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非保守系统广义 Raitzin 正则方程的形式不变性与非 Noether 守恒量*

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研究非保守系统广义 Raitzin 正则方程的形式不变性与非 Noether 守恒量.列出系统的 Raitzin 正则方程.提出在无限小变换下系统形式不变性的定义和判据.给出系统的形式不变性是 Lie 对称性的充要条件.建立 Hojman 守恒定理,并举例说明结果的应用.

关键词:非保守系统,Raitzin 正则方程,形式不变性,非 Noether 守恒量

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1. 引　言

1948 年由 Jacobi 和 Ostrogradsky 提出的以含有高阶导数的 Lagrange 函数为基础建立起来的理论称为广义经典动力学.它在物理学、数学和力学中具有重要的理论价值和实用意义.在物理学方面,特别在力学和场论中研究带高阶导数的 Lagrange 力学,例如关于带二阶导数的电磁理论[1];在数学方面,近代微分几何方法的描述已有较好的结果[2].在工程中,如研究定梁轴弯曲形状时,问题化为寻求梁的总位能的极小.此处 Lagrange 函数中含有二阶导数[3].

动力学系统的守恒量,不仅具有数学重要性,而且表现为深刻的物理规律,它已成为近代分析力学的一个重要研究方向.寻求守恒量的主要方法有:Noether 理论[4],Lie 对称性[5],积分因子理论[6],Mei 方法[7]和 Hojman 方法[8].近年来,人们将这些方法推广到广义经典动力学,已取得一系列重要成果[9—18].

1961 年西班牙学者 Raitzin 提出新的正则变量 r 和 s,建立了新的运动微分方程,被称为 Raitzin 正则方程[19].事实上,有些动力学问题应用该方程求解要比用 Hamilton 方程简单些[20—22].因此,在分析力学中对 Raitzin 正则方程的研究具有重要意义.1990 年作者将该方程推广到广义经典动力学,得到广义 Raitzin 正则方程[23].目前,对它的研究已有进展[24].

本文应用 Mei 方法寻求非保守系统广义 Raitzin 正则方程的非 Noether 守恒量.

2. 广义 Raitzin 正则方程

假设广义力学系统的位形由 n 个广义坐标 q_1,q_2,…,q_n 确定.Lagrange 函数为

$$L = L(t, q^j_{(0)}, q^j_{(1)}, \cdots, q^j_{(\delta)}) \quad (j = 1,2,\cdots,n), \tag{1}$$

其中

$$q^j_{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} q^j \quad (j = 1,2,\cdots,n; i = 0,1,2,\cdots,\delta). \tag{2}$$

在低维空间中,广义非势力为 $Q^j = Q^j(t, q^l_{(0)}, q^l_{(1)})$,而在高维空间中,广义非势力为

$$Q^j = Q^j(t, q^l_{(0)}, q^l_{(1)}, \cdots, q^l_{(\delta)}) \quad (j, l = 1,2,\cdots,n), \tag{3}$$

广义非保守系统的 Lagrange 方程为

$$\sum_{i=0}^{\delta} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\frac{\partial L}{\partial q^j_{(i)}}\right) + Q^j = 0 \quad (j = 1,2,\cdots,n), \tag{4}$$

广义动量为

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$$p_j^{(i)} = \sum_{l=0}^{\delta-i-1} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\partial L}{\partial q^j_{(l+i+1)}}\right)$$

$$(i = 0,1,\cdots,\delta-1;j = 1,2,\cdots,n). \quad (5)$$

引入 Raitzin 正则变量和函数

$$r_j^{(i)} = p_j^{(i+1)} = -\sum_{l=1}^{\delta-i} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l}\left(\frac{\partial L}{\partial q^j_{(l+i)}}\right),$$

$$s^j_{(i)} = \dot{q}^j_{(i)},$$

$$R(t,s^j_{(i)},r_j^{(i)}) = L - r_j^{(i)} q^j_{(i)}$$

$$(i = 0,1,\cdots,\delta-1;j = 1,2,\cdots,n). \quad (6)$$

于是,Raitzin 正则方程为

$$s^j_{(i)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j^{(i)}},$$

$$r_j^{(i)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^j_{(i)}} - \widetilde{Q}^j_{(i)}, \quad (7)$$

其中 $\widetilde{Q}^j_{(i)}$ 为 $t, s^l_{(\beta)}, r_l^{(\beta)}$ 的函数.

