第 55 卷 第 12 期 2006 年 12 月
1000-3290/2006/55(12)/6214-08

物 理 学 报
ACTA PHYSICA SINICA

Vol.55,No.12,December,2006

构造非线性发展方程精确解的一种方法*

套格图桑† 斯仁道尔吉

(内蒙古师范大学数学科学学院,呼和浩特 010022)

(2006 年 3 月 9 日收到;2006 年 7 月 31 日收到修改稿)

在双曲正切函数法、齐次平衡法、辅助方程法的基础上引入非线性发展方程的一个新形式解和新辅助方程,并利用符号计算系统 Mathematica 构造了 Benjamin-Bona-Mahoney(BBM) 方程和修正的 BBM 方程的新精确孤立波解.这种方法在寻找其他非线性发展方程的新精确解方面具有普遍意义.

关键词:新辅助方程,形式解,非线性发展方程,精确孤立波解

PACC:0340K,0290

1. 引 言

在孤立子理论的研究领域内构造非线性发展方程(组)的精确解长期以来一直得到数学、物理学家的关注.到目前为止人们已经发展了基于计算机代数的许多直接方法,如双曲正切函数法[1]、齐次平衡法[2]、Jacobi 椭圆函数展开法[3]、辅助方程法[4]等.文献[5,6]用 Riccati 方程与多种形式解相结合的方法构造了精确解.文献[7,8]用不同的方法构造了 Benjamin-Bona-Mahoney(BBM)方程和修正的 BBM(mBBM)方程的精确解.本文在文献[1—8]的基础上给出新辅助方程与非线性发展方程的一种新形式解相结合的方法,借助符号计算系统 Mathematica 构造了 BBM 方程和 mBBM 方程的新精确解.这种方法构造非线性发展方程(组)的精确解具有普遍意义.

2. 新辅助方程的应用步骤及其新形式解

对于给定的非线性发展方程(以 $1+1$ 维非线性发展方程为例)

$$H(u,u_x,u_t,u_{xx},u_{xt},u_{tt},\cdots)=0, \tag{1}$$

进行行波变换 $u(x,t)=u(\xi)$,$\xi=x+\omega t$ 后得到常微分方程

$$G(u,u_\xi,u_{\xi\xi},u_{\xi\xi\xi},\cdots)=0. \tag{2}$$

假设方程(2)具有下列一种新形式解:

$$u(\xi)=g_0+\sum_{j=1}^{m}\left(\frac{z(\xi)}{1+z^2(\xi)}\right)^{j-1}\times\left(\frac{g_j z(\xi)}{1+z^2(\xi)}+\frac{f_j(1-z^2(\xi))}{1+z^2(\xi)}\right). \tag{3}$$

式中,g_i $(i=0,1,2,\cdots,m)$,f_j $(j=1,2,\cdots,m)$为待定常数.这里,m 为由领头项分析法[9]确定的自然数,$z(\xi)$是下列新辅助方程的解:

$$\left(\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right)^2=(1+z(\xi))^2(az^2(\xi)+bz(\xi)+c). \tag{4}$$

我们得到了方程(4)的双曲函数解、三角函数解、指数函数解和双周期 Weierstrass 椭圆函数解.

2.1. 双曲函数解

方程(4)的双曲函数解为

$$z(\xi)=\frac{-b+2\sqrt{c}\left(\sqrt{c}-\sqrt{M}\tanh\left(\frac{1}{2}\sqrt{M}\xi\right)\right)}{a-\left(\sqrt{c}-\sqrt{M}\tanh\left(\frac{1}{2}\sqrt{M}\xi\right)\right)^2}\quad(M>0,c\geqslant 0), \tag{5}$$

$$z(\xi)=\frac{-c+\left(\sqrt{a}-\sqrt{M}\tanh\left(\frac{1}{2}\sqrt{M}\xi\right)\right)^2}{b-2\sqrt{a}\left(\sqrt{a}-\sqrt{M}\tanh\left(\frac{1}{2}\sqrt{M}\xi\right)\right)}\quad(M>0,a\geqslant 0), \tag{6}$$

* 国家自然科学基金(批准号:10461006)、内蒙古自治区高等学校科学研究基金(批准号:NJ02035)、内蒙古自治区自然科学基金(批准号:200408020103)和内蒙古师范大学自然科学研究计划(批准号:QN005023)资助的课题.

