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事件空间中非完整非保守系统的守恒量存在定理及其逆定理*

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提出了事件空间中非完整非保守系统守恒定律构成的一般途径.首先,列写系统的运动微分方程,给出积分因子的定义.其次,详细地研究了守恒量存在的必要条件,并建立了事件空间中非完整非保守系统的守恒量存在定理及其逆定理.最后,举例说明结果的应用.

关键词:事件空间,非完整非保守系统,积分因子,守恒定理

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1. 引 言

Synge 在他的著作中研究了各种空间(包括事件空间)中的完整保守系统动力学[1].这种研究不仅有几何意义,而且有力学意义.Rumyantsev 发展了这一思想,得到事件空间中非完整系统带乘子的参数方程[2].梅凤翔研究了参数方程的场积分方法[3].

力学系统守恒量的研究,不仅具有数学的重要性,而且表现为深刻的物理规律,它已成为近代分析力学中的一个重要研究方向.Mei 研究了事件空间中非完整系统的 Noether 对称性和 Lie 对称性[4],Li 给出了事件空间中变质量非完整系统的守恒律[5],Mei 和 Xu 等人分别建立了事件空间中完整系统的 Noether 对称性,Lie 对称性,形式不变性[6]及 Hojman 守恒量[7].

1984 年,Djukic′提出了构造保守力学系统守恒律的积分因子方法[8].该方法与上述几种著名方法相比,限制条件少、容易运算、有广泛的应用价值,在分析力学的积分理论中非常有发展前途.近年来,这种方法的研究已有进展[9—13].

本文通过运动微分方程积分因子的概念来寻求事件空间中非完整非保守系统的守恒量.

2. 系统的运动方程

事件空间中非完整系统的 Routh 型方程[4]为

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Lambda}{\partial x'_\alpha}-\frac{\partial\Lambda}{\partial x_\alpha}=P_\alpha+\lambda_\beta\frac{\partial F_\beta}{\partial x'_\alpha}$$
$$(\alpha=1,2,\cdots,n+1),\tag{1}$$

式中 $x_1=t,x_{s+1}=q_s,(s=1,2,\cdots,n),\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau}=x'_\alpha,\lambda_\beta$ 为约束乘子.而 $\Lambda(x_\alpha,x'_\alpha)$ 为事件空间中的 Lagrange 函数,它与位形空间中的 Lagrange 函数 L 有关系,即

$$\Lambda(x_\alpha,x'_\alpha)=x'_1L\left(x_\alpha,\frac{x'_{s+1}}{x'_1}\right),\tag{2}$$

P_α 为事件空间中的非势广义力,它与位形空间中的非势广义力 Q_s 有关系,即

$$P_1=-Q''_sx'_{s+1},$$
$$P_{s+1}=x'_1Q''_s\left(x_\alpha,\frac{x'_{s+1}}{x'_1}\right),(s=1,2,\cdots,n),\tag{3}$$

F_β 为事件空间中的非完整约束,它与位形空间中非完整约束有关系,即

$$F_\beta(x_\alpha,x'_\alpha)=f_\beta\left(x_\alpha,\frac{x'_{s+1}}{x'_1}\right)=0.\tag{4}$$

式中采用 Einstein 求和约定,全文下同.

由方程(1)和(4)可求出 λ_β 作为 x_α,x'_α 的函数,于是方程(1)可写为

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$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}=P_\alpha+R_\alpha,(\alpha=1,2,\cdots,n+1),\tag{5}$$

其中

$$R_\alpha=R_\alpha(x_\alpha,x'_\alpha)=\lambda_\beta(x_\alpha,x'_\alpha)\frac{\partial F_\beta}{\partial x'_\alpha}.\tag{6}$$

方程(5)称为事件空间中与非完整系统(1)和(4)相应的完整系统的运动方程.只要对初始条件施加非完整约束(4)的限制,则非完整系统(1)和(4)的积分,可通过相应完整系统(5)求得.

