\omega b^{(2)2}\coth kh_3\right]\frac{\cosh 2k(z - z_3)}{\sinh 2kh_3}\sin 2(kx - \omega t) \quad (z_3 \leqslant z \leqslant z_2 + \eta^{(2)}), \tag{65}$$

$$\eta_{\mathrm{II}}^{(1)} = d^{(1)}\cos 2(kx - \omega t), \tag{66}$$

$$\eta_{\mathrm{II}}^{(2)} = d^{(2)}\cos 2(kx - \omega t), \tag{67}$$

式中

$$d^{(1)} = \frac{l_1 a_{22} - l_2 a_{12}}{a_{11} a_{22} - a_{12} a_{21}},$$
$$d^{(2)} = \frac{l_2 a_{11} - l_1 a_{21}}{a_{11} a_{22} - a_{12} a_{21}}, \tag{68}$$

$$a_{11} = g(\rho^{(1)} - \rho^{(2)}) + \frac{2\omega^2}{k}(\rho^{(1)}\coth 2kh_1 + \rho^{(2)}\coth 2kh_2), \tag{69}$$

$$a_{12} = a_{21} = -\frac{2\rho^{(2)}\omega^2}{k\sinh 2kh_2}, \tag{70}$$

$$a_{22} = g(\rho^{(2)} - \rho^{(3)}) + \frac{2\omega^2}{k}(\rho^{(2)}\coth 2kh_2 + \rho^{(3)}\coth 2kh_3), \tag{71}$$

$$\begin{aligned} l_1 = & -\rho^{(1)}\omega^2 b^{(1)2}\Big[\frac{1}{4}(\coth kh_1)^2 + \coth kh_1 \cdot \coth 2kh_1 - \frac{3}{4}\Big] \\ & + \rho^{(2)}\omega^2\Big\{ b^{(1)}\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2}\coth 2kh_2 \\ & - b^{(2)}\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2 \cdot \sinh 2kh_2} \\ & + \frac{1}{4}\Big(\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2}\Big)^2 - \frac{3}{4}b^{(1)2}\Big\}, \end{aligned} \tag{72}$$

$$\begin{aligned} l_2 = & -\rho^{(2)}\omega^2\Big\{ b^{(1)}\frac{b^{(1)}\cosh kh_2 - b^{(2)}}{\sinh kh_2 \cdot \sinh 2kh_2} \\ & - b^{(2)}\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2}\coth 2kh_2 \\ & + \frac{1}{4}\Big(\frac{b^{(1)} - b^{(2)}\cosh kh_2}{\sinh kh_2}\Big)^2 - \frac{3}{4}b^{(2)2}\Big\} \\ & + \rho^{(3)}\omega^2 b^{(2)2}\Big[\frac{1}{4}(\coth kh_3)^2 + \coth kh_3 \cdot \coth 2kh_3 - \frac{3}{4}\Big]. \end{aligned} \tag{73}$$

这样,我们可以得到二阶解为

$$\Phi^{(i)} = \Phi_{\mathrm{I}}^{(i)} + \Phi_{\mathrm{II}}^{(i)} \qquad (i = 1,2,3), \tag{74}$$

$$\eta^{(i)} = \eta_{\mathrm{I}}^{(i)} + \eta_{\mathrm{II}}^{(i)} \qquad (i = 1,2). \tag{75}$$

由(68)—(73)式可以看出,$d^{(1)}$ 和 $d^{(2)}$ 依赖于各层流体的厚度及密度.若已知 $b^{(1)}$ 和 $b^{(2)}$,则 $d^{(1)}$ 和 $d^{(2)}$ 完全可由(68)式解出,且在解的表达式中含有 $b^{(1)2}$,$b^{(2)2}$ 和 $b^{(1)}b^{(2)}$,因而二阶方程的解描述了界面波二阶非线性修正及两界面波的非线性相互作用.这说明二阶解由一阶解和二阶非线性修正项及两界面波的二阶非线性相互作用所确定,并且依赖于各层流体的厚度及密度.