† E-mail:tgts@imnu.edu.cn

$$z(\xi)=\frac{b-2\sqrt{c}\left(\sqrt{c}+\sqrt{M}\coth\left(\frac{1}{2}\sqrt{M}\xi\right)\right)}{-a+\left(\sqrt{c}+\sqrt{M}\coth\left(\frac{1}{2}\sqrt{M}\xi\right)\right)^{2}}\quad(M>0,c\geqslant 0),\tag{7}$$

$$z(\xi)=\frac{-c+\left(\sqrt{a}-\sqrt{M}\coth\left(\frac{1}{2}\sqrt{M}\xi\right)\right)^{2}}{b+2\sqrt{a}\left(\sqrt{a}+\sqrt{M}\coth\left(\frac{1}{2}\sqrt{M}\xi\right)\right)}\quad(M>0,a\geqslant 0),\tag{8}$$

$$z(\xi)=\frac{b-2c-2\sqrt{cM}(\tanh(\sqrt{M}\xi)\pm \mathrm{i}\,\mathrm{sech}(\sqrt{M}\xi))}{-a+[\sqrt{c}+\sqrt{M}(\tanh(\sqrt{M}\xi)\pm \mathrm{i}\,\mathrm{sech}(\sqrt{M}\xi))]^{2}}\quad(M>0,c\geqslant 0),\tag{9}$$

$$z(\xi)=\frac{b-\sqrt{c}\left[2\sqrt{c}+\sqrt{M}\left(\coth\left(\frac{1}{4}\sqrt{M}\xi\right)+\tanh\left(\frac{1}{4}\sqrt{M}\xi\right)\right)\right]}{-a+\frac{1}{4}\left[2\sqrt{c}+\sqrt{M}\left(\coth\left(\frac{1}{4}\sqrt{M}\xi\right)+\tanh\left(\frac{1}{4}\sqrt{M}\xi\right)\right)\right]^{2}}\quad(M>0,c\geqslant 0),\tag{10}$$

$$z(\xi)=\frac{b-\dfrac{2\sqrt{c}\left(R\sqrt{c}+\sqrt{M(R^{2}-P^{2})}+P\sqrt{c}\cosh(\sqrt{M}\xi)+P\sqrt{M}\sinh(\sqrt{M}\xi)\right)}{R+P\cosh(\sqrt{M}\xi)}}{-a+\left[\dfrac{R\sqrt{c}+\sqrt{M(R^{2}-P^{2})}+P\sqrt{c}\cosh(\sqrt{M}\xi)+P\sqrt{M}\sinh(\sqrt{M}\xi)}{R+P\cosh(\sqrt{M}\xi)}\right]^{2}}$$
$$(M>0,c\geqslant 0,R^{2}-P^{2}\geqslant 0),\tag{11}$$

$$z(\xi)=\frac{b-\dfrac{2\sqrt{c}\left(R\sqrt{c}-\sqrt{M(R^{2}+P^{2})}+P\sqrt{M}\cosh(\sqrt{M}\xi)+P\sqrt{c}\sinh(\sqrt{M}\xi)\right)}{R+P\sinh(\sqrt{M}\xi)}}{-a+\left[\dfrac{R\sqrt{c}-\sqrt{M(R^{2}+P^{2})}+P\sqrt{M}\cosh(\sqrt{M}\xi)+P\sqrt{c}\sinh(\sqrt{M}\xi)}{R+P\sinh(\sqrt{M}\xi)}\right]^{2}}$$
$$(M>0,c\geqslant 0),\tag{12}$$