3. 积分因子与守恒量存在定理

3.1. 积分因子

如果不变式$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right]\xi_\alpha$恒等地变为

$$\begin{aligned}&\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right]\xi_\alpha\\ \equiv&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial \Lambda}{\partial x'_\alpha}\xi_\alpha+G\right]+\mu_\alpha\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right],\end{aligned}\tag{7}$$

式中G,μ_α是τ,x_β,x'_β的函数.则称$\xi_\alpha=\xi_\alpha(\tau,x_\beta,x'_\beta)$为运动方程(5)的积分因子.

3.2. 守恒量存在定理

合并(5)式和(7)式,得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial \Lambda}{\partial x'_\alpha}\xi_\alpha+G\right]\\ &=-\mu_\alpha\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right].\end{aligned}\tag{8}$$

定理1 如果函数ξ_α是方程(5)的积分因子,则对应于非完整系统(1)和(4)的完整系统(5)存在守恒量(初积分)为

$$I=\frac{\partial \Lambda}{\partial x'_\alpha}\xi_\alpha+G=\mathrm{const},(\alpha=1,2,\cdots,n+1).\tag{9}$$

证明 将(9)式对τ求导数,并考虑方程(5)和(8),有

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\tau}&=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\partial \Lambda}{\partial x'_\alpha}\xi_\alpha+G\right]\\ &=-\mu_\alpha\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right]=0.\end{aligned}\tag{10}$$

对于已知的动力学系统,如果函数ξ_α是方程(5)的积分因子,则每一组函数ξ_α,G和μ_α必定满足必要条件(8).这个条件可写为

$$\frac{\partial \Lambda}{\partial x'_\alpha}\xi'_\alpha+\frac{\partial \Lambda}{\partial x_\alpha}\xi_\alpha+(P_\alpha+R_\alpha)\xi_\alpha+G'+\Phi=0,\tag{11}$$

其中

$$\Phi=\mu_\alpha\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \Lambda}{\partial x'_\alpha}-\frac{\partial \Lambda}{\partial x_\alpha}-P_\alpha-R_\alpha\right].\tag{12}$$

显然,如果函数组ξ_α,G和μ_α满足必要条件(11),并使(9)式的等号右边成为常数时,有下面的定理.

定理2 对于满足必要条件(11)的每个非奇异函数组ξ_α,G和μ_α存在与非完整系统(1)和(4)相应的完整系统(5)的守恒量(9).

通过方程(11)的积分,可以给出一个守恒量的非奇异函数组ξ_α,G和μ_α.在任何情况下,如果得到方程(11)的任意解,其中不包含任何积分常数,我们称此解为方程(11)的一个函数解.根据定理2,可以得到相应完整系统(5)的一个守恒量(初积分).

4. Killing 方程

利用积分因子理论寻求动力学系统的守恒量,关键在于找到函数$\xi_\alpha=\xi_\alpha(\tau,x_\varepsilon,x'_\varepsilon)$,$G=G(\tau,x_\alpha,x'_\alpha)$和$\mu_\alpha=\mu_\alpha(\tau,x_\varepsilon,x'_\varepsilon)$.通常将必要条件(11)展开,由于函数$\xi_\alpha$,$G$和$\mu_\alpha$不依赖$x''_\varepsilon$,因此含$x''_\varepsilon$的项的系数和不含$x''_\varepsilon$的项分别等于零,可以将方程(11)分离成$(n+2)$个线性偏微分方程:

