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非线性非完整系统 Raitzin 正则方程的 Hojman 守恒定理*

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利用时间不变的无限小变换下的 Lie 对称性,研究非线性非完整系统 Raitzin 正则方程的 Hojman 守恒定理.列出系统的运动微分方程.建立时间不变的无限小变换下的确定方程.给出系统的 Hojman 守恒定理,并举例说明结果的应用.

关键词:非线性非完整系统,Raitzin 正则方程,Lie 对称性,确定方程,Hojman 守恒定理

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1. 引 言

力学系统的守恒量,不仅具有数学重要性,而且表现为深刻的物理规律,它已成为近代分析力学的一个重要研究方向.寻求守恒量的主要方法有 Noether 理论[1],Lie 对称性[2],积分因子理论[3],Hojman 方法[4]和梅氏方法[5].近年来对这类方法的研究已得到重要进展[6—23].

1961 年 Raitzin 提出了新型正则变量 r,s,建立了一类新运动微分方程,被称为 Raitzin 正则方程[24].经实际运算得知,有些动力学问题应用该方程求解要比用 Hamiliton 正则方程简单些.因此,在分析力学中,Raitzin 正则方程具有重要的理论意义和应用价值.目前,已取得一些研究成果[22—27].

本文利用 Hojman 方法来寻求非线性非完整系统 Raitzin 正则方程的非 Noether 守恒量.

2. 非线性非完整系统的 Raitzin 正则方程

研究 N 个质点组成的力学系统,它的位形由 n 个广义坐标 $q_1,q_2,\cdots,q_n$ 确定.系统的运动受有 g 个理想双面 Chetaev 型非完整约束

$$f_\beta(t,q,\dot q)=0,(\beta=1,2,\cdots,g). \tag{1}$$

非完整约束(1)加在虚位移 δq_j 上的限制为 Appell-Chetaev 条件

$$\frac{\partial f_\beta}{\partial \dot q_j}\delta q_j=0, \tag{2}$$

系统的运动方程可表为 Routh 形式[28]

$$\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial \dot q_j}-\frac{\partial L}{\partial q_j}=Q_j+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_j},\quad (j=1,2,\cdots,n), \tag{3}$$

其中 L 为系统的 Lagrange 函数;Q_j 为非势广义力;λ_β 为约束乘子.

在运动微分方程积分之前,可由方程(1),(3)求出 λ_β 为 $t,q,\dot q$ 的函数.这样,方程(3)可写为

$$\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial \dot q_j}-\frac{\partial L}{\partial q_j}=Q_j+\Lambda_j,\quad (j=1,2,\cdots,n), \tag{4}$$

其中

$$\Lambda_j=\Lambda_j(t,q,\dot q)=\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_j}. \tag{5}$$

它是非完整约束力,已表为 $t,q,\dot q$ 的函数.方程(4)称为对应于非完整系统(1),(3)的完整系统的运动方程.只要对初始条件施加非完整约束(1)的限制,

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则非完整系统(1),(3)的积分可通过相应完整系统(4)求得[28].

2.1. 相应完整系统的 Raitzin 正则方程

1961 年 Raitzin 提出新型正则变量[24]

$$r_j = \frac{\partial L}{\partial q_j}, \qquad s_j = \dot{q}_j, \tag{6}$$

并引入函数

$$R(t,r,s) = L - r_j q_j, (j = 1,2,\cdots,n), \tag{7}$$

此处 $L = L(t,q,\dot{q})$ 为 Lagrange 函数. 于是有

$$\begin{aligned} &\frac{\partial R}{\partial s_j} = \frac{\partial L}{\partial \dot{q}_j}, \\ &\frac{\partial R}{\partial r_j} = - q_j, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} = \dot{p}_j, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j} = - \dot{q}_j = - s_j. \end{aligned} \tag{8}$$

将(8)式代入方程(4),得到相应完整系统的 Raitzin 正则方程

$$\begin{aligned} s_j &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \\ r_j &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \widetilde{Q}_j + \widetilde{\Lambda}_j, \end{aligned} \tag{9}$$

其中记号 ~ 表示 $q,\dot{q}$ 用 t,r,s 代换所得的表达式.