$$z(\xi)=\frac{b+\dfrac{4(a-b)\sqrt{c}\cosh(\sqrt{M}\xi)}{\pm 2\mathrm{i}\sqrt{M}+2\sqrt{c}\cosh(\sqrt{M}\xi)-2\sqrt{M}\sinh(\sqrt{M}\xi)}}{-a+\left[\dfrac{2(a-b)\cosh(\sqrt{M}\xi)}{\pm 2\mathrm{i}\sqrt{M}+2\sqrt{c}\cosh(\sqrt{M}\xi)-2\sqrt{M}\sinh(\sqrt{M}\xi)}\right]^{2}}$$
$$(M>0,c\geqslant 0,a-b>0),\tag{13}$$

$$z(\xi)=\frac{b-\dfrac{4(a-b)\sqrt{c}\sinh(\sqrt{M}\xi)}{\pm 2\sqrt{M}+2\sqrt{M}\cosh(\sqrt{M}\xi)-2\sqrt{c}\sinh(\sqrt{M}\xi)}}{-a+\left[\dfrac{2(a-b)\sinh(\sqrt{M}\xi)}{\pm 2\sqrt{M}-2\sqrt{M}\cosh(\sqrt{M}\xi)+2\sqrt{c}\sinh(\sqrt{M}\xi)}\right]^{2}}$$
$$(M>0,c\geqslant 0,a-b>0),\tag{14}$$

$$z(\xi)=\frac{-(a-b)(b-2c)+[b^{2}-a(b+2c)]\cosh(\sqrt{M}\xi)+2a\sqrt{cM}\sinh(\sqrt{M}\xi)}{2a^{2}-3ab+b^{2}+[-b^{2}+a(b+2c)]\cosh(\sqrt{M}\xi)-2a\sqrt{Mc}\sinh(\sqrt{M}\xi)}$$
$$(M>0,c\geqslant 0),\tag{15}$$

2.2. 三角函数解

方程(4)的三角函数解为

$$z(\xi)=\frac{b-2\sqrt{c}\left(\sqrt{c}-\sqrt{-M}\tan\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)}{-a+\left(-\sqrt{c}+\sqrt{-M}\tan\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)^{2}}\quad(M<0,c\geqslant 0),\tag{16}$$

$$z(\xi)=\frac{-c+\left(\sqrt{a}+\sqrt{-M}\tan\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)^2}{b-2\sqrt{a}\left(\sqrt{a}+\sqrt{-M}\tan\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)}\quad(M<0,a\geqslant0),\tag{17}$$

$$z(\xi)=\frac{-b+2\sqrt{c}\left(\sqrt{c}+\sqrt{-M}\cot\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)}{a-\left(\sqrt{c}+\sqrt{-M}\cot\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)^2}\quad(M<0,c\geqslant0),\tag{18}$$

$$z(\xi)=\frac{-c+\left(\sqrt{a}-\sqrt{-M}\cot\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)^2}{b-2\sqrt{a}\left(\sqrt{a}-\sqrt{-M}\cot\left(\frac{1}{2}\sqrt{-M}\xi\right)\right)}\quad(M<0,a\geqslant0),\tag{19}$$

$$z(\xi)=\frac{b-2c+2\sqrt{-Mc}\left(\tan(\sqrt{-M}\xi)\pm\sec(\sqrt{-M}\xi)\right)}{-a+\left[\sqrt{c}-\sqrt{-M}\left(\tan(\sqrt{-M}\xi)\pm\sec(\sqrt{-M}\xi)\right)\right]^2}\quad(M<0,c\geqslant0),\tag{20}$$

$$z(\xi)=\frac{b+\sqrt{c}\left[-2\sqrt{c}+\sqrt{-M}\left(-\cot\left(\frac{1}{4}\sqrt{-M}\xi\right)+\tan\left(\frac{1}{4}\sqrt{-M}\xi\right)\right)\right]}{-a+\frac{1}{4}\left[2\sqrt{c}+\sqrt{-M}\left(\cot\left(\frac{1}{4}\sqrt{-M}\xi\right)-\tan\left(\frac{1}{4}\sqrt{-M}\xi\right)\right)\right]^2}\quad(M<0,c\geqslant0),\tag{21}$$