$$\begin{aligned}&\frac{\partial \Lambda}{\partial x_\alpha}\xi_\alpha+\frac{\partial \Lambda}{\partial x'_\alpha}\left[\frac{\partial \xi_\alpha}{\partial \tau}+\frac{\partial \xi_\alpha}{\partial x_\varepsilon}x'_\varepsilon\right]+(P_\alpha+R_\alpha)\xi_\alpha\\ &+\frac{\partial G}{\partial \tau}+\frac{\partial G}{\partial x_\alpha}x'_\alpha+\mu_\alpha\left[\frac{\partial^2 \Lambda}{\partial x'_\alpha\partial \tau}+\frac{\partial^2 \Lambda}{\partial x'_\alpha\partial x_\varepsilon}x'_\varepsilon\right.\\ &\left.-P_\alpha-R_\alpha\right]=0,\end{aligned}\tag{13}$$

$$\frac{\partial \Lambda}{\partial x'_\alpha}\frac{\partial \xi_\alpha}{\partial x'_\varepsilon}+\frac{\partial G}{\partial x'_\varepsilon}+\mu_\alpha\frac{\partial^2 \Lambda}{\partial x'_\alpha\partial x'_\varepsilon}=0,$$

$$(\alpha,\varepsilon=1,2,\cdots,n+1).\tag{14}$$

方程(13)和(14)是关于$(2n+3)$个未知函数ξ_α,G和μ_α的$(n+2)$个方程.因此,方程组的解法具有大的自由度.最合乎逻辑的解法是首先指定$(n+2)$个函数,然后对其余的$(n+1)$个未知函数解方程组.通常把函数G和μ_α视为指定的函数.

5. 逆定理

对于相应完整系统(5).如果已知系统的初积

分,于是从积分因子的逆定理可以找到函数 G,μ_α 和积分因子 ξ_α.

积分因子理论建立的动力学系统的初积分(9)与对应的积分因子 ξ_α 和函数 G,μ_α 之间的关系必须同必要条件(11)相容.这表明必要条件(11)与方程(9),(13)和(14)等价.

假设相应的完整系统(5)有一个初积分

$$I = I(\tau, x_\alpha, x'_\alpha) = \text{const}. \tag{15}$$

从(9)式计算 $\frac{\partial G}{\partial x'_\varepsilon}$,有

$$\frac{\partial G}{\partial x'_\varepsilon} = \frac{\partial I}{\partial x'_\varepsilon} - \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x'_\varepsilon}\xi_\alpha - \frac{\partial \Lambda}{\partial x'_\alpha}\frac{\partial \xi_\alpha}{\partial x'_\varepsilon}, \tag{16}$$

将(16)式代入方程(14),得

$$\frac{\partial \Lambda}{\partial x'_\alpha \partial x'_\varepsilon}\xi_\alpha - \frac{\partial I}{\partial x'_\varepsilon} - \mu_\alpha \frac{\partial^2 \Lambda}{\partial x'_\alpha \partial x'_\varepsilon} = 0. \tag{17}$$

其次,令初积分(15)等于守恒量(9),即

$$\frac{\partial \Lambda}{\partial x'_\alpha}\xi_\alpha + G = I. \tag{18}$$

从方程(17)和(18)可以得到 ξ_α,G 和 μ_α.于是,有下面的定理.

定理 3(逆定理)　如果相应完整系统(5)有一个初积分(15),则与积分(15)相应的积分因子 ξ_α 和函数 G,μ_α 由方程(17)和(18)确定.

6. 举　例

已知力学系统在位形空间中的 Lagrange 函数为

$$L = \frac{1}{2}m(\dot q_1^2 + \dot q_2^2 + \dot q_3^2) - mgq_3, \tag{19}$$

非完整约束为

$$f = \dot q_1^2 + \dot q_2^2 - \dot q_3^2 = 0. \tag{20}$$

在事件空间中试求系统的守恒量.

首先,研究正问题.求系统的守恒量.

第一步,建立事件空间中系统的运动方程.

