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Whittaker 方程的场方法*

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用场方法来求解 Whittaker 方程.将一个场变量取作为其余场变量和时间的函数并对这个函数建立基本偏微方程.如能求得它的完全积分,那么 Whittaker 方程的解可由解代数方程来得到.

关键词:场变量,基本偏微分方程,场方法,积分

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1. 引 言

20 世纪 30 年代 Tolman 提出如下问题:是否存在一组方程,它们不能由一个 Lagrange 函数来得到.著名经典力学家 Whittaker 给出一组方程来回答这个问题,这就是后来被称之为 Whittaker 方程[1,2].这个方程引起众多物理学家和力学家的重视,并在经典力学发展史上,特别是在 Lagrange 力学逆问题的研究中占有重要位置.文献[3]指出,对称性和第一积分是常微分方程的两个基本结构.

对称性主要有 Noether 对称性[4—7],Lie 对称性[8—18]和形式不变性[9—22].Whittaker 方程的求解,在数学上没有什么困难.场积分方法已在振动理论,完整非保守系统和非完整系统运动方程的积分上取得重要进展[23,24].场积分方法应该说是一种近代的力学方法.本文期望用这种力学方法来求解 Whittaker 方程.

2. Whittaker 方程的求解

Whittaker 在回答 Tolman 所提问题时,给出方程[2]

$$\ddot{q}_1 = q_1, \ \ddot{q}_2 = \dot{q}_1. \tag{1}$$

Whittaker 确信这组方程不能由任何 Laglange 函数导出来.用常系数线性微分方程理论,不难求解方程(1).

下面用场积分方法这一力学方法来解方程(1).

令

$$x_1 = q_1, x_2 = q_2, x_3 = \dot{q}_1, x_4 = \dot{q}_2, \tag{2}$$

则方程(1)表为

$$\dot{x}_1 = x_3, \dot{x}_2 = x_4, \dot{x}_3 = x_1, \dot{x}_4 = x_3. \tag{3}$$

根据场方法[23,24],设

$$x_2 = u(t, x_1, x_3, x_4), \tag{4}$$

则基本偏微分方程为

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_1}x_3 + \frac{\partial u}{\partial x_3}x_1 + \frac{\partial u}{\partial x_4}x_3 - x_4 = 0. \tag{5}$$

令其解有形式

$$u = x_2 = f_1(t) + f_2(t)x_1 + f_3(t)x_3 + f_4(t)x_4,$$

将其代入方程(5),比较自由项以及含 x_1, x_3 和 x_4 的项,得到

$$\dot{f}_1 = 0, \dot{f}_2 + f_3 = 0, \dot{f}_3 + f_2 + f_4 = 0, \dot{f}_4 - 1 = 0.$$

由此解得待定函数 f_1, f_2, f_3 和 f_4,即

$$f_1 = C_1, f_2 = C_2\exp(t) + C_3\exp(-t) - t - C_4,$$

$$f_3 = -C_2\exp(t) + C_3\exp(-t) + 1, f_4 = t + C_4,$$

其中 C_1, C_2, C_3, C_4 为任意常数.因此有

$$\begin{aligned} u = x_2 = {} & C_1 + \{C_2\exp(t) + C_3\exp(-t) \\ & - t - C_4\}x_1 + \{-C_2\exp(t) \\ & + C_3\exp(-t) + 1\}x_3 + (t + C_4)x_4. \end{aligned} \tag{6}$$

令初始条件为

$$x_s(0) = x_{s0}, (s = 1, \cdots, 4), \tag{7}$$

将(7)式代入(6)式并消去 C_1,得

$$\begin{aligned} u = x_2 = {} & x_{20} - (C_2 + C_3 - C_4)x_{10} \\ & - (-C_2 + C_3 + 1)x_{30} - C_4x_{40} \\ & + \{C_2\exp(t) + C_3\exp(-t) - t - C_4\}x_1 \end{aligned}$$

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$$+\{-C_2\exp(t)+C_3\exp(-t)+1\}x_3$$
$$+(t+C_4)x_4. \tag{8}$$

于是有

$$\frac{\partial u}{\partial C_2}=-x_{10}+x_{30}+x_1\exp(t)-x_3\exp(t)=0,$$
$$\frac{\partial u}{\partial C_3}=-x_{10}-x_{30}+x_1\exp(-t)+x_3\exp(-t)=0,$$
$$\frac{\partial u}{\partial C_4}=x_{10}-x_{40}-x_1+x_4=0. \tag{9}$$

解代数方程(9),得

$$x_1=\frac{1}{2}\{(x_{10}-x_{30})\exp(-t)+(x_{10}+x_{30})\exp(t)\},$$
$$x_3=\frac{1}{2}\{(x_{10}+x_{30})\exp(t)-(x_{10}-x_{30})\exp(-t)\},$$
$$x_4=x_{40}-x_{10}+\frac{1}{2}\{(x_{10}-x_{30})\exp(-t)$$
$$+(x_{10}+x_{30})\exp(t)\}, \tag{10}$$

将(10)式代入(8)式,得

$$x_2=x_{20}+(x_{40}-x_{10})t-x_{30}$$
$$+\frac{1}{2}\{(x_{10}+x_{30})\exp(t)$$
$$-(x_{10}-x_{30})\exp(-t)\}. \tag{11}$$

(10),(11)式就是 Whittaker 方程(1)在初值(7)下的解.

3. 结　　论

解力学问题通常用数学方法.反过来,解数学问题也可以利用力学方法.本文就是利用场方法这一力学方法来解 Whittaker 方程.

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A field method for solving Whittaker equations*

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Abstract

We present a field method for solving the Whittaker equations. One of the field variables is considered as a function of other variables and time, and the fundamental partial differential equation for the function is established. If the complete integral of the partial differential equation can be obtained, the solution of the Whittaker equations are obtained by solving the algebraic equations.

Keywords: field variable, fundamental partial differential equation, field method, integral

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