$$z(\xi)=\frac{b+\dfrac{4(a-b)\sqrt{c}\cos(\sqrt{-M}\xi)}{\pm2\sqrt{-M}+2\sqrt{c}\cos(\sqrt{-M}\xi)+2\sqrt{-M}\sin(\sqrt{-M}\xi)}}{-a+\left[\dfrac{2(a-b)\sqrt{c}\cos(\sqrt{-M}\xi)}{\pm2\sqrt{-M}+2\sqrt{c}\cos(\sqrt{-M}\xi)+2\sqrt{-M}\sin(\sqrt{-M}\xi)}\right]^2}\quad(M<0,c\geqslant0),\tag{22}$$

$$z(\xi)=\frac{b-\dfrac{4(a-b)\sqrt{c}\sin(\sqrt{-M}\xi)}{\pm2\sqrt{-M}+2\sqrt{M}\cos(\sqrt{-M}\xi)-2\sqrt{c}\sin(\sqrt{-M}\xi)}}{-a+\left[\dfrac{2(a-b)\sin(\sqrt{-M}\xi)}{\pm2\sqrt{-M}+2\sqrt{M}\cos(\sqrt{-M}\xi)-2\sqrt{c}\sin(\sqrt{-M}\xi)}\right]^2}\quad(M<0,c\geqslant0),\tag{23}$$

$$z(\xi)=\frac{-(a-b)(b-2c)+[b^2-a(b+2c)]\cos(\sqrt{-M}\xi)-2a\sqrt{-cM}\sin(\sqrt{-M}\xi)}{2a^2-3ab+b^2+[-b^2+a(b+2c)]\cos(\sqrt{-M}\xi)+2a\sqrt{-Mc}\sin(\sqrt{-M}\xi)}$$
$$(M<0,c\geqslant0),\tag{24}$$

其中 $M=b-a-c$.

2.3. 指数函数解

方程(4)的指数函数解为

$$z(\xi)=\frac{\exp(\sqrt{a}\xi)}{1-\exp(\sqrt{a}\xi)}\quad(b=c=0,a>0),\tag{25}$$

$$z(\xi)=-1+\exp(\sqrt{c}\xi)\quad(a=b=0,c>0),\tag{26}$$

2.4. 双周期 Weierstrass 椭圆函数解

方程(4)的双周期 Weierstrass 椭圆函数解为

$$z(\xi)=\wp\left(\frac{\sqrt{b}}{2}\xi,h_1,h_2\right)\quad(a=0,c=-2b,$$
$$b>0,h_1=12,h_2=8).\tag{27}$$

将(3),(4)式代入(2)式,得到 $z(\xi)$的一个多项式. 令 $z(\xi)$的系数为零,可得到一个 g_i ($i=0,1,2,\cdots,m$), f_j ($j=1,2,\cdots,m$), a,b,c,ω 为未知量的非线性代数方程组. 借助符号计算系统 Mathematica 或 Maple 求出该方程组的解,再把该非线性代数方程组的每一组解与(5)—(27)式一起代回(3)式,可得到非线性发展方程(组)的精确解.

3. 方法的应用

3.1. BBM 方程的精确解

考虑文献[7,8]讨论的 BBM 方程[7,8]

$$u_t+c_0u_x+uu_x+\beta u_{xxt}=0,\tag{28}$$

将 $u(x,t)=u(\xi)$, $\xi=x+\omega t$ 代入(28)式并对 ξ 积

分一次后得到下列方程：

$$\alpha u + \frac{1}{2}u^2 + \beta\omega u'' = 0$$
$$(\alpha = \omega + c_0). \tag{29}$$

由领头项分析法，得到平衡常数 $m = 2$. 因此取方程(29)的解

$$u(\xi) = g_0 + \frac{g_1 z(\xi)}{1 + z^2(\xi)} + \frac{f_1(1 - z^2(\xi))}{1 + z^2(\xi)} + \frac{z(\xi)}{1 + z^2(\xi)}\left(\frac{g_2 z(\xi)}{1 + z^2(\xi)} + \frac{f_2(1 - z^2(\xi))}{1 + z^2(\xi)}\right). \tag{30}$$

将(4)和(30)式一起代入(29)式，得到 $z(\xi)$ 的一个多项式. 令 $z(\xi)$ 的系数为零，可得非线性代数方程组(限于篇幅未列出).

