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完整力学系统的高阶运动微分方程

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从质点系的牛顿动力学方程出发,引入系统的高阶速度能量,导出完整力学系统的高阶 Lagrange 方程、高阶 Nielsen 方程以及高阶 Appell 方程,并证明了完整系统三种形式的高阶运动微分方程是等价的.结果表明,完整系统高阶运动微分方程揭示了系统运动状态的改变与力的各阶变化率之间的联系,这是牛顿动力学方程以及传统分析力学方程不能直接反映的.因此,完整系统高阶运动微分方程是对牛顿动力学方程及传统 Lagrange 方程、Nielsen 方程、Appell 方程等二阶运动微分方程的进一步补充.

关键词:高阶速度能量, 高阶 Lagrange 方程, 高阶 Nielsen 方程, 高阶 Appell 方程

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1. 引 言

力学中描述物体运动的动力学方程通常都是二阶运动微分方程.在已知物体受力及初始条件的前提下,根据牛顿动力学方程原则上可以确定物体的运动规律.但实际的动力学问题比较复杂,物体往往受变力作用而运动.如果已知的不是物体所受的力,而是物体受的力随时间的变化率,甚至是物体受的力随时间的高阶变化率,则无法直接用牛顿动力学方程所反映的二阶运动微分方程得到物体的运动规律.因此,探索如何根据力随时间的变化率以及力随时间的高阶变化率确定物体的运动规律,是很有必要的.

在根据力随时间的变化率研究物体运动规律方面,人们已建立了加速度随时间的变化率(急动度)[1—3]、力随时间的变化率(力变率)[4]、猝量方程[5,6]、加速度能定理[6]等概念及规律.1991 年,Mei 等人提出了完整系统关于广义速度的 Lagrange 方程[7].最近,文献[8]从牛顿运动方程出发给出了该方程的一种推导方法,并称该方程为三阶 Lagrange 方程.可见,根据力变率研究完整系统的运动规律已受到人们的关注.本文从质点系的牛顿动力学方程出发,进一步引入系统的高阶速度能量,导出完整力学系统的高阶 Lagrange 方程、高阶 Nielsen 方程以及高阶 Appell 方程,它们是与力随时间的高阶变化率相对应的完整系统的高阶运动微分方程.人们在非完整力学和广义经典力学意义上也曾得到广义 Euler-Lagrange 方程、广义 Nielsen 方程及高阶非完整系统的 Appell 方程[7,9,10],但它们本质上只是关于广义力的运动微分方程.本文最后证明了完整系统三种形式的高阶运动微分方程是等价的.

2. 完整系统高阶 Lagrange 方程的导出

考虑一个由 N 个质点组成的完整、理想力学体系,对第 i 个质点的牛顿动力学方程等号两边同时对时间求 m 阶导数,然后移项,得

$$\overset{(m)}{\boldsymbol{F}}_i+\overset{(m)}{\boldsymbol{R}}_i-m_i\overset{(m+2)}{\boldsymbol{r}_i}=0$$
$$(i=1,2,\cdots,N;m=0,1,2,\cdots),\tag{1}$$

式中 $\boldsymbol{F}_i$ 和 $\boldsymbol{R}_i$ 分别为作用于第 i 个质点上的主动力合力和约束反力合力,它们一般是位矢 $\boldsymbol{r}_i$、速度 $\dot{\boldsymbol{r}}_i$ 和时间 t 的函数.用第 i 个质点坐标空间的虚位移 $\delta\boldsymbol{r}_i$ 标乘(1)式,并对 i 求和,则得

$$\sum_{i=1}^{N}(\overset{(m)}{\boldsymbol{F}}_i+\overset{(m)}{\boldsymbol{R}}_i-m_i\overset{(m+2)}{\boldsymbol{r}_i})\cdot\delta\boldsymbol{r}_i=0.\tag{2}$$