4.讨 论

若已知内波波动周期及初始振幅 $b^{(1)}$,由频散关系(53)及(55)式可以求得波数 k 及 $b^{(2)}$,再利用(68)式可解出 $d^{(1)}$ 或 $d^{(2)}$,于是就得到了内波波面位移的一阶解及二阶解.一阶解 $\eta_{\mathrm{I}}^{(1)}$ 及 $\eta_{\mathrm{I}}^{(2)}$ 就是通常的线性解,而二阶方程的解 $\eta_{\mathrm{II}}^{(1)}$ 及 $\eta_{\mathrm{II}}^{(2)}$ 反映了二阶非线性修正及非线性相互作用.这具体体现在:一阶解波面形状是正弦波,而由于这种非线性修正及非线性相互作用,使得二阶解的波形虽然仍呈周期性变化,但已不再是正弦波,且与正弦波相比,其波峰变得较陡,波谷变得较平坦.此外,一阶解与二阶解都依赖于各层流体的密度及厚度.

若取 $\rho^{(1)} \ll \rho^{(2)}$,$\rho^{(1)} \ll \rho^{(3)}$ 及 $h_1 \to \infty$,此时上层可看成是大气,$z = z_1$ 可看成是自由表面,问题简化为文献[10]讨论的情形.图2为两层密度成层流体内波示意图,经坐标平移并对前面所得一阶解及二阶方程的解化简可得一阶方程的解为

$$\Phi_{\mathrm{I}}^{(2)} = \frac{\omega}{k}\frac{b^{(1)}\cosh kz - b^{(2)}\cosh k(z - k_2)}{\sinh kh_2} \times \sin(kx - \omega t), \tag{76}$$

$$\Phi_{\mathrm{I}}^{(3)} = \frac{\omega}{k}\frac{b^{(2)}\cosh k(z + h_3)}{\sinh kh_3}\sin(kx - \omega t), \tag{77}$$

$$\eta_{\mathrm{I}}^{(1)} = b^{(1)}\cos(kx - \omega t), \tag{78}$$

$$\eta_{\mathrm{I}}^{(2)} = b^{(2)}\cos(kx - \omega t), \tag{79}$$

即

$$\Phi_{\mathrm{I}}^{(2)} = (A^{(1)}\cosh kz + B^{(1)}\sinh kz)\sin(kx - \omega t), \tag{80}$$

$$\Phi_{\mathrm{I}}^{(3)} = D^{(1)}\cosh k(z + h_3)\sin(kx - \omega t). \tag{81}$$

这里

$$\begin{aligned} A^{(1)} &= \frac{1}{\omega}\Big(\frac{\rho^{(3)}}{\rho^{(2)}}\frac{\omega^2}{k}\coth kh_3 - \frac{\rho^{(3)} - \rho^{(2)}}{\rho^{(2)}}g\Big)b^{(2)}, \\ B^{(1)} &= \frac{\omega}{k}b^{(2)}, \\ D^{(1)} &= \frac{\omega}{k\sinh kh_3}b^{(2)}, \\ b^{(1)} &= \Big[\frac{1}{g}\Big(\frac{\rho^{(3)}}{\rho^{(2)}}\frac{\omega^2}{k}\coth kh_3 - \frac{\rho^{(3)} - \rho^{(2)}}{\rho^{(2)}}g\Big) \times \coth kh_2 + \frac{\omega^2}{gk}\sinh kh_2\Big]b^{(2)}. \end{aligned} \tag{82}$$

频散关系(53)式经化简后,可得

$$\omega^4\Big(\coth kh_2 \coth kh_3 + \frac{\rho^{(2)}}{\rho^{(3)}}\Big)$$

$$- \omega^2 (\coth kh_2 + \coth kh_3) gk + \frac{\rho^{(3)} - \rho^{(2)}}{\rho^{(3)}} g^2 k^2 = 0. \quad (83)$$

将(78)—(83)式与文献[10]中(30)—(40)式对比，不难看出，(78)—(83)式即为文献[10]中一阶解的结果.

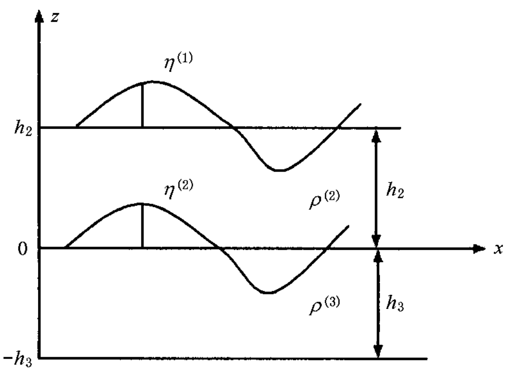

图 2　两层密度成层流体内波示意图

同理，可得二阶方程的解为

$$\Phi_{\mathrm{II}}^{(2)} = (A^{(2)} \cosh 2kz + B^{(2)} \sinh 2kz) \sin(kx - \omega t), \quad (84)$$