令 $x_1 = t$,$x_2 = q_1$,$x_3 = q_2$,$x_4 = q_3$ 则有

$$\Lambda = \frac{1}{2}\frac{m}{x'_1}[(x'_2)^2 + (x'_3)^2 + (x'_4)^2] - mgx'_1 x'_4, \tag{21}$$

$$F = \frac{1}{(x'_1)^2}[(x'_2)^2 + (x'_3)^2 - (x'_4)^2] = 0, \tag{22}$$

$$P_1 = P_2 = P_3 = P_4 = 0. \tag{23}$$

运动方程(1)的后面三个方程给出:

$$m\left(\frac{x'_2}{x'_1}\right)' = \lambda \frac{2x'_2}{(x'_1)^2},$$

$$m\left(\frac{x'_3}{x'_1}\right)' = \lambda \frac{2x'_3}{(x'_1)^2},$$

$$m\left(\frac{x'_4}{x'_1}\right)' + mgx'_1 = -\lambda \frac{2x'_4}{(x'_1)^2}. \tag{24}$$

由方程(22)和(24)解得

$$\lambda = -\frac{mg(x'_1)^3}{4x'_4}. \tag{25}$$

将(25)式代入方程(24),有

$$m\left(\frac{x'_2}{x'_1}\right)' = -\frac{mgx'_1 x'_2}{2x'_4},$$

$$m\left(\frac{x'_3}{x'_1}\right)' = -\frac{mgx'_1 x'_3}{2x'_4},$$

$$m\left(\frac{x'_4}{x'_1}\right)' + mgx'_1 = \frac{mgx'_1}{2}. \tag{26}$$

或写为

$$\frac{mx''_2}{x'_1} - \frac{mx'_2 x''_1}{(x'_1)^2} = -\frac{mgx'_1 x'_2}{2x'_4},$$

$$\frac{mx''_3}{x'_1} - \frac{mx'_3 x''_1}{(x'_1)^2} = -\frac{mgx'_1 x'_3}{2x'_4},$$

$$\frac{mx''_4}{x'_1} - \frac{mx'_4 x''_1}{(x'_1)^2} + mgx'_1 = \frac{mgx'_1}{2}. \tag{27}$$

对照方程(5),得

$$P_2 + R_2 = -\frac{mgx'_1 x'_2}{2x'_4}, P_3 + R_3 = -\frac{mgx'_1 x'_3}{2x'_4},$$

$$P_4 + R_4 = \frac{1}{2}mgx'_1. \tag{28}$$

第二步,建立必要条件(11)并求解.

$$\begin{aligned}
&- mgx'_1\xi_4 + \frac{m}{x'_1}x'_2\xi'_2 + \frac{m}{x'_1}x'_3\xi'_3 + \frac{m}{x'_1}x'_4\xi'_4 \\
&+ \left\{-\frac{m}{2(x'_1)^2}[(x'_2)^2 + (x'_3)^2 + (x'_4)^2] - mgx_4\right\}\xi'_1 \\
&+ \left(-\frac{mgx'_1 x'_2}{2x'_4}\right)\xi_2 + \left(-\frac{mgx'_1 x'_3}{2x'_4}\right)\xi_3 + \frac{1}{2}mgx'_1\xi_4 \\
&+ G' + \mu_2\left[\frac{x''_2}{x'_1} - \frac{x'_2 x''_1}{(x'_1)^2} + \frac{gx'_1 x'_2}{2x'_4}\right] \\
&+ \mu_3\left[\frac{x''_3}{x'_1} - \frac{x'_3 x''_1}{(x'_1)^2} + \frac{gx'_1 x'_3}{2x'_4}\right] \\
&+ \mu_4\left[\frac{x''_4}{x'_1} - \frac{x'_4 x''_1}{(x'_1)^2} + gx'_1 - \frac{gx'_1}{2}\right] = 0.
\end{aligned} \tag{29}$$