假设系统是非奇异的,则由方程(7)可求得所有的 $\dot{s}^j_{(i)}$ 和 $\dot{r}_j^{(i)}$,记作

$$\dot{s}^j_{(i)} = g^j_{(i)}(t,\boldsymbol{s},\boldsymbol{r}),$$

$$r_j^{(i)} = h_j^{(i)}(t,\boldsymbol{s},\boldsymbol{r})$$

$$(i = 0,1,\cdots,\delta-1;j = 1,2,\cdots,n). \quad (8)$$

3. 系统的 Lie 对称性

取时间和正则变量的无限小变换

$$t^* = t + \Delta t,$$

$$s^{*j}_{(i)}(t^*) = s^j_{(i)}(t) + \Delta s^j_{(i)},$$

$$r_j^{*(i)}(t^*) = r_j^{(i)}(t) + \Delta r_j^{(i)}. \quad (9)$$

或其展开式

$$t^* = t + \varepsilon\tau^0(t,\boldsymbol{s},\boldsymbol{r})$$

$$s^{*j}_{(i)}(t^*) = s^j_{(i)}(t) + \varepsilon\xi^j_{(i)}(t,\boldsymbol{s},\boldsymbol{r}),$$

$$r_j^{*(i)}(t^*) = r_j^{(i)}(t) + \varepsilon\eta_j^{(i)}(t,\boldsymbol{s},\boldsymbol{r}). \quad (10)$$

其中 ε 为无限小参数,$\tau^0, \xi^j_{(i)}, \eta_j^{(i)}$ 为无限小生成元.

引入无限小变换生成元向量

$$X^{(0)} = \tau^0\frac{\partial}{\partial t} + \xi^j_{(i)}\frac{\partial}{\partial s^j_{(i)}} + \eta_j^{(i)}\frac{\partial}{\partial r_j^{(i)}}, \quad (11)$$

它的一次扩展为

$$X^{(1)} = X^{(0)} + (\dot{\xi}^j_{(i)} - \dot{s}^j_{(i)}\dot{\tau}^0)\frac{\partial}{\partial \dot{s}^j_{(i)}} + (\dot{\eta}_j^{(i)} - \dot{r}_j^{(i)}\dot{\tau}^0)\frac{\partial}{\partial \dot{r}_j^{(i)}}. \quad (12)$$

根据常微分方程不变性的判据,在变换(10)下,方程(8)的不变性由下式表示

$$\dot{\xi}^j_{(i)} - \dot{s}^j_{(i)}\dot{\tau}^0 = X^{(0)}(g^j_{(i)}),$$

$$\dot{\eta}_j^{(i)} - \dot{r}_j^{(i)}\dot{\tau}^0 = X^{(0)}(h_j^{(i)}). \quad (13)$$

称(13)式为 Lie 对称性变换的确定方程.

定义 1 如果无限小变换(10)的生成元 $\tau^0, \xi^j_{(i)}, \eta_j^{(i)}$ 满足确定方程(13),则对应的变换为 Lie 对称性变换.

4. 系统的形式不变性

定义 2 在无限小变换(10)下,如果非保守系统 Raitzin 正则方程(7)的动力学函数 $R, \widetilde{Q}^j_{(i)}$ 保持不变,即

$$s^j_{(i)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j^{(i)}}, r_j^{(i)} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial s^j_{(i)}} - \widetilde{Q}^{*j}_{(i)}, \quad (14)$$

则称这种不变性为方程(7)的形式不变性.