假设系统是非奇异,则由方程(9)可解出所有的 $\dot{s}_j$ 和 $\dot{r}_j$,记作

$$\begin{aligned} \dot{s}_j &= g_j(t,r,s), \\ \dot{r}_j &= h_j(t,r,s), (j = 1,2,\cdots,n). \end{aligned} \tag{10}$$

2.2. 非完整系统的 Raitzin 正则方程

利用 Raitzin 变量(6),非完整约束(1)可写为

$$\tilde{f}_\beta(t,r,s) = 0, (\beta = 1,2,\cdots,g). \tag{11}$$

再将(8)式代入方程(3),则由方程(1),(3)表示的非完整系统,其 Raitzin 正则方程为

$$\begin{aligned} s_j &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j}, \\ r_j &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_j} - \widetilde{Q}_j - \lambda_\beta \frac{\partial \tilde{f}_\beta}{\partial s_j}. \end{aligned} \tag{12}$$

3. 无限小变换与相应完整系统的确定方程

引入时间不变的特殊无限小变换

$$\begin{aligned} &t^* = t, s_j^*(t^*) = s_j(t) + \Delta s_j, \\ &r_j^*(t^*) = r_j(t) + \Delta r_j. \end{aligned} \tag{13}$$

它们的展开形式为

$$\begin{aligned} &t^* = t, \\ &s_j^*(t^*) = s_j(t) + \varepsilon\xi_j(t,r,s), \\ &r_j^*(t^*) = r_j(t) + \varepsilon\eta_j(t,r,s), \end{aligned} \tag{14}$$

其中 ε 为无限小参数,ξ_j 和 η_j 为无限小生成元.

引入无限小生成元向量

$$X^{(0)} = \xi_j \frac{\partial}{\partial s_j} + \eta_j \frac{\partial}{\partial r_j}, \tag{15}$$

它的一次扩展

$$X^{(1)} = X^{(0)} + \dot{\xi}_j \frac{\partial}{\partial \dot{s}_j} + \dot{\eta}_j \frac{\partial}{\partial \dot{r}_j}. \tag{16}$$

根据微分方程在无限小变换群下的不变性理论知,方程(10)在无限小变换(14)下的不变性表为

$$\begin{aligned} &\dot{\xi}_j = X^{(0)}(g_j), \dot{\eta}_j = X^{(0)}(h_j), \\ &(j = 1,2,\cdots,n), \end{aligned} \tag{17}$$

其中

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_j \frac{\partial}{\partial s_j} + h_j \frac{\partial}{\partial r_j}. \tag{18}$$

称方程(17)为方程(10)的确定方程.

定义 1　如果无限小变换(14)的生成元 ξ_j 和 η_j 满足确定方程(17),则相应的对称性称为方程(10)的 Lie 对称性.

Lie 对称性不一定导致守恒量,下面的定理给出 Lie 对称性导致守恒量的条件及守恒量的形式.

4. 相应完整系统的 Hojman 守恒定理

定理 1　对相应完整系统(9),如果无限小变换(14)的生成元 ξ_j 和 η_j 满足确定方程(17),且存在某函数 $\mu = \mu(r,s)$,使得

$$\frac{1}{\mu}\frac{\partial}{\partial s_j}(\mu g_j) + \frac{1}{\mu}\frac{\partial}{\partial r_j}(\mu h_j) = 0, \tag{19}$$

则系统存在 Lie 对称性的非 Noether 守恒量,即

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial s_j}(\mu\xi_j) + \frac{1}{\mu}\frac{\partial}{\partial r_j}(\mu\eta_j) = \text{const}. \tag{20}$$

证明　将(20)式对时间 t 求导数,有

$$\begin{aligned} \frac{\mathrm{d}I_{\mathrm{H}}}{\mathrm{d}t} = &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s_j}\right)\xi_j + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r_j}\right)\eta_j + \frac{1}{\mu}\frac{\partial\mu}{\partial s_j}\dot{\xi}_j \\ &+ \frac{1}{\mu}\frac{\partial\mu}{\partial r_j}\dot{\eta}_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi_j}{\partial s_j} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta_j}{\partial r_j}. \end{aligned} \tag{21}$$