将 $f_1 = 0$，$\omega + c_0 = \alpha$ 与符号计算系统 Mathematica 应用于该方程组求出如下解：

$$g_0 = -3(c_0 + \omega),\quad a = c = -\frac{c_0 + \omega}{16\beta\omega},\quad b = \frac{c_0 + \omega}{8\beta\omega},\quad g_2 = 12(c_0 + \omega),\quad f_2 = g_1 = 0; \tag{31}$$

$$g_0 = -\frac{3}{2}(c_0 + \omega),\quad a = c = -\frac{c_0 + \omega}{8\beta\omega},\quad b = 0,\quad g_2 = f_2 = 0,\quad g_1 = -3(c_0 + \omega); \tag{32}$$

$$g_0 = -\frac{5}{4}(c_0 + \omega),\quad a = c = \frac{3(c_0 + \omega)}{64\beta\omega},\quad b = \frac{c_0 + \omega}{32\beta\omega},\quad g_2 = g_1 = 3(c_0 + \omega),\quad f_2 = 0; \tag{33}$$

$$g_0 = -\frac{3}{4}(c_0 + \omega),\quad a = c = -\frac{3(c_0 + \omega)}{64\beta\omega},\quad b = -\frac{c_0 + \omega}{32\beta\omega},\quad g_2 = g_1 = -3(c_0 + \omega),\quad f_2 = 0; \tag{34}$$

$$g_0 = -\frac{1}{2}(c_0 + \omega),\quad a = c = \frac{c_0 + \omega}{8\beta\omega},\quad b = 0,\quad g_2 = f_2 = 0,\quad g_1 = 3(c_0 + \omega); \tag{35}$$

$$g_0 = c_0 + \omega,\quad a = c = \frac{c_0 + \omega}{16\beta\omega},\quad b = -\frac{c_0 + \omega}{8\beta\omega},\quad g_2 = -12(c_0 + \omega),\quad g_1 = f_2 = 0. \tag{36}$$

将(31)—(36)式分别与(5)和(14)式一起代入(30)式(其中 $f_1 = 0$)，可得方程(28)的一组精确孤立波解

$$u_1(x,t) = -3(c_0 + \omega)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c_0 + \omega}{-\beta\omega}}(x + \omega t)\right)\quad ((c_0 + \omega)\beta\omega < 0),$$

$$u_2(x,t) = \frac{6(c_0 + \omega)}{-1 - 3\cosh\left(\sqrt{\frac{c_0 + \omega}{-\beta\omega}}(x + \omega t)\right) + 2\sqrt{2}\sinh\left(\sqrt{\frac{c_0 + \omega}{-\beta\omega}}(x + \omega t)\right)}\quad ((c_0 + \omega)\beta\omega < 0),$$

$$u_3(x,t) = \frac{(c_0 + \omega)\left(2 - 7\cosh\left(\sqrt{\frac{c_0 + \omega}{\beta\omega}}(x + \omega t)\right) + 4\sqrt{3}\sinh\left(\sqrt{\frac{c_0 + \omega}{\beta\omega}}(x + \omega t)\right)\right)}{\left(-2\cosh\left(\frac{1}{2}\sqrt{\frac{c_0 + \omega}{\beta\omega}}(x + \omega t)\right) + \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{\frac{c_0 + \omega}{\beta\omega}}(x + \omega t)\right)\right)^2}$$
$$((c_0 + \omega)\beta\omega > 0),$$

$$u_4(x,t) = -\frac{3(c_0 + \omega)}{\left(-2\cosh\left(\frac{1}{2}\sqrt{\frac{c_0 + \omega}{-\beta\omega}}(x + \omega t)\right) + \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{\frac{c_0 + \omega}{-\beta\omega}}(x + \omega t)\right)\right)^2}$$
$$((c_0 + \omega)\beta\omega < 0),$$