根据理想约束的定义,有

$$\sum_{i=1}^{N}\boldsymbol{R}_i\cdot\delta\boldsymbol{r}_i=\sum_{i=1}^{N}\boldsymbol{R}_i\cdot\delta\dot{\boldsymbol{r}}_i=\cdots$$
$$=\sum_{i=1}^{N}\boldsymbol{R}_i\cdot\delta\overset{(m)}{\boldsymbol{r}_i}=0,\tag{3}$$

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$$\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{R}}_i\cdot\delta\boldsymbol{r}_i=0. \tag{4}$$

将(4)式代入(2)式,得

$$\sum_{i=1}^{N}(\overset{(m)}{\boldsymbol{F}}_i-m_i\overset{(m+2)}{\boldsymbol{r}}_i)\cdot\delta\boldsymbol{r}_i=0. \tag{5}$$

引入 s 个广义坐标 $q_\alpha(\alpha=1,2,\cdots,s)$,则有

$$\boldsymbol{r}_i=\boldsymbol{r}_i(q_\alpha,t),\delta\boldsymbol{r}_i=\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\delta q_\alpha. \tag{6}$$

将(6)式代入(5)式,整理得

$$\sum_{\alpha=1}^{s}\left[-\sum_{i=1}^{N}m_i\overset{(m+2)}{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}+\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\right]\delta q_\alpha=0. \tag{7}$$

令

$$P_\alpha=\sum_{i=1}^{N}m_i\overset{(m+2)}{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}, \tag{8}$$

则(8)式可进一步表示为

$$\begin{aligned}P_\alpha&=\sum_{i=1}^{N}m_i\overset{(m+2)}{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\right)\\&\quad-\sum_{i=1}^{N}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}.\end{aligned} \tag{9}$$

由(6)式中 $\boldsymbol{r}_i$ 的表示形式,可求得

$$\dot{\boldsymbol{r}}_i=\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\dot{q}_\alpha+\frac{\partial\boldsymbol{r}_i}{\partial t}, \tag{10}$$

$$\begin{aligned}\ddot{\boldsymbol{r}}_i&=\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\ddot{q}_\alpha+\sum_{\beta=1}^{s}\sum_{\alpha=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha}\dot{q}_\alpha\dot{q}_\beta\\&\quad+2\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha\partial t}\dot{q}_\alpha+\frac{\partial^2\boldsymbol{r}_i}{\partial t^2},\end{aligned} \tag{11}$$

$$\begin{aligned}\dddot{\boldsymbol{r}}_i&=\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\dddot{q}_\alpha+3\sum_{\beta=1}^{s}\sum_{\alpha=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha}\ddot{q}_\alpha\dot{q}_\beta\\&\quad+3\sum_{\alpha=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\alpha\partial t}\ddot{q}_\alpha+\sum_{\gamma=1}^{s}\sum_{\beta=1}^{s}\sum_{\alpha=1}^{s}\frac{\partial^3\boldsymbol{r}_i}{\partial q_\gamma\partial q_\beta\partial q_\alpha}\dot{q}_\alpha\dot{q}_\beta\dot{q}_\gamma\\&\quad+3\sum_{\beta=1}^{s}\sum_{\alpha=1}^{s}\frac{\partial^3\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha\partial t}\dot{q}_\alpha\dot{q}_\beta\\&\quad+3\sum_{\alpha=1}^{s}\frac{\partial^3\boldsymbol{r}_i}{\partial q_\alpha\partial t^2}\dot{q}_\alpha+\frac{\partial^3\boldsymbol{r}_i}{\partial t^3},\end{aligned} \tag{12}$$

$\cdots\cdots$,

$$\begin{aligned}\overset{(m)}{\boldsymbol{r}}_i&=\sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\overset{(m)}{q}_\alpha+m\left[\sum_{\beta=1}^{s}\sum_{\alpha=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha}\overset{(m-1)}{q}_\alpha\dot{q}_\beta\right.\\&\quad\left.+\sum_{\alpha=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\alpha\partial t}\overset{(m-1)}{q}_\alpha\right]+\cdots.\end{aligned} \tag{13}$$