由方程(10)和(15),确定方程(17)可写为

$$\begin{aligned}\dot{\xi}_j &= \xi_k \frac{\partial g_j}{\partial s_k} + \eta_k \frac{\partial g_j}{\partial r_k},\\ \dot{\eta}_j &= \xi_k \frac{\partial h_j}{\partial s_k} + \eta_k \frac{\partial h_j}{\partial r_k},\\ &(k, j = 1, 2, \cdots, n).\end{aligned} \tag{22}$$

于是,有

$$\begin{aligned}\frac{\partial \dot{\xi}_j}{\partial s_j} &= \frac{\partial \xi_k}{\partial s_j}\frac{\partial g_j}{\partial s_k} + \frac{\partial \eta_k}{\partial s_j}\frac{\partial g_j}{\partial r_k}\\ &\quad + \xi_k \frac{\partial^2 g_j}{\partial s_k \partial s_j} + \eta_k \frac{\partial^2 g_j}{\partial r_k \partial s_j},\\ \frac{\partial \dot{\eta}_j}{\partial r_j} &= \frac{\partial \xi_k}{\partial r_j}\frac{\partial h_j}{\partial s_k} + \frac{\partial \eta_k}{\partial r_j}\frac{\partial h_j}{\partial r_k}\\ &\quad + \xi_k \frac{\partial^2 h_j}{\partial s_k \partial r_j} + \eta_k \frac{\partial^2 h_j}{\partial r_k \partial r_j}.\end{aligned} \tag{23}$$

注意到

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_j}{\partial s_j} &= \frac{\partial^2 \xi_j}{\partial s_j \partial s_k} g_k + \frac{\partial^2 \xi_j}{\partial s_j \partial r_k} h_k + \frac{\partial^2 \xi_j}{\partial s_j \partial t},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \eta_j}{\partial r_j} &= \frac{\partial^2 \eta_j}{\partial r_j \partial s_k} g_k + \frac{\partial^2 \eta_j}{\partial r_j \partial r_k} h_k + \frac{\partial^2 \eta_j}{\partial r_j \partial t}.\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{\xi}_j &= \frac{\partial \xi_j}{\partial s_k} g_k + \frac{\partial \xi_j}{\partial r_k} h_k + \frac{\partial \xi_j}{\partial t},\\ \dot{\eta}_j &= \frac{\partial \eta_j}{\partial s_k} g_k + \frac{\partial \eta_j}{\partial r_k} h_k + \frac{\partial \eta_j}{\partial t}.\end{aligned} \tag{25}$$

于是,有

$$\begin{aligned}\frac{\partial \dot{\xi}_j}{\partial s_j} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_j}{\partial s_j} + \frac{\partial \xi_j}{\partial s_k}\frac{\partial g_k}{\partial s_j} + \frac{\partial \xi_j}{\partial r_k}\frac{\partial h_k}{\partial s_j},\\ \frac{\partial \dot{\eta}_j}{\partial r_j} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \eta_j}{\partial r_j} + \frac{\partial \eta_j}{\partial s_k}\frac{\partial g_k}{\partial r_j} + \frac{\partial \eta_j}{\partial r_k}\frac{\partial h_k}{\partial r_j}.\end{aligned} \tag{26}$$

将方程(23)代入方程(26),得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_j}{\partial s_j} &= \frac{\partial \eta_k}{\partial s_j}\frac{\partial g_j}{\partial r_k} - \frac{\partial \xi_j}{\partial r_k}\frac{\partial h_k}{\partial s_j}\\ &\quad + \xi_k \frac{\partial^2 g_j}{\partial s_k \partial s_j} + \eta_k \frac{\partial^2 g_j}{\partial r_k \partial s_j},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \eta_j}{\partial r_j} &= \frac{\partial \xi_k}{\partial r_j}\frac{\partial h_j}{\partial s_k} - \frac{\partial \eta_j}{\partial s_k}\frac{\partial g_k}{\partial r_j}\\ &\quad + \xi_k \frac{\partial^2 h_j}{\partial s_k \partial r_j} + \eta_k \frac{\partial^2 h_j}{\partial r_k \partial r_j}.\end{aligned} \tag{27}$$