在无限小变换(10)下,展开 $R^*, \widetilde{Q}^{*j}_{(i)}$,有

$$R^* = R + \varepsilon[X^{(0)}(R)] + 0(\varepsilon^2),$$

$$\widetilde{Q}^{*j}_{(i)} = \widetilde{Q}^j_{(i)} + \varepsilon[X^{(0)}(\widetilde{Q}^j_{(i)})] + 0(\varepsilon^2). \quad (15)$$

于是可得到下面的判据.

判据 对 Raitzin 正则方程(7),若 $R, \widetilde{Q}^j_{(i)}$ 和无限小生成元 $\tau^0, \xi^j_{(i)}, \eta_j^{(i)}$ 满足下面的关系式

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r_j^{(i)}}[X^{(0)}(R)] = 0, \quad (16)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s^j_{(i)}}[X^{(0)}(R)] - X^{(0)}[\widetilde{Q}^j_{(i)}] = 0, \quad (17)$$

则相应的不变性称为方程(7)的形式不变性.

证明 将关系式(15)代入方程(14)的第二式,忽略 ε^2 及以上高阶小项,并注意方程(7),有

$$r_j^{(i)} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial s^j_{(i)}} + \widetilde{Q}^{*j}_{(i)}$$

$$= r_j^{(i)} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s^j_{(i)}}\{R + \varepsilon X^{(0)}[R]\} + \widetilde{Q}^j_{(i)} + \varepsilon X^{(0)}[\widetilde{Q}^j_{(i)}]$$

$$= r_j^{(i)} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^j_{(i)}} + \widetilde{Q}^j_{(i)} - \varepsilon\left\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s^j_{(i)}}[X^{(0)}(R)] - X^{(0)}[\widetilde{Q}^j_{(i)}]\right\} = 0, \quad (18)$$

同理,有

$$s^j_{(i)} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R^*}{\partial r_j^{(i)}} = s^j_{(i)} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r_j^{(i)}} + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r_j^{(i)}}[X^{(0)}(R)] = 0. \quad (19)$$

5. 形式不变性与 Lie 对称性

为研究 Raitzin 正则方程的形式不变性与 Lie 对称性,做如下计算:

$$X^{(0)}(R)=\tau^0\frac{\partial R}{\partial t}+\xi^j_{(i)}\frac{\partial R}{\partial s^j_{(i)}}+\eta^{(i)}_j\frac{\partial R}{\partial r^{(i)}_j},$$

$$\begin{aligned}\frac{\partial}{\partial s^j_{(i)}}[X^{(0)}(R)]&=\frac{\partial\tau^0}{\partial s^j_{(i)}}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial s^j_{(i)}}\\&+\frac{\partial\xi^l_{(\beta)}}{\partial s^j_{(i)}}\frac{\partial R}{\partial s^l_{(\beta)}}+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial s^j_{(i)}}\\&+\frac{\partial\eta^{(\beta)}_l}{\partial s^j_{(i)}}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial s^j_{(i)}}.\end{aligned}\tag{20}$$

$$\begin{aligned}\frac{\partial}{\partial r^{(i)}_j}[X^{(0)}(R)]&=\frac{\partial\tau^0}{\partial r^{(i)}_j}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial r^{(i)}_j}\\&+\frac{\partial\xi^l_{(\beta)}}{\partial r^{(i)}_j}\frac{\partial R}{\partial s^l_{(\beta)}}+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial r^{(i)}_j}\\&+\frac{\partial\eta^{(\beta)}_l}{\partial r^{(i)}_j}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial r^{(i)}_j},\end{aligned}\tag{21}$$

$$X^{(0)}[g^j_{(i)}]=\tau^0\frac{\partial g^j_{(i)}}{\partial t}+\xi^l_{(\beta)}\frac{\partial g^j_{(i)}}{\partial s^l_{(\beta)}}+\eta^{(\beta)}_l\frac{\partial g^j_{(i)}}{\partial r^{(\beta)}_l},\tag{22}$$