其中(13)式未写出之项不含 $\overset{(m)}{q}_\alpha$,$\overset{(m-1)}{q}_\alpha$.由(10)—(13)式可得

$$\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\overset{(m)}{q}_\alpha}=\frac{\partial\overset{(m-1)}{\boldsymbol{r}}_i}{\partial\overset{(m-1)}{q}_\alpha}=\frac{\partial\overset{(m-2)}{\boldsymbol{r}}_i}{\partial\overset{(m-2)}{q}_\alpha}=\cdots=\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha}=\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}, \tag{14}$$

$$\begin{aligned}&\frac{\partial\overset{(m)}{\boldsymbol{r}}_i}{\partial\overset{(m-1)}{q}_\alpha}=m\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_\alpha},\frac{\partial\overset{(m-1)}{\boldsymbol{r}}_i}{\partial\overset{(m-2)}{q}_\alpha}=(m-1)\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_\alpha},\cdots,\\&\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha}=2\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_\alpha}.\end{aligned} \tag{15}$$

将(14)式代入(9)式,得

$$P_\alpha=\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\frac{\partial\overset{(m+1)}{\boldsymbol{r}}_i}{\partial\overset{(m+1)}{q}_\alpha}\right)-\sum_{i=1}^{N}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}. \tag{16}$$

再将(16)式右边第二项适当化简, 并将(15)式代入,有

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}&=\sum_{\beta=1}^{s}\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha}\dot{q}_\beta+\frac{\partial^2\boldsymbol{r}_i}{\partial t\partial q_\alpha}\\&=\frac{\partial\dot{\boldsymbol{r}}_i}{\partial q_\alpha}=\frac{1}{m+1}\frac{\partial\overset{(m+1)}{\boldsymbol{r}}_i}{\partial\overset{(m)}{q}_\alpha}.\end{aligned} \tag{17}$$

将(17)式代入(16)式,可得

$$\begin{aligned}P_\alpha&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial\overset{(m+1)}{q}_\alpha}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\overset{(m+1)}{\boldsymbol{r}}_i\right)\right]\\&\quad-\frac{1}{m+1}\frac{\partial}{\partial\overset{(m)}{q}_\alpha}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\overset{(m+1)}{\boldsymbol{r}}_i\right).\end{aligned} \tag{18}$$

定义 1　将

$$S_m=\sum_{i=1}^{N}\frac{1}{2}m_i\overset{(m+1)}{\boldsymbol{r}}_i\cdot\overset{(m+1)}{\boldsymbol{r}}_i$$

称为系统的 m 阶速度能量,通常写成

$$\begin{aligned}&S_m=S_m(q_\alpha,\dot{q}_\alpha,\ddot{q}_\alpha,\cdots,\overset{(m)}{q}_\alpha,\overset{(m+1)}{q}_\alpha,t)\\&(\alpha=1,2,\cdots,s;m=0,1,2,\cdots).\end{aligned} \tag{19}$$

当 $m=0$ 时,(19)式为零阶速度能量 $S_0=T$,正是熟知的系统的动能;当 $m=1$ 时,(19)式为一阶速度能量

$$S_1=\sum_{i=1}^{N}\frac{1}{2}m_i\ddot{\boldsymbol{r}}_i\cdot\ddot{\boldsymbol{r}}_i$$

正是系统的加速度能量.可见,系统的 m 阶速度能量是对系统动能及系统加速度能量的进一步扩充.