$$\Phi_{\mathrm{II}}^{(3)} = D^{(2)} \cosh 2k(z + h_3) \sin(kx - \omega t), \quad (85)$$

$$\eta_{\mathrm{II}}^{(1)} = d^{(1)} \cos 2(kx - \omega t), \quad (86)$$

$$\eta_{\mathrm{II}}^{(2)} = d^{(2)} \cos 2(kx - \omega t), \quad (87)$$

式中

$$A^{(2)} = -\frac{1}{2\omega}(\alpha_1 d^{(2)} + \alpha_2 b^{(2)2}),$$

$$B^{(2)} = \beta_1 d^{(2)} - \frac{1}{2} k\beta_2 b^{(2)2}, \quad (88)$$

$$D^{(2)} = \frac{\omega d^{(2)} - \frac{1}{2} k^2 \alpha_3 b^{(2)2} \cosh kh_3}{k \sinh 2kh_3}.$$

这里

$$\alpha_1 = \frac{\rho^{(3)} - \rho^{(2)}}{\rho^{(2)}} g - \frac{\rho^{(3)}}{\rho^{(2)}} \frac{2\omega^2 \cosh 2kh_3}{k \sinh 2kh_3}, \quad (89)$$

$$\alpha_2 = \frac{\rho^{(3)}}{\rho^{(2)}} \Big[k\omega \alpha_3 \cosh kh_3 \coth 2kh_3 - \frac{1}{2}\omega^2 \Big(1 - \frac{1}{2\sinh^2 kh_3}\Big)\Big] + \frac{1}{4}(3\omega^2 - k^2 \alpha_4^2), \quad (90)$$

$$\alpha_3 = \frac{\omega}{k \sinh kh_3}, \quad (91)$$

$$\alpha_4 = \frac{1}{\omega}\Big(\frac{\rho^{(3)}}{\rho^{(2)}} \frac{\omega^2}{k} \coth kh_3 - \frac{\rho^{(3)} - \rho^{(2)}}{\rho^{(2)}} g\Big). \quad (92)$$

这里 $d^{(1)}$，$d^{(2)}$ 仍由(68)—(73)式给出，只要在其中取 $\rho^{(1)} = 0$ 即可. 将(84)—(92)式与文献[10]中的(44)—(52)式对比可知，此即为文献[10]中二阶解的结果.

以上讨论表明，文献[10]导出的理论结果为本文的特殊情形.

5. 结　　论

以小振幅波理论为基础，利用摄动方法研究了三层密度成层状态下的界面内波，求得了三层密度成层状态下各层流体速度势的二阶解及界面内波波面位移的二阶 Stokes 解. 结果表明：一阶解为正弦波解，与传统线性理论的结果相一致，且两界面波振幅相互调制，反映了一阶解之间的线性作用关系. 二阶方程的解反映了二阶非线性修正及两界面波之间的非线性相互作用. 从而二阶解由一阶解和二阶非线性修正项及两界面波之间的非线性相互作用所确定，描述了各界面波间二阶非线性相互作用. 经过这种非线性相互作用，使得界面波形发生变化，与一阶解的正弦波相比，其波形仍呈周期性变化，但波峰变得较陡，波谷变得较平坦. 一阶解与二阶解都依赖于各层流体的密度及厚度. 此外，注意到由文献[10]导出的上边界为自由表面的两层流体系统表面波及内波的一阶与二阶 Stokes 解是本研究的特例.

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Second-order Stokes solutions for internal waves in three-layer density-stratified fluid*

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Abstract

In this paper, internal waves in three-layer stratified fluid are investigated by using a perturbation method, and the second-order asymptotic solutions of the velocity potentials and the second-order Stokes solutions of the associated elevations of the interfacial waves are presented based on the small amplitude wave theory. As expected, the first-order solutions are consistent with ordinary linear theoretical results, and the second-order solutions describe the second-order modification on the linear theory and the interactions between the two interfacial waves. Both the first-order and second-order solutions derived depend on the depths and densities of the three-layer fluid. It is also noted that the solutions obtained from the present work include the theoretical results derived by Umeyama as special cases.

Keywords: three-layer density-stratified fluid, internal wave, second-order Stokes solutions, small amplitude wave theory

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