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求解微分方程的 Hojman 方法*

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首先,将 Hojman 用于求解二阶微分方程组守恒量的方法推广并应用于一阶微分方程组,特别是奇数维微分方程组的积分问题.然后,证明 Hojman 定理是本文定理的特殊情形.最后,举例说明结果的应用.

关键词:微分方程, Hojman 定理, 守恒量, 积分
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1. 引 言

Hojman 提出的用时间不变的特殊无限小变换,既不用 Lagrange 函数也不用 Hamilton 函数,而直接导出二阶微分方程组守恒量的方法[1],被广泛应用于各类约束力学系统的积分问题,如相空间运动微分方程的非 Noether 守恒量[2],完整力学系统[3—7],非完整力学系统[8—11], Birkhoff 系统[12,13],广义 Hamilton 系统[14,15],以及时间可变的 Hojman 定理[16]等.以上大多研究涉及偶数维微分方程.实际上, Hojman 方法对奇数维微分方程也可应用.本文证明一般一阶微分方程组的 Hojman 定理,并举例说明定理的应用.

2. 一阶微分方程组的 Hojman 定理

研究一阶微分方程组

$$\dot{x}_i = f_i(t, \boldsymbol{x}) \quad (i = 1, \cdots, m), \tag{1}$$

其中 m 既可为偶数也可为奇数.取时间不变的特殊无限小变换

$$t^* = t, \; x_i^*(t^*) = x_i(t) + \varepsilon\xi_i(t, \boldsymbol{x}), \tag{2}$$

方程(1)在变换(2)下的 Lie 对称性确定方程为

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_i = \frac{\partial f_i}{\partial x_j}\xi_j \quad (i, j = 1, \cdots, m), \tag{3}$$

其中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + f_i\frac{\partial}{\partial x_i}. \tag{4}$$

定理 对于满足方程(3)的生成元 ξ_i,若存在某函数 $\mu = \mu(t, \boldsymbol{x})$ 使得

$$\frac{\partial f_i}{\partial x_i} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{5}$$

则方程(1)有 Hojman 守恒量

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial x_i}(\mu\xi_i) = \mathrm{const}. \tag{6}$$

证明 因

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_i\right) - \frac{\partial\xi_i}{\partial x_j}\frac{\partial f_j}{\partial x_i}, \tag{7}$$

将(3)和(5)式代入(7)式,得

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_i}{\partial x_i} = -\xi_j\frac{\partial}{\partial x_j}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\right), \tag{8}$$

于是

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial x_i}\xi_i\right) - \xi_j\frac{\partial}{\partial x_j}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu\right)\\ &= \frac{\partial}{\partial t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial x_i}\xi_i\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial x_i}\xi_i\right)f_j\\ &\quad - \xi_j\frac{\partial}{\partial x_j}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial t}\right) - \xi_j\frac{\partial}{\partial x_j}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial x_i}f_i\right)\\ &= \frac{1}{\mu}\frac{\partial\mu}{\partial x_i}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_i - \frac{\partial f_i}{\partial x_j}\xi_j\right) = 0.\end{aligned} \tag{9}$$

证毕.

上述 Hojman 定理可用于求一阶微分方程组的积分,不论方程的维数是偶数还是奇数.应用这个定理的主要困难是求解方程(3)和(5).若能找到一部分积分,则可以使方程降阶;若能找到全部积分,则找到了方程的解.

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3. 关于原 Hojman 定理

现在证明,Hojman 原来的定理是上述定理的特殊情形.

研究二阶方程组

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \cdots, n). \tag{10}$$

令

$$x_s = q_s, x_{n+s} = \dot{q}_s, \tag{11}$$

则方程(10)可以表示为一阶方程组

$$\begin{aligned} &\dot{x}_s = x_{n+s}, \\ &\dot{x}_{n+s} = \alpha_s(t, x_k, x_{n+k}), \\ &(s, k = 1, \cdots, n). \end{aligned} \tag{12}$$

确定方程(3)给出

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \xi_{n+s}, \\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_{n+s} = \frac{\partial \alpha_s}{\partial x_k}\xi_k + \frac{\partial \alpha_s}{\partial x_{n+k}}\xi_{n+k}, \end{aligned} \tag{13}$$

即

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial x_k}\xi_k + \frac{\partial \alpha_s}{\partial x_{n+k}}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{14}$$

方程(5)给出

$$\frac{\partial \alpha_s}{\partial x_{n+s}} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{15}$$

而守恒量(6)式给出

$$\begin{aligned} I_{\mathrm{H}} &= \frac{1}{\mu}\frac{\partial}{\partial x_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial x_{n+s}}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) \\ &= \mathrm{const}. \end{aligned} \tag{16}$$

(14)—(16)式就是 Hojman 原来定理的结果.这就证明,原 Hojman 定理是本文定理的特例.原 Hojman 定理仅用于偶数维方程,而本文定理还可用于奇数维方程.