将(18)式代入(7)式,得

$$\begin{aligned}&\sum_{\alpha=1}^{s}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S_m}{\partial\overset{(m+1)}{q}_\alpha}\right)+\frac{1}{m+1}\frac{\partial S_m}{\partial\overset{(m)}{q}_\alpha}\right.\\&\quad\left.+\sum_{i=1}^{N}\overset{(m)}{\boldsymbol{F}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\right]\delta q_\alpha=0.\end{aligned} \tag{20}$$

又令

$$Q_\alpha^m = \sum_{i=1}^N \overset{(m)}{\boldsymbol F}_i \cdot \frac{\partial \boldsymbol r_i}{\partial q_\alpha}, \tag{21}$$

称 Q_α^m 为 m 阶广义主动力变率.对于完整系统，q_1，q_2，…，q_s 彼此独立,由(20)式可得

$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial S_m}{\partial \overset{(m+1)}{q_\alpha}}\right) - \frac{1}{m+1}\frac{\partial S_m}{\partial \overset{(m)}{q_\alpha}} = Q_\alpha^m$$

$$(\alpha = 1,2,\cdots,s;m = 0,1,2,\cdots). \tag{22}$$

(22)式就是描述完整力学系统运动的 $m+2$ 阶微分方程.特别地,当 $m=0$ 时,(22)式为

$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\mathrm dT}{\mathrm d\dot q_\alpha}\right) - \frac{\mathrm dT}{\mathrm dq_\alpha} = Q_\alpha$$

$$(\alpha = 1,2,\cdots,s). \tag{23}$$

(23)式正是传统分析力学中的 Lagrange 方程,其中 $Q_\alpha = Q_\alpha^0 = \sum_{i=1}^N \boldsymbol F_i \cdot \frac{\partial \boldsymbol r_i}{\partial q_\alpha}$ 为广义主动力;当 $m=1$ 时,(22)式为

$$\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial S_1}{\partial \ddot q_\alpha}\right) - \frac{1}{2}\frac{\partial S_1}{\partial \dot q_\alpha} = Q_\alpha^1 (\alpha = 1,2,\cdots,s), \tag{24}$$

(24)式正是文献[7]提出的完整系统关于广义速度的 Lagrange 方程(亦即文献[8]所推导的三阶 Lagrange 方程),其中 $Q_\alpha^1 = \sum_{i=1}^N \dot{\boldsymbol F}_i \cdot \frac{\partial \boldsymbol r_i}{\partial q_\alpha}$ 为一阶广义主动力变率.(22)式可称为完整系统高阶 Lagrange 方程,它是对传统 Lagrange 方程及三阶 Lagrange 方程在理论上的进一步扩充.

3. 完整系统高阶 Nielsen 方程的导出

将(14)式代入(8)式,则(8)式还可写为

$$\begin{aligned}P_\alpha &= \sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \frac{\partial \boldsymbol r_i}{\partial q_\alpha} = \sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \frac{\partial \overset{(m+1)}{\boldsymbol r}_i}{\partial \overset{(m+1)}{q_\alpha}} \\ &= \frac{\partial}{\partial \overset{(m+1)}{q_\alpha}}\left(\sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \overset{(m+1)}{\boldsymbol r}_i\right) \\ &\quad - \sum_{i=1}^N m_i \frac{\partial \overset{(m+2)}{\boldsymbol r}_i}{\partial \overset{(m+1)}{q_\alpha}} \cdot \overset{(m+1)}{\boldsymbol r}_i,\end{aligned} \tag{25}$$