$$u_5(x,t)=\frac{2(c_0+\omega)\left(2-3\cosh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)+2\sqrt{2}\sinh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}{1+3\cosh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)-2\sqrt{2}\sinh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)}$$
$$((c_0+\omega)\beta\omega>0),$$

$$u_6(x,t)=-(c_0+\omega)\left(-2+\cosh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)$$
$$((c_0+\omega)\beta\omega>0),$$

$$u_7(x,t)=-\frac{27(c_0+\omega)}{\left(5\cosh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{-\beta\omega}}(x+\omega t)\right)-4\sinh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{-\beta\omega}}(x+\omega t)\right)\right)^2}$$
$$((c_0+\omega)\beta\omega<0),$$

$$u_8(x,t)=\frac{(c_0+\omega)\left(2-1351\cosh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)+780\sqrt{3}\sinh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}{\left(26\cosh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)-15\sqrt{3}\sinh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)^2}$$
$$((c_0+\omega)\beta\omega>0),$$

$$u_9(x,t)=-\frac{3(c_0+\omega)}{\left(26\cosh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{-\beta\omega}}(x+\omega t)\right)-15\sqrt{3}\sinh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{-\beta\omega}}(x+\omega t)\right)\right)^2}$$
$$((c_0+\omega)\beta\omega<0),$$

$$u_{10}(x,t)=-\frac{(c_0+\omega)\left(-81+41\cosh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)-40\sinh\left(\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}{\left(5\cosh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)-4\sinh\left(\frac{1}{2}\sqrt{\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)^2}$$
$$((c_0+\omega)\beta\omega>0).$$

3.2. mBBM 方程的精确解

下面构造 mBBM 方程[7,8]的精确解.

$$u_t+c_0u_x+\alpha u^2u_x+\beta u_{xxt}=0, \tag{37}$$

将 $u(x,t)=u(\xi)$, $\xi=x+\omega t$ 代入(37)式并对 ξ 积分一次后得到如下方程：

$$\lambda u+\frac{1}{3}\alpha u^3+\beta\omega u''=0\quad(\omega+c_0=\lambda). \tag{38}$$

由领头项分析法,得到平衡常数 $m=1$. 因此,取方程(38)的解

$$u(\xi)=g_0+\frac{g_1z(\xi)}{1+z^2(\xi)}+\frac{f_1(1-z^2(\xi))}{1+z^2(\xi)}. \tag{39}$$

将(4)和(39)式一起代入(38)式,得到 $z(\xi)$ 的一个多项式. 令 $z(\xi)$ 的系数为零,可得如下非线性代数方程组：

$$6\lambda f_1-24c\beta\omega f_1+2\alpha f_1^3+6\lambda g_0+6\alpha f_1^2g_0+6\alpha f_1g_0^2+2\alpha g_0^3+3b\beta\omega g_1+6c\beta\omega g_1=0,$$

$$-36b\beta\omega f_1-72c\beta\omega f_1+6\lambda g_1+6a\beta\omega g_1+12b\beta\omega g_1-30c\beta\omega g_1+6\alpha f_1^2g_1+12\alpha f_1g_0g_1+6\alpha g_0^2g_1=0,$$

$$6\lambda f_1-48a\beta\omega f_1-96b\beta\omega f_1+24c\beta\omega f_1-6\alpha f_1^3+18\lambda g_0-6\alpha f_1^2g_0+6\alpha f_1g_0^2+6\alpha g_0^3+18a\beta\omega g_1-27b\beta\omega g_1-72c\beta\omega g_1+6\alpha f_1g_1^2+6\alpha g_0g_1^2=0,$$

$$-120a\beta\omega f_1+120c\beta\omega f_1+12\lambda g_1-24a\beta\omega g_1-72b\beta\omega g_1-24c\beta\omega g_1-12\alpha f_1^2g_1+12\alpha g_0^2g_1+2\alpha g_1^3=0,$$

$$-6\lambda f_1-24a\beta\omega f_1+96b\beta\omega f_1+48c\beta\omega f_1+6\alpha f_1^3+18\lambda g_0-6\alpha f_1^2g_0-6\alpha f_1g_0^2+6\alpha g_0^3-72a\beta\omega g_1-27b\beta\omega g_1$$