4. 算 例

用 Hojman 定理求解三阶方程

$$(t+1)^3\dddot{x} - 3(t+1)^2\ddot{x} + 4(t+1)\dot{x} - 4x = 0. \tag{17}$$

令

$$x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}, \tag{18}$$

则方程(17)可以表示为一阶方程组

$$\begin{aligned} &\dot{x}_1 = x_2, \dot{x}_2 = x_3, \\ &\dot{x}_3 = \frac{3}{t+1}x_3 - \frac{4}{(t+1)^2}x_2 + \frac{4}{(t+1)^3}x_1. \end{aligned} \tag{19}$$

Lie 对称性的确定方程(3)给出

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 = \xi_2, \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = \xi_3, \\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 = \frac{3}{t+1}\xi_3 - \frac{4}{(t+1)^2}\xi_2 \\ &\qquad + \frac{4}{(t+1)^3}\xi_1. \end{aligned} \tag{20}$$

方程(5)给出

$$\frac{3}{t+1} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0. \tag{21}$$

方程(20)有解

$$\xi_1 = t+1, \xi_2 = 1, \xi_3 = 0, \tag{22}$$

$$\begin{aligned} &\xi_1 = (t+1)\ln(t+1), \\ &\xi_2 = \ln(t+1) + 1, \\ &\xi_3 = (t+1)^{-1}, \end{aligned} \tag{23}$$

$$\begin{aligned} &\xi_1 = (t+1)^4, \\ &\xi_2 = 4(t+1)^3, \\ &\xi_3 = 12(t+1)^2. \end{aligned} \tag{24}$$

可以找到方程(21)的如下解:

$$\mu = (t+1)^{-3}, \tag{25}$$

$$\begin{aligned} \mu = {} & (t+1)^{-3} \times \frac{1}{9}(t+1)^{-1} \\ & \times \{x_1[12\ln(t+1) + 8] - x_2(t+1) \\ & \times [12\ln(t+1) - 1] + x_3(t+1)^2 \\ & \times [3\ln(t+1) - 1]\}, \end{aligned} \tag{26}$$

$$\begin{aligned} \mu = {} & (t+1)^{-3} \times \frac{1}{3}(t+1)^{-1}\{-4x_1 \\ & + 4(t+1)x_2 - (t+1)^2x_3\}, \end{aligned} \tag{27}$$

$$\begin{aligned} \mu = {} & (t+1)^{-3} \times \frac{1}{9}(t+1)^{-3}\{x_1(t+1)^{-1} \\ & - x_2 + x_3(t+1)\}. \end{aligned} \tag{28}$$

将(22),(26)式代入守恒量(6)式得

$$\begin{aligned} I_{\mathrm{H}} = {} & 9(t+1)\{x_1[12\ln(t+1) + 8] \\ & - x_2(t+1)[12\ln(t+1) - 1] \\ & + x_3(t+1)^2[3\ln(t+1) - 1]\}^{-1} \\ = {} & \frac{1}{C_1}, \end{aligned} \tag{29}$$

将(23),(27)式代入(6)式得

$$\begin{aligned} I_{\mathrm{H}} = {} & 3(t+1)\{-4x_1 + 4(t+1)x_2 \\ & - (t+1)^2x_3\}^{-1} \\ = {} & \frac{1}{C_2}. \end{aligned} \tag{30}$$

将(24),(28)式代入(6)式得

$$I_{\mathrm{H}} = 9(t+1)^3\{x_1(t+1)^{-1} - x_2$$

$$+ x_3(t+1)\}^{-1}$$

$$= \frac{1}{C_3}, \tag{31}$$

由(29)—(31)式解得

$$x_1 = x = C_1(t+1) + C_2(t+1) \times \ln(t+1) + C_3(t+1)^4, \tag{32}$$

这就是方程(17)的通解.

为得到上述结果,也可以适当选取 ξ_1,ξ_2,ξ_3 为 t 和 x_1,x_2,x_3 的函数.若仅用(22)—(25)式,则得到平凡守恒量.

5.结　　论

本文将 Hojman 定理推广并应用于解一般的一阶微分方程组.原 Hojman 定理是本文定理的一个特殊情形.原 Hojman 定理不能应用于解奇数维微分方程,本文的定理则可以.应用本文定理的关键是适当选取生成元 ξ_i 和函数 μ,使得可以找到非平凡的守恒量.

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Hojman method for solving differential equations*

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Abstract

First, the method presented by Hojman for finding the conserved quantity of the system of second order differential equations is generalized and applied to the system of first order differential equations, particularly to the odd-dimensional system for finding the integral in this paper. Next, it is proved that the Hojman theorem is a special case of the theorem given in this paper. Finally, an example is given to illustrate the application of the result.

Keywords: differential equation, Hojman theorem, conserved quantity, integral

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