$$X^{(0)}[h^{(i)}_j]=\tau^0\frac{\partial h^{(i)}_j}{\partial t}+\xi^l_{(\beta)}\frac{\partial h^{(i)}_j}{\partial s^l_{(\beta)}}+\eta^{(\beta)}_l\frac{\partial h^{(i)}_j}{\partial r^{(\beta)}_l},\tag{23}$$

$$X^{(0)}[\tilde{Q}^j_{(i)}]=\tau^0\frac{\partial\tilde{Q}^j_{(i)}}{\partial t}+\xi^l_{(\beta)}\frac{\partial\tilde{Q}^j_{(i)}}{\partial s^l_{(\beta)}}+\eta^{(\beta)}_l\frac{\partial\tilde{Q}^j_{(i)}}{\partial r^{(\beta)}_l},\tag{24}$$

将方程(16)与方程(13)的第一式相减,并代入上述相应的方程,得

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial r^{(i)}_j}[X^{(0)}(R)]-\dot{\xi}^j_{(i)}+\dot{s}^j_{(i)}\dot{\tau}^0+X^{(0)}[g^j_{(i)}]\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{\partial\tau^0}{\partial r^{(i)}_j}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial r^{(i)}_j}+\frac{\partial\xi^l_{(\beta)}}{\partial r^{(i)}_j}\frac{\partial R}{\partial s^l_{(\beta)}}\\&+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial r^{(i)}_j}+\frac{\partial\eta^{(\beta)}_l}{\partial r^{(i)}_j}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial r^{(i)}_j}\Big\}\\&-\dot{\xi}^j_{(i)}+\dot{s}^j_{(i)}\dot{\tau}^0+\tau^0\frac{\partial g^j_{(i)}}{\partial t}+\xi^l_{(\beta)}\frac{\partial g^j_{(i)}}{\partial s^l_{(\beta)}}\\&+\eta^{(\beta)}_l\frac{\partial g^j_{(i)}}{\partial r^{(\beta)}_l},\end{aligned}\tag{25}$$

同理,有

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial s^j_{(i)}}[X^{(0)}(R)]-X^{(0)}[\tilde{Q}^j_{(i)}]\\&-\dot{\eta}^{(i)}_j+\dot{r}^{(i)}_j\dot{\tau}^0+X^{(0)}(h^{(i)}_j)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{\partial\tau^0}{\partial s^j_{(i)}}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial s^j_{(i)}}+\frac{\partial\xi^l_{(\beta)}}{\partial s^j_{(i)}}\frac{\partial R}{\partial s^l_{(\beta)}}\\&+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial s^j_{(i)}}+\frac{\partial\eta^{(\beta)}_l}{\partial s^j_{(i)}}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial s^j_{(i)}}\Big\}\\&-\tau^0\frac{\partial\tilde{Q}^j_{(i)}}{\partial t}-\xi^l_{(\beta)}\frac{\partial\tilde{Q}^j_{(i)}}{\partial s^l_{(\beta)}}-\eta^{(\beta)}_l\frac{\partial\tilde{Q}^j_{(i)}}{\partial r^{(\beta)}_l}\\&-\dot{\eta}^{(i)}_j+\dot{r}^{(i)}_j\dot{\tau}^0+\tau^0\frac{\partial h^{(i)}_j}{\partial t}+\xi^l_{(\beta)}\frac{\partial h^{(i)}_j}{\partial s^l_{(\beta)}}\\&+\eta^{(\beta)}_l\frac{\partial h^{(i)}_j}{\partial r^{(\beta)}_l}.\end{aligned}\tag{26}$$