$$
\begin{aligned}
&+18c\beta\omega g_1-6\alpha f_1g_1^2+6\alpha g_0g_1^2\\
=&0,\\
&72a\beta\omega f_1+36b\beta\omega f_1+6\lambda g_1\\
&-30a\beta\omega g_1+12b\beta\omega g_1+6c\beta\omega g_1\\
&+6\alpha f_1^2g_1-12\alpha f_1g_0g_1+6\alpha g_0^2g_1\\
=&0,\\
&-6\lambda f_1+24a\beta\omega f_1-2\alpha f_1^3+6\lambda g_0\\
&+6\alpha f_1^2g_0-6\alpha f_1g_0^2+2\alpha g_0^3\\
&6a\beta\omega g_1+3b\beta\omega g_1\\
=&0.
\end{aligned}
$$

将 $\omega+c_0=\lambda$ **和符号计算系统** Mathematica**,应用于该非线性代数方程组,求出如下解：**

$$
\begin{aligned}
&a=\frac{c_0+\omega}{8\beta\omega},\\
&b=-\frac{c_0+\omega}{4\beta\omega},\\
&c=\frac{c_0+\omega}{8\beta\omega},\\
&g_0=f_1=0,\\
&g_1=\mp\frac{2\sqrt{3(c_0+\omega)}}{\sqrt{-\alpha}}\\
&\qquad(\alpha(c_0+\omega)<0);
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&a=-\frac{3(c_0+\omega)}{16\beta\omega},\\
&b=-\frac{c_0+\omega}{8\beta\omega},\\
&c=-\frac{3(c_0+\omega)}{16\beta\omega},\\
&g_0=-\frac{\sqrt{3(c_0+\omega)}}{\sqrt{-2\alpha}},\\
&f_1=0,\\
&g_1=\mp\frac{\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}}\\
&\qquad(\alpha(c_0+\omega)<0);
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&a=-\frac{c_0+\omega}{4\beta\omega},\\
&b=\frac{c_0+\omega}{2\beta\omega},\\
&c=-\frac{c_0+\omega}{4\beta\omega},\\
&g_0=g_1=0,\\
&f_1=\mp\frac{\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}}\qquad(\alpha(c_0+\omega)<0).
\end{aligned}
\tag{42}
$$

将(40)—(42)式分别与(5)和(15)式一起代入(39)式,可得方程(37)的如下一组精确孤立波解：

$$
u_{11}^{\pm}(x,t)=\pm\frac{2\sqrt{3(c_0+\omega)}\tanh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)}{\sqrt{-\alpha}\left(1+\tanh^2\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)\right)}\qquad(\beta\omega(c_0+\omega)>0,\alpha(c_0+\omega)<0),
$$

$$
u_{12}^{\pm}(x,t)=\pm\frac{\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}\left(-2\cosh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)+\sqrt{3}\sinh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}
$$
$$
(\beta\omega(c_0+\omega)<0,\alpha(c_0+\omega)<0),
$$

$$
u_{13}^{\pm}(x,t)=\pm\frac{\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}}\operatorname{sech}\left(\frac{\sqrt{c_0+\omega}}{\sqrt{-\beta\omega}}(x+\omega t)\right)\qquad(\beta\omega(c_0+\omega)<0,\alpha(c_0+\omega)<0),
$$

$$
u_{14}^{\pm}(x,t)=\mp\frac{2\sqrt{3(c_0+\omega)}\left(2\cosh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)-\sinh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)\right)}{\sqrt{-\alpha}\left(5\cosh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)-4\sinh\left(\frac{\sqrt{c_0+\omega}}{\sqrt{2\beta\omega}}(x+\omega t)\right)\right)}
$$
$$
\times\left(\cosh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)-2\sinh\left(\frac{\sqrt{c_0+\omega}}{2\sqrt{2\beta\omega}}(x+\omega t)\right)\right)
$$
$$
(\beta\omega(c_0+\omega)>0,\alpha(c_0+\omega)<0),
$$