将(15)式代入(25)式化简,并结合定义 1,得

$$\begin{aligned}P_\alpha &= \frac{\partial}{\partial \overset{(m+1)}{q_\alpha}}\left(\sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \overset{(m+1)}{\boldsymbol r}_i\right) \\ &\quad - (m+2)\sum_{i=1}^N m_i \frac{\partial \dot{\boldsymbol r}_i}{\partial q_\alpha} \cdot \overset{(m+1)}{\boldsymbol r}_i \\ &= \frac{\partial}{\partial \overset{(m+1)}{q_\alpha}}\left(\sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \overset{(m+1)}{\boldsymbol r}_i\right) \\ &\quad - \frac{m+2}{m+1}\sum_{i=1}^N m_i \frac{\partial \overset{(m+1)}{\boldsymbol r}_i}{\partial \overset{(m)}{q_\alpha}} \cdot \overset{(m+1)}{\boldsymbol r}_i \\ &= \frac{\partial}{\partial \overset{(m+1)}{q_\alpha}}\left[\frac{\mathrm d}{\mathrm dt}\left(\sum_{i=1}^N \frac12 m_i \overset{(m+1)}{\boldsymbol r}_i \cdot \overset{(m+1)}{\boldsymbol r}_i\right)\right] \\ &\quad - \frac{m+2}{m+1}\frac{\partial}{\partial \overset{(m)}{q_\alpha}}\left(\sum_{i=1}^N \frac12 m_i \overset{(m+1)}{\boldsymbol r}_i \cdot \overset{(m+1)}{\boldsymbol r}_i\right) \\ &= \frac{\partial \dot S_m}{\partial \overset{(m+1)}{q_\alpha}} - \frac{m+2}{m+1}\frac{\partial S_m}{\partial \overset{(m)}{q_\alpha}}.\end{aligned} \tag{26}$$

将(26)式及(21)式代入(7)式,并考虑到完整系统 $q_1,q_2,\cdots,q_s$ 彼此独立,可得

$$\frac{\partial \dot S_m}{\partial \overset{(m+1)}{q_\alpha}} - \frac{m+2}{m+1}\frac{\partial S_m}{\partial \overset{(m)}{q_\alpha}} = Q_\alpha^m$$

$$(\alpha = 1,2,\cdots,s;m = 0,1,2,\cdots). \tag{27}$$

(27)式是描述完整力学系统运动的 $m+2$ 阶微分方程的另一形式,称为完整系统高阶 Nielsen 方程.当 $m=0$ 时,(27)式为

$$\frac{\partial \dot T}{\partial \dot q_\alpha} - 2\frac{\partial T}{\partial q_\alpha} = Q_\alpha \quad (\alpha = 1,2,\cdots,s), \tag{28}$$

(28)式正是传统分析力学中的 Nielsen 方程;当 $m=1$ 时,(27)式为

$$\frac{\partial \dot S_1}{\partial \ddot q_\alpha} - \frac32\frac{\partial S_1}{\partial \dot q_\alpha} = Q_\alpha^1 \quad (\alpha = 1,2,\cdots,s), \tag{29}$$

(29)式可称为完整系统三阶 Nielsen 方程.

4. 完整系统高阶 Appell 方程的导出

将(14)式代入(8)式,并结合定义 1,(8)式又可写为

$$\begin{aligned}P_\alpha &= \sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \frac{\partial \boldsymbol r_i}{\partial q_\alpha} = \sum_{i=1}^N m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \frac{\partial \overset{(m+2)}{\boldsymbol r}_i}{\partial \overset{(m+2)}{q_\alpha}} \\ &= \frac{\partial}{\partial \overset{(m+2)}{q_\alpha}}\left(\sum_{i=1}^N \frac12 m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \overset{(m+2)}{\boldsymbol r}_i\right) = \frac{\partial S_{m+1}}{\partial \overset{(m+2)}{q_\alpha}},\end{aligned} \tag{30}$$

按定义 1,(30)式中的

$$S_{m+1} = \sum_{i=1}^N \frac12 m_i \overset{(m+2)}{\boldsymbol r}_i \cdot \overset{(m+2)}{\boldsymbol r}_i$$

为系统的 $m+1$ 阶速度能量.将(30)式及(21)式代入(7)式,由于完整系统 $q_1,q_2,\cdots,q_s$ 彼此独立,可得

$$\frac{\partial S_{m+1}}{\partial \overset{(m+2)}{q}_{\alpha}} = Q_{\alpha}^{m}\quad(\alpha = 1,2,\cdots,s;m = 0,1,2,\cdots). \tag{31}$$