由此得如下结果

定理 1 对于 Raitzin 正则方程(7),形式不变性为 Lie 对称性的充要条件是无限小生成元 τ^0,$\xi^j_{(i)}$,$\eta^{(i)}_j$ 满足

$$\begin{aligned}&\dot{\xi}^j_{(i)}-\dot{s}^j_{(i)}\dot{\tau}^0\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{\partial\tau^0}{\partial r^{(i)}_j}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial r^{(i)}_j}+\frac{\partial\xi^l_{(\beta)}}{\partial r^{(i)}_j}\frac{\partial R}{\partial s^l_{(\beta)}}\\&+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial r^{(i)}_j}+\frac{\partial\eta^{(\beta)}_l}{\partial r^{(i)}_j}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial r^{(i)}_j}\Big\}\\&+\tau^0\frac{\partial g^j_{(i)}}{\partial t}+\xi^l_{(\beta)}\frac{\partial g^j_{(i)}}{\partial s^l_{(\beta)}}+\eta^{(\beta)}_l\frac{\partial g^j_{(i)}}{\partial r^{(\beta)}_l},\end{aligned}\tag{27}$$

$$\begin{aligned}&\dot{\eta}^{(i)}_j-\dot{r}^{(i)}_j\dot{\tau}^0\\&=\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{\partial\tau^0}{\partial s^j_{(i)}}\frac{\partial R}{\partial t}+\tau^0\frac{\partial^2R}{\partial t\partial s^j_{(i)}}+\frac{\partial\xi^l_{(\beta)}}{\partial s^j_{(i)}}\frac{\partial R}{\partial s^l_{(\beta)}}\\&+\xi^l_{(\beta)}\frac{\partial^2R}{\partial s^l_{(\beta)}\partial s^j_{(i)}}+\frac{\partial\eta^{(\beta)}_l}{\partial s^j_{(i)}}\frac{\partial R}{\partial r^{(\beta)}_l}+\eta^{(\beta)}_l\frac{\partial^2R}{\partial r^{(\beta)}_l\partial s^j_{(i)}}\Big\}\\&-\tau^0\frac{\partial\tilde{Q}^j_{(i)}}{\partial t}-\xi^l_{(\beta)}\frac{\partial\tilde{Q}^j_{(i)}}{\partial s^l_{(\beta)}}-\eta^{(\beta)}_l\frac{\partial\tilde{Q}^j_{(i)}}{\partial r^{(\beta)}_l}\\&+\tau^0\frac{\partial h^{(i)}_j}{\partial t}+\xi^l_{(\beta)}\frac{\partial h^{(i)}_j}{\partial s^l_{(\beta)}}+\eta^{(\beta)}_l\frac{\partial h^{(i)}_j}{\partial r^{(\beta)}_l}.\end{aligned}\tag{28}$$

6. 形式不变性与 Hojman 守恒定理

对 Raitzin 正则方程(7),在时间不变的特殊无限小变换下,形式不变性通过 Lie 对称性间接导出 Hojman 守恒定理.

取时间不变的特殊无限小变换

$$t^{*} = t,\quad s^{*j}_{(i)}(t^{*}) = s^{j}_{(i)}(t) + \varepsilon\xi^{j}_{(i)}(t,s,r),$$
$$r^{*(i)}_{j}(t^{*}) = r^{(i)}_{j}(t) + \eta^{(i)}_{j}(t,s,r). \tag{29}$$

此时,形式不变性为 Lie 对称性的充要条件(27)和(28)有形式

$$\dot{\xi}^{j}_{(i)} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial\xi^{l}_{(\beta)}}{\partial r^{(i)}_{j}}\frac{\partial R}{\partial s^{l}_{(\beta)}} + \xi^{l}_{(\beta)}\frac{\partial^2 R}{\partial s^{l}_{(\beta)}\partial r^{(i)}_{j}} + \frac{\partial\eta^{(\beta)}_{l}}{\partial r^{(i)}_{j}}\frac{\partial R}{\partial r^{(\beta)}_{l}} + \eta^{(\beta)}_{l}\frac{\partial^2 R}{\partial r^{(\beta)}_{l}\partial r^{(i)}_{j}}\right\} + \xi^{l}_{(\beta)}\frac{\partial g^{j}_{(i)}}{\partial s^{l}_{(\beta)}} + \eta^{(\beta)}_{l}\frac{\partial g^{j}_{(i)}}{\partial r^{(\beta)}_{l}}, \tag{30}$$