其次,将方程(23)和(27)代入方程(21),有

$$\begin{aligned}\frac{\mathrm{d}I_{\mathrm{H}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)\xi_j + \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)\eta_j\\ &\quad + \frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big(\xi_k \frac{\partial g_j}{\partial s_k} + \eta_k \frac{\partial g_j}{\partial r_k}\Big)\\ &\quad + \frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big(\xi_k \frac{\partial h_j}{\partial s_k} + \eta_k \frac{\partial h_j}{\partial r_k}\Big) + \xi_k \frac{\partial^2 g_j}{\partial s_k \partial s_j}\\ &\quad + \eta_k \frac{\partial^2 g_j}{\partial r_k \partial s_j} + \xi_k \frac{\partial^2 h_j}{\partial s_k \partial r_j} + \eta_k \frac{\partial^2 h_j}{\partial r_k \partial r_j}\\ &= \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)\dot{s}_k \xi_j + \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)\dot{r}_k \xi_j\\ &\quad + \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)\dot{s}_k \eta_j + \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)\dot{r}_j \eta_j\\ &\quad + \frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big(\xi_k \frac{\partial g_j}{\partial s_k} + \eta_k \frac{\partial g_j}{\partial r_k}\Big)\\ &\quad + \frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big(\xi_k \frac{\partial h_j}{\partial s_k} + \eta_k \frac{\partial h_j}{\partial r_k}\Big) + \xi_k \frac{\partial^2 g_j}{\partial s_k \partial s_j}\\ &\quad + \eta_k \frac{\partial^2 g_j}{\partial r_k \partial s_j} + \xi_k \frac{\partial^2 h_j}{\partial s_k \partial r_j} + \eta_k \frac{\partial^2 h_j}{\partial r_k \partial r_j}.\end{aligned} \tag{28}$$

由条件(19),得

$$\begin{aligned}\frac{\partial^2 g_j}{\partial s_j \partial s_k} + \frac{\partial^2 h_j}{\partial r_j \partial s_k} &= -\frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)g_j - \frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\frac{\partial g_j}{\partial s_k}\\ &\quad - \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)h_j - \frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\frac{\partial h_j}{\partial s_k},\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial^2 g_j}{\partial s_j \partial r_k} + \frac{\partial^2 h_j}{\partial r_j \partial r_k} &= -\frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)g_j - \frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\frac{\partial g_j}{\partial r_k}\\ &\quad - \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)h_j - \frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\frac{\partial h_j}{\partial r_k}.\end{aligned} \tag{30}$$

考虑到

$$\begin{aligned}\frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)g_k \xi_j &= \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)g_j \xi_k,\\ \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)g_k \eta_j &= \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)g_j \eta_k,\\ \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial s_j}\Big)h_k \xi_j &= \frac{\partial}{\partial s_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)h_j \xi_k,\\ \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)h_k \eta_j &= \frac{\partial}{\partial r_k}\Big(\frac{1}{\mu}\frac{\partial \mu}{\partial r_j}\Big)h_j \eta_k,\end{aligned} \tag{31}$$

现将方程(29),(30)和(31)代入方程(28),得

$$\frac{\mathrm{d}I_{\mathrm{H}}}{\mathrm{d}t} = 0, \text{即 } I_{\mathrm{H}} = \mathrm{const}. \tag{32}$$

因此,相应完整系统(9)存在形如(20)式的 Lie 对称性的非 Noether 守恒量.

由(20)式表示的守恒量不依赖于系统的 Lagrange 函数或 Raitzin 函数,也不依赖于 Lie 对称性的结构方程或 Noether 等式.而唯一的依赖于系统的 Lie 对称变换.它是 Hojman 给出的守恒定理[4] 对于 Raitzin 正则方程的推广.

定理 2　对于满足确定方程(17)的无限小生成元 ξ_j 和 η_j,如果使得

$$\frac{\partial g_j}{\partial s_j} + \frac{\partial h_j}{\partial r_j} = 0, \tag{33}$$

则相应完整系统(9)存在 Lie 对称性的非 Noether 守恒量,即

$$I'_{\mathrm{H}} = \frac{\partial \xi_j}{\partial s_j} + \frac{\partial \eta_j}{\partial r_j} = \mathrm{const}. \tag{34}$$