(31)式是描述完整力学系统运动的 $m+2$ 阶微分方程的又一形式,称为完整系统高阶 Appell 方程. 当 $m=0$ 时,(31)式为

$$\frac{\partial S_{1}}{\partial \ddot{q}_{\alpha}} = Q_{\alpha}\quad(\alpha = 1,2,\cdots,s), \tag{32}$$

(32)式正是传统分析力学中的 Appell 方程;当 $m=1$ 时,(31)式为

$$\frac{\partial S_{2}}{\partial \dddot{q}_{\alpha}} = Q_{\alpha}^{1}\quad(\alpha = 1,2,\cdots,s), \tag{33}$$

(33)式可称为完整系统三阶 Appell 方程.

5. 完整系统三种形式的高阶运动微分方程的等价性

将(19)式代入(27)式左边,化简,得

$$\begin{aligned}
&\frac{\partial \dot{S}_{m}}{\partial \overset{(m+1)}{q}_{\alpha}} - \frac{m+2}{m+1}\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \frac{\partial}{\partial \overset{(m+1)}{q}_{\alpha}}\left[\sum_{\beta=1}^{s}\sum_{k=0}^{m}\frac{\partial S_{m}}{\partial \overset{(k)}{q}_{\beta}}\overset{(k+1)}{q}_{\beta} + \frac{\partial S_{m}}{\partial t}\right] - \frac{m+2}{m+1}\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \left[\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}} + \sum_{k=0}^{m}\sum_{\beta=1}^{s}\frac{\partial}{\partial \overset{(m+1)}{q}_{\alpha}}\left(\frac{\partial S_{m}}{\partial \overset{(k)}{q}_{\beta}}\right)\overset{(k+1)}{q}_{\beta} + \frac{\partial}{\partial \overset{(m+1)}{q}_{\alpha}}\left(\frac{\partial S_{m}}{\partial t}\right)\right]\\
&\quad - \frac{m+2}{m+1}\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \left[\sum_{k=0}^{m}\sum_{\beta=1}^{s}\frac{\partial}{\partial \overset{(k)}{q}_{\beta}}\left(\frac{\partial S_{m}}{\partial \overset{(m+1)}{q}_{\alpha}}\right)\overset{(k+1)}{q}_{\beta} + \frac{\partial}{\partial t}\left(\frac{\partial S_{m}}{\partial \overset{(m+1)}{q}_{\alpha}}\right)\right]\\
&\quad + \left(1 - \frac{m+2}{m+1}\right)\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}S_{m}}{\mathrm{d}\overset{(m+1)}{q}_{\alpha}}\right) - \frac{1}{m+1}\frac{\mathrm{d}S_{m}}{\mathrm{d}\overset{(m)}{q}_{\alpha}}.
\end{aligned} \tag{34}$$

按定义 1,利用(14)式和(17)式,(31)式左边可化为

$$\begin{aligned}
\frac{\partial S_{m+1}}{\partial \overset{(m+2)}{q}_{\alpha}} &= \sum_{i=1}^{N} m_{i}\overset{(m+2)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+2)}{\boldsymbol{r}}_{i}}{\partial \overset{(m+2)}{q}_{\alpha}}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m+1)}{q}_{\alpha}}\right)\\
&\quad - \sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{r}_{i}}{\partial q_{\alpha}}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \overset{(m+1)}{q}_{\alpha}}\left(\sum_{i=1}^{N}\frac{1}{2}m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\right)\\
&\quad - \frac{1}{m+1}\sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \overset{(m+1)}{q}_{\alpha}}\left(\sum_{i=1}^{N}\frac{1}{2}m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\right)\\
&\quad - \frac{1}{m+1}\frac{\partial}{\partial \overset{(m)}{q}_{\alpha}}\left(\sum_{i=1}^{N}\frac{1}{2}m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S_{m}}{\partial \overset{(m+1)}{q}_{\alpha}}\right) - \frac{1}{m+1}\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}.
\end{aligned} \tag{35}$$