$$\dot{\eta}^{(i)}_{j} = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial\xi^{l}_{(\beta)}}{\partial s^{j}_{(i)}}\frac{\partial R}{\partial s^{l}_{(\beta)}} + \xi^{l}_{(\beta)}\frac{\partial^2 R}{\partial s^{l}_{(\beta)}\partial s^{j}_{(i)}} + \frac{\partial\eta^{(\beta)}_{l}}{\partial s^{j}_{(i)}}\frac{\partial R}{\partial r^{(\beta)}_{l}} + \eta^{(\beta)}_{l}\frac{\partial^2 R}{\partial r^{(\beta)}_{l}\partial s^{j}_{(i)}}\right\} - \xi^{l}_{(\beta)}\frac{\partial\widetilde{Q}^{j}_{(i)}}{\partial s^{l}_{(\beta)}} - \eta^{(\beta)}_{l}\frac{\partial\widetilde{Q}^{j}_{(i)}}{\partial r^{(\beta)}_{l}} + \xi^{l}_{(\beta)}\frac{\partial h^{j}_{(i)}}{\partial s^{l}_{(\beta)}} + \eta^{(\beta)}_{l}\frac{\partial h^{j}_{(i)}}{\partial r^{(\beta)}_{l}}. \tag{31}$$

定理 2 如果无限小变换(10)的生成元 $\xi^{j}_{(i)}$ $\eta^{(i)}_{j}$ 满足条件(30)和(31),并且存在某函数 $\mu(t,s^{j}_{(i)},r^{(i)}_{j})$ 使下式成立:

$$\frac{1}{\mu}\frac{\partial}{\partial s^{j}_{(i)}}(\mu g^{j}_{(i)}) + \frac{1}{\mu}\frac{\partial}{\partial r^{(i)}_{j}}(\mu h^{(i)}_{j}) = 0. \tag{32}$$

则广义 Raitzin 正则方程(7)的形式不变性导致 Hojman 守恒量

$$I = \frac{1}{\mu}\frac{\partial}{\partial s^{j}_{(i)}}(\mu\xi^{j}_{(i)}) + \frac{1}{\mu}\frac{\partial}{\partial r^{(i)}_{j}}(\mu\eta^{(i)}_{j}) = \text{const}. \tag{33}$$

证明 由文献[24]知.若 $\xi^{j}_{(i)}$,$\eta^{(i)}_{j}$ 是 Lie 对称性的无限小生成元,则广义 Raitzin 正则方程(7)存在守恒量(33).又根据定理的假设条件,$\xi^{j}_{(i)}$,$\eta^{(i)}_{j}$ 满足关系式(30)和(31).因此,方程(7)不仅是形式不变的而且也是 Lie 对称的.

7. 举 例

假设力学系统的 Lagrange 函数为

$$L = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\ddot{y}^2, \tag{34}$$

非势广义力为

$$Q = -\ddot{y}. \tag{35}$$

试求系统的非 Noether 守恒量.

解 首先建立 Raitzin 正则方程.