证明　将(34)式对时间 t 求导数,并考虑方程(27),有

$$\begin{aligned}\frac{\mathrm{d}I'_{\mathrm{H}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_j}{\partial s_j} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \eta_j}{\partial r_j} \\ &= \frac{\partial \eta_k}{\partial s_j}\frac{\partial g_j}{\partial r_k} - \frac{\partial \eta_j}{\partial s_k}\frac{\partial g_k}{\partial r_j} \\ &\quad + \frac{\partial \xi_k}{\partial r_j}\frac{\partial h_j}{\partial s_k} - \frac{\partial \xi_j}{\partial r_k}\frac{\partial h_k}{\partial s_j} \\ &\quad + \xi_k \frac{\partial}{\partial s_k}\left(\frac{\partial g_j}{\partial s_j} + \frac{\partial h_j}{\partial r_j}\right) \\ &\quad + \eta_k \frac{\partial}{\partial r_k}\left(\frac{\partial g_j}{\partial s_j} + \frac{\partial h_j}{\partial r_j}\right).\end{aligned} \tag{35}$$

由于

$$\begin{aligned}&\frac{\partial \eta_k}{\partial s_j}\frac{\partial g_j}{\partial r_k} = \frac{\partial \eta_j}{\partial s_k}\frac{\partial g_k}{\partial r_j}, \\ &\frac{\partial \xi_k}{\partial r_j}\frac{\partial h_j}{\partial s_k} = \frac{\partial \xi_j}{\partial r_k}\frac{\partial h_k}{\partial s_j}, \\ &\qquad (j,k = 1,2,\cdots,n).\end{aligned} \tag{36}$$

将方程(33)和(36)代入(35)式,则得

$$\frac{\mathrm{d}I'_{\mathrm{H}}}{\mathrm{d}t} = 0, \text{即 } I'_{\mathrm{H}} = \mathrm{const}. \tag{37}$$

5. 非完整系统的 Hojman 守恒定理

非完整约束(11)在无限小变换(14)下的不变性归结为如下的限制方程:

$$X^{(0)}(\tilde{f}_\beta(t,r,s)) = 0, \ (\beta = 1,2,\cdots,g), \tag{38}$$

或写为

$$\begin{aligned}&\xi_j \frac{\partial \tilde{f}_\beta}{\partial s_j} + \eta_j \frac{\partial \tilde{f}_\beta}{\partial r_j} = 0, \\ &(j = 1,2,\cdots,n; \beta = 1,2,\cdots,g).\end{aligned} \tag{39}$$

定义 2　如果无限小生成元 ξ_j, η_j 满足确定方程(17)和限制方程(39),则相应的对称性称为非完整系统(11),(12)的 Lie 对称性.

定理 3　对非完整系统(11),(12),如果无限小生成元 ξ_j, η_j 满足确定方程(17)和限制方程(39),且存在某函数 $\mu = \mu(r,s)$,使得方程(19)成立,则系统存在形如(20)式的非 Noether 守恒量.

6. 举　　例

已知力学系统的 Lagrange 函数为

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3, \tag{40}$$

非线性非完整约束为

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0. \tag{41}$$

试求系统的非 Noether 守恒量.

第一步:建立相应完整系统的 Raitzin 正则方程.

方程(3)给出

$$\begin{aligned}&m\ddot{q}_1 = 2\lambda\dot{q}_1, m\ddot{q}_2 = 2\lambda\dot{q}_2, \\ &m\ddot{q}_3 + mg = -2\lambda\dot{q}_3.\end{aligned} \tag{42}$$

由方程(41),(42)解得

$$\lambda = -\frac{mg}{4\dot{q}_3}, \tag{43}$$

于是,有

$$\begin{aligned}&m\ddot{q}_1 = -\frac{mg\dot{q}_1}{2\dot{q}_3}, m\ddot{q}_2 = -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ &m\ddot{q}_3 + mg = \frac{1}{2}mg.\end{aligned} \tag{44}$$

对照方程(4),有

$$\begin{aligned}&Q_1 + \Lambda_1 = -\frac{mg\dot{q}_1}{2\dot{q}_3}, \\ &Q_2 + \Lambda_2 = -\frac{mg\dot{q}_2}{2\dot{q}_3}, \\ &Q_3 + \Lambda_3 = \frac{1}{2}mg.\end{aligned} \tag{45}$$