$$u_{15}^{\pm}(x,t)=\pm\frac{\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}\left(-26\cosh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)+15\sqrt{3}\sinh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}$$

$$(\beta\omega(c_0+\omega)<0,\alpha(c_0+\omega)<0),$$

$$u_{16}^{\pm}(x,t)=\pm\frac{3\sqrt{6(c_0+\omega)}}{\sqrt{-\alpha}\left(5\cosh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)-4\sinh\left(\sqrt{-\frac{c_0+\omega}{\beta\omega}}(x+\omega t)\right)\right)}$$

$$(\beta\omega(c_0+\omega)<0,\alpha(c_0+\omega)<0).$$

4. 结 论

本文给出了新辅助方程与非线性发展方程的一种新形式解相结合的方法，借助符号计算系统Mathematica构造了BBM方程和mBBM方程的许多精确解. 其中形如$u_1(x,t)$，$u_6(x,t)$，$u_{13}^{\pm}(x,t)$的精确解以前已得到，其余精确解是本文新得到的. 下面给出的形式解与辅助方程(4)相结合，也能得到非线性发展方程的新精确解.

$$u(\xi)=g_0+\sum_{j=1}^{m}\left(\frac{1-z^2(\xi)}{1+z^2(\xi)}\right)^{j-1}\times\left(\frac{g_j z(\xi)}{1+z^2(\xi)}+\frac{f_j(1-z^2(\xi))}{1+z^2(\xi)}\right). \quad (43)$$

另外，本文给出的形式解(3)和(43)式结合其他辅助方程，仍然能得到非线性发展方程(组)的新精确解.

[1] Parkes E J，Duffy B R 1996 *Comput*. *Phys*. *Commun*. **98** 288

[2] Wang M L 1995 *Phys*. *Lett*. A **199** 169

[3] Parkes E J，Duffy B R，Abbott P C 2001 *Phys*. *Lett*. A **295** 280

[4] Sirendaoreji，Sun J 2003 *Phys*. *Lett*. A **309** 387

[5] Chen Y，Li B，Zhang H Q 2003 *Chin*. *Phys*. **12** 940

[6] Chen Y，Yan Z Y，Li B *et al* 2003 *Chin*. *Phys*. **12** 1

[7] Liu S K，Chen H，Fu Z T *et al* 2003 *Acta Phys*. *Sin*. **52** 1842 (in Chinese)[刘式适、陈 华、付遵涛等 2003 物理学报 **52** 1842]

[8] Taogetusang，Sirendaoerji 2004 *Acta Phys*. *Sin*. **53** 4052 (in Chinese)[套格图桑、斯仁道尔吉 2004 物理学报 **53** 4052]

[9] Malfliet W 1992 *Am*. *J*. *Phys*. **60** 650

The method for constructing the exact solutions to the nonlinear evolution equation*

Taogetusang† Sirendaoerji

(*College of Mathematical Science*, *Inner Mongolia Normal University*, *Huhhot* 010022, *China*)

(Received 9 March 2006; revised manuscript received 31 July 2006)

Abstract

Based on tanh-function method, homogeneous balance method and auxiliary equation, a new auxiliary equation was introduced in the paper, meanwhile, a new invariance solution was obtained, and a new exact solitary wave solution for Benjamin-Bona-Mahoney(BBM) equation and modified BBM equation was constructed using the symbolic calculation system of Mathematica. The method introduced in the paper has general significance in searching for exact solutions to the nonlinear developing equation.

Keywords: new auxiliary equation, invariance solution, nonlinear developing equation, exact solitary wave solution

PACC: 0340K, 0290

* Project supported by the National Natural Science Foundation of China (Grant No. 10461006), the Science Research Foundation of Institution of Higher Education of Inner Mongolia Autonomous Region, China (Grant No. NJ02035), the Natural Science Foundation of Inner Mongolia Autonomous Region, China (Grant No. 200408020103) and the Natural Science Research Program of Inner Mongolia Normal University, China (Grant No. QN005023).

† E-mail: tgts@imnu.edu.cn