令

$$q^{1}_{(1)} = \dot{y},\ q^{1}_{(2)} = \ddot{y},\ q^{1}_{(1)} = s^{1}_{(0)},$$
$$q^{1}_{(2)} = s^{1}_{(1)},\ \widetilde{Q}^{1}_{(0)} = 0,\ \widetilde{Q}^{1}_{(1)} = -s^{1}_{(1)}. \tag{36}$$

于是

$$L = \frac{1}{2}(q^{1}_{(1)})^2 + \frac{1}{2}(q^{1}_{(2)})^2,$$
$$r^{(0)}_{1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{1}_{(1)}} - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial q^{1}_{(2)}} = s^{1}_{(1)} - \dot{s}^{1}_{(2)},$$
$$r^{(1)}_{1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^{1}_{(2)}} = \dot{s}^{1}_{(1)}, \tag{37}$$

$$R = \frac{1}{2}(s^{1}_{(0)})^2 + \frac{1}{2}(s^{(0)}_{1})^2 - r^{(0)}_{1}q^{1}_{(0)} - r^{(1)}_{1}q^{1}_{(1)}. \tag{38}$$

方程(7)给出

$$r^{(0)}_{1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^{1}_{(0)}} - \widetilde{Q}^{1}_{(0)} = s^{1}_{(1)},$$
$$r^{(1)}_{1} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^{1}_{(1)}} - \widetilde{Q}^{1}_{(1)} = s^{1}_{(2)} + s^{1}_{(1)},$$
$$s^{1}_{(0)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r^{(0)}_{1}} = q^{1}_{(1)},$$
$$s^{1}_{(1)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r^{(1)}_{1}} = q^{1}_{(2)}. \tag{39}$$

由方程(8),得

$$\dot{r}^{(0)}_{1} = \dot{s}^{1}_{(1)} = r^{(1)}_{1} = h^{(0)}_{1},$$
$$\dot{r}^{(1)}_{1} = \dot{s}^{1}_{(2)} + \dot{s}^{1}_{(1)} = r^{(1)}_{1} = h^{(1)}_{1},$$
$$\dot{s}^{1}_{(0)} = \dot{q}^{1}_{(1)} = s^{1}_{(1)} = g^{1}_{(0)},$$
$$\dot{s}^{1}_{(1)} = \dot{q}^{1}_{(2)} = \dot{s}^{1}_{(1)} = r^{(1)}_{1} = g^{1}_{(1)}. \tag{40}$$

其次,由方程(13),Lie 对称性的确定方程为

$$\dot{\xi}^{1}_{(0)} = \xi^{1}_{(1)},\ \dot{\xi}^{1}_{(1)} = \eta^{(1)}_{1}\ (\dot{\xi}^{1}_{(1)} = \dot{\xi}^{1}_{(2)}),$$
$$\dot{\eta}^{(0)}_{1} = \eta^{(1)}_{1},\ \dot{\eta}^{(1)}_{1} = \eta^{(1)}_{1}. \tag{41}$$

取

$$\xi^{1}_{(0)} = 1,\ \xi^{1}_{(1)} = 0\ (\xi^{1}_{(2)} = 0),$$
$$\eta^{(0)}_{1} = 1,\ \eta^{(1)}_{1} = 0, \tag{42}$$

经验证,生成元(42)满足确定方程(41)及条件(30)和(31).因此,系统(40)不仅是形式不变的而且也是 Lie 对称的.

最后,求系统的守恒量.

方程(32)给出

$$\mu = (2s^{1}_{(0)} - s^{1}_{(2)})^2 \mathrm{e}^{-t}, \tag{43}$$

于是由(33)式得守恒量

$$I = 4(2s^{1}_{(0)} - s^{1}_{(2)})^{-1}, \tag{44}$$

或

$$I = 4(2\dot{y} - \dddot{y})^{-1}. \tag{45}$$

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Form invariance and non-Noether conserved quantity of generalized Raitzin's canonical equations of non-conservative system*

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Abstract

The form invariance and the non-Noether conserved quantity of generalized Raitzin's canonical equations of non-conservative system are studied. The definition and criterion of the form invariance in the system under infinitesimal transformations are proposed. The necessary and sufficient condition under which the form invariance is a Lie symmetry is given. The Hojman theorem is established. Finally，an example is given to illustrate the application of the result.

Keywords：non-conservative system，Raitzin's canonical equation，form invariance，non-Noether conserved quantity

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