引入 Raitzin 正则变量及函数

$$\begin{aligned}&r_1 = \frac{\partial L}{\partial q_1} = 0, r_2 = \frac{\partial L}{\partial q_2} = 0, r_3 = \frac{\partial L}{\partial q_3} = -mg, \\ &s_1 = \dot{q}_1, s_2 = \dot{q}_2, s_3 = \dot{q}_3.\end{aligned} \tag{46}$$

$$R = L - r_j q_j = \frac{1}{2}m(s_1^2 + s_2^2 + s_3^2). \tag{47}$$

于是得到相应完整系统的 Raitzin 正则方程为

$$\begin{aligned}r_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_1} - \tilde{Q}_1 - \tilde{\Lambda}_1 \\ &= m\dot{s}_1 + \frac{1}{2}mg\frac{s_1}{s_3}, \\ r_2 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_2} - \tilde{Q}_2 - \tilde{\Lambda}_2\end{aligned}$$

$$= m\dot{s}_2 + \frac{1}{2} mg \frac{s_2}{s_3},$$

$$r_3 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s_3} - \widetilde{Q}_3 - \widetilde{\Lambda}_3$$

$$= m\dot{s}_3 - \frac{1}{2} mg . \quad (48)$$

$$s_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_1} = 0,$$

$$s_2 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_2} = 0,$$

$$s_3 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_3} = 0. \quad (49)$$

方程(10)给出

$$\dot{s}_1 = -\frac{1}{2} g \frac{s_1}{s_3} = g_1,$$

$$\dot{s}_2 = -\frac{1}{2} g \frac{s_2}{s_3} = g_2,$$

$$\dot{s}_3 = -\frac{1}{2} g = g_3 . \quad (50)$$

$$\dot{r}_1 = h_1 = 0, \dot{r}_2 = h_2 = 0,$$

$$\dot{r}_3 = h_3 = 0. \quad (51)$$

第二步 建立确定方程,由方程(17)得

$$\dot{\xi}_1 = -\frac{g}{2s_3}\xi_1 + \frac{s_1 g}{2s_3^2}\xi_3,$$

$$\dot{\xi}_2 = -\frac{g}{2s_3}\xi_2 + \frac{s_2 g}{2s_3^2}\xi_3,$$

$$\dot{\xi}_3 = 0, \dot{\eta}_1 = 0, \dot{\eta}_2 = 0, \dot{\eta}_3 = 0. \quad (52)$$

第三步 建立限制方程.

约束方程(41)可由 Raitzin 变量表示为

$$\tilde{f} = s_1^2 + s_2^2 - s_3^2 = 0, \quad (53)$$

限制方程(39)给出

$$\xi_1 s_1 + \xi_2 s_2 - \xi_3 s_3 = 0. \quad (54)$$

取

$$\xi_1 = -s_2, \xi_2 = s_1, \xi_3 = 0,$$

$$\eta_1 = 0, \eta_2 = 0, \eta_3 = 0. \quad (55)$$

经验证方程(55)满足确定方程(52)和限制方程(54).因此相应的对称性是 Lie 对称的.

第四步 求守恒量.

由方程(19)解得

$$\mu = \frac{1}{g_j} = \frac{1}{g_1 + g_2 + g_3}$$

$$= -\frac{2s_3}{g(s_1 + s_2 + s_3)}, \quad (56)$$

于是守恒量(20)给出

$$I_{\mathrm{H}} = \frac{s_1 - s_2}{s_1 + s_2 + s_3} = \mathrm{const}. \quad (57)$$

或写为

$$I_{\mathrm{H}} = \frac{\dot{q}_1 - \dot{q}_2}{\dot{q}_1 + \dot{q}_2 + \dot{q}_3} = \mathrm{const}. \quad (58)$$

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Hojman's conservation theorems for Raitzin's canonical equations of motion of nonlinear nonholonomic systems*

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Abstract

Using the Lie symmetry under infinitesimal transformations in which the time is not variable，the Hojman's conservation theorems for Raitzin's canonical equations of motion of nonlinear nonholonomic systems are studied. The differential equations of the systems and the determining equations of Lie symmetry under infinitesimal transformations are given. The Hojman's conservation theorems of the systems are established. Finally，we give an example to illustrate the application of the result.

Keywords：nonlinear nonholonmic system，Raitzin's canonical equation，Lie symmetry，determining equation，Hojman's conservation theorem

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