利用(14)式和(15)式,(27)式左边可化为

$$\begin{aligned}
&\frac{\partial \dot{S}_{m}}{\partial \overset{(m+1)}{q}_{\alpha}} - \frac{m+2}{m+1}\frac{\partial S_{m}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \sum_{i=1}^{N} m_{i}\overset{(m+2)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m+1)}{q}_{\alpha}} + \sum_{i=1}^{N} m_{i}\frac{\partial \overset{(m+2)}{\boldsymbol{r}}_{i}}{\partial \overset{(m+1)}{q}_{\alpha}}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\\
&\quad - \frac{m+2}{m+1}\sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \sum_{i=1}^{N} m_{i}\overset{(m+2)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+2)}{\boldsymbol{r}}_{i}}{\partial \overset{(m+2)}{q}_{\alpha}} + (m+2)\sum_{i=1}^{N} m_{i}\frac{\partial \dot{\boldsymbol{r}}_{i}}{\partial q_{\alpha}}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\\
&\quad - \frac{m+2}{m+1}\sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \frac{\partial}{\partial \overset{(m+2)}{q}_{\alpha}}\left(\sum_{i=1}^{N}\frac{1}{2}m_{i}\overset{(m+2)}{\boldsymbol{r}}_{i}\cdot\overset{(m+2)}{\boldsymbol{r}}_{i}\right)\\
&\quad + \frac{m+2}{m+1}\sum_{i=1}^{N} m_{i}\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m)}{q}_{\alpha}}\cdot\overset{(m+1)}{\boldsymbol{r}}_{i}\\
&\quad - \frac{m+2}{m+1}\sum_{i=1}^{N} m_{i}\overset{(m+1)}{\boldsymbol{r}}_{i}\cdot\frac{\partial \overset{(m+1)}{\boldsymbol{r}}_{i}}{\partial \overset{(m)}{q}_{\alpha}}\\
&= \frac{\partial S_{m+1}}{\partial \overset{(m+2)}{q}_{\alpha}}.
\end{aligned} \tag{36}$$

从(34)式、(35)式及(36)式可见,(22)式、(27)式及(31)式完全等价.

6. 结　　论

系统高阶速度能量的引入,是对系统动能和系统加速度能量的进一步扩充,其必要性首先从完整系统三阶 Appell 方程(33)式即可看到. 完整系统高

阶 Lagrange 方程及高阶 Nielsen 方程揭示了变力作用下系统运动状态的改变与力的各阶变化率之间的联系，与 m 阶广义主动力变率相联系的是描述系统运动状态的 m 阶速度能量的改变，这是牛顿动力学方程以及传统分析力学方程无法直接反映的.因此，由牛顿动力学方程导出的完整系统高阶运动微分方程是对牛顿动力学方程及传统 Lagrange 方程、Nielsen 方程、Appell 方程的进一步补充.由于完整系统高阶运动微分方程由牛顿动力学方程导出，又包容了传统的分析力学方程，所以它们本质上是完全相容的.

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Higher-order differential equations of motion of a holonomic mechanical system

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Abstract

Starting from Newtonian kinetic equations of a particle system, the energy of higher order-velocity of the system is introduced; higher-order Lagrange equations, higher-order Nielsen equations and higher-order Appell equations of a holonomic mechanical system are derived, from which we prove that the three kinds of higher-order differential equations of motion of the holonomic system are equivalent to each other. The result indicates that the higher-order differential equations of motion of the holonomic system reveal the relationship between the changes of the system's motion state and the rate of change of force at every order, which cannot be obtained by using Newtonian kinetic equations and the traditional analytical mechanical equations. Therefore, the higher-order differential equations of motion of the holonomic system are a complement to the second-order differential equations of motion, including Newtonian kinetic equations and the traditional Lagrange equations, Nielsen equations and Appell equations.

Keywords: energy of higher-order velocity, higher-order Lagrange equations, higher-order Nielsen equations, higher-order Appell equations

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