必要条件(29)有解

$$\begin{aligned}
&\xi_1 = 1, \xi_2 = \xi_3 = \xi_4 = 0, \\
&\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0, \\
&G_1 = 0.
\end{aligned} \tag{30}$$

$$\xi_4 = 1, \xi_1 = \xi_2 = \xi_3 = 0,$$

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0,$$

$$G_2 = \frac{1}{2} mgx_1 . \tag{31}$$

将方程(30)和(31)分别代入(9)式,得守恒量为

$$I_1 = -\frac{m}{2(x_1')^2}[(x_2')^2 + (x_3')^2 + (x_4')^2] - mgx_4 = \text{const}, \tag{32}$$

$$I_2 = \frac{mx_4'}{x_1'} + \frac{1}{2} mgx_1 = \text{const}. \tag{33}$$

其次,研究逆问题.

假设系统有初积分(33),方程(17)和(18)分别给出

$$\begin{aligned} &\frac{m}{(x_1')^3}[(x_2')^2 + (x_3')^2 + (x_4')^2]\xi_1 \\ &+ \left[-\frac{mx_2'}{(x_1')^2}\right]\xi_2\left[-\frac{mx_3'}{(x_1')^2}\right]\xi_3 \\ &+ \left[-\frac{mx_4'}{(x_1')^2}\right]\xi_4 + \frac{mx_4'}{(x_1')^2} \\ &- \mu_1\left\{\frac{m}{(x_1')^3}[(x_2')^2 + (x_3')^2 + (x_4')^2]\right\} \\ &- \mu_2\left[-\frac{mx_2'}{(x_1')^2}\right] - \mu_3\left[-\frac{mx_3'}{(x_1')^2}\right] \\ &- \mu_4\left[-\frac{mx_4'}{(x_1')^2}\right] = 0, \\ &\left[-\frac{mx_2'}{(x_1')^2}\right]\xi_1 + \frac{m}{x_1'}\xi_2 = \left[-\frac{mx_2'}{(x_1')^2}\right]\mu_1 + \frac{m}{x_1'}\mu_2, \\ &\left[-\frac{mx_3'}{(x_1')^2}\right]\xi_1 + \frac{m}{x_1'}\xi_3 = \left[-\frac{mx_3'}{(x_1')^2}\right]\mu_1 + \frac{m}{x_1'}\mu_3, \\ &\left[-\frac{mx_4'}{(x_1')^2}\right]\xi_1 + \frac{m}{x_1'}\xi_4 = \left[-\frac{mx_2'}{(x_1')^2}\right]\mu_1 + \frac{m}{x_1'}\mu_4. \end{aligned} \tag{34}$$

$$\begin{aligned} &\xi_1\left\{\frac{m}{2(x_1')^2}[(x_2')^2 + (x_3')^2 + (x_4')^2] - mgx_4\right\} \\ &+ \xi_2\left(\frac{mx_2'}{x_1'}\right) + \xi_3\left(\frac{mx_3'}{x_1'}\right) + \xi_4\left(\frac{mx_4'}{x_1'}\right) + G \\ &= \frac{mx_4'}{x_1'} + \frac{1}{2} mgx_1 . \end{aligned} \tag{35}$$

当取 $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0, G = \frac{1}{2} mgx_4$ 时,由方程(34)和(35)得

$$\xi_1 = \xi_2 = \xi_3 = 0, \xi_4 = 1. \tag{36}$$

这与文献[4]的结果一致.

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Existence theorem and its converse of conserved quantities for the nonholonomic nonconservative systems in the event space *

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Abstract

A general approach to the construction of conservation laws for nonholonomic nonconservative dynamical systems in the event space was presented. Firstly, the differential equations of notion of systems are written, and the definition of integrating factors is given. Next, the necessary conditions for the existence of the conserved quantities are studied in detail. Finally, the existence theorem and its converse of conserved quantities for the nonholonomic nonconservative dynamical systems in the event space are established, and an example is given to illustrate the application of the result.

Keywords: event space, nonholonomic nonconservative system, integrating factor, conservation theorem
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