第 55 卷 第 8 期 2006 年 8 月
1000-3290/2006/55(08)/3813-04

物 理 学 报
ACTA PHYSICA SINICA

Vol.55, No.8, August, 2006

变质量 Birkhoff 系统的 Lie 对称性和非 Noether 守恒量

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(2005 年 5 月 30 日收到;2006 年 4 月 25 日收到修改稿)

采用嵌入质量法建立了变质量系统的 Birkhoff 方程.根据 Lie 对称性理论给出了变质量 Birkhoff 系统的 Lie 对称性确定方程,得到了系统的 Lie 对称直接导致非 Noether 守恒量的存在条件和形成.举例说明结果的应用.

关键词:变质量, Birkhoff 系统, Lie 对称性, 非 Noether 守恒量

PACC:0320

1. 引　　言

力学系统的对称性和守恒量之间有着密切的联系,寻求力学系统守恒量的近代方法,主要有 Noether 方法、Lie 方法和形式不变性方法.关于三类对称性直接和间接导致的守恒量研究,已取得了许多有意义的研究成果[1—7].1992 年以来,梅凤翔等[8]建立了 Birkhoff 系统动力学,构造了理论框架. Birkhoff 系统动力学是近代经典力学的一个重要分支,是数学物理学科中的一个近代发展方向.由于空间技术和其他工业技术的进步,变质量系统动力学的研究变得越来越重要.文献[9,10]讨论了变质量系统的 Birkhoff 表示,表明变质量系统能纳入 Birkhoff 系统.本文建立变质量 Birkhoff 系统,研究系统的 Lie 对称性及其导致的非 Noether 守恒量,为经典 Birkhoff 动力学应用于现代物理学提供了桥梁.

2. 变质量 Birkhoff 系统

设力学系统由 N 个质点组成,在时刻 t,第 i 个质点的质量为 $m_i(i=1,\cdots,N)$;在瞬时 $t+\mathrm{d}t$,由质点分离(或并入)的微粒质量为 $\mathrm{d}m_i$.设系统的位形由 n 个 Birkhoff 坐标 $a^\mu(\mu=1,\cdots,n)$确定,这里质量是依赖于时间和坐标的函数,$m_i=m_i(t,\boldsymbol{a})$.

采取嵌入质量法,构造变质量 Birkhoff 系统的 Birkhoff 函数和 Birkhoff 函数组

$$\begin{aligned}\tilde{B}^* &= \tilde{B}^*(t,\boldsymbol{a}) = B^*(m_i(t,\boldsymbol{a}),t,\boldsymbol{a}),\\ \tilde{R}_\nu^* &= \tilde{R}_\nu^*(t,\boldsymbol{a}) = R_\nu^*(m_i(t,\boldsymbol{a}),t,\boldsymbol{a})\\ &(\nu=1,\cdots,2n).\end{aligned} \tag{1}$$

于是有

$$\begin{aligned}\frac{\partial\tilde{B}^*}{\partial a^\mu} &= \frac{\partial B^*}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial B^*}{\partial a^\mu},\\ \frac{\partial\tilde{R}_\nu^*}{\partial a^\mu} &= \frac{\partial R_\nu^*}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}+\frac{\partial R_\nu^*}{\partial a^\mu},\\ \frac{\partial\tilde{R}_\nu^*}{\partial t} &= \frac{\partial R_\nu^*}{\partial m_i}\frac{\partial m_i}{\partial t}+\frac{\partial R_\nu^*}{\partial t}.\\ &(\nu,\mu=1,\cdots,2n;i=1,\cdots,N).\end{aligned} \tag{2}$$

变质量系统的 Birkhoff 方程表示为

$$\left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu-\left(\frac{\partial\tilde{B}^*}{\partial a^\mu}+\frac{\partial\tilde{R}_\mu^*}{\partial t}\right)=0$$
$$(\mu,\nu=1,\cdots,2n). \tag{3}$$

用方程(3)描述的力学系统称为变质量 Birkhoff 系统.

方程(3)可表示为

$$\tilde{\Omega}_{\mu\nu}^*\dot{a}^\nu-\left(\frac{\partial\tilde{B}^*}{\partial a^\mu}+\frac{\partial\tilde{R}_\mu^*}{\partial t}\right)=0\qquad(\mu,\nu=1,\cdots,2n), \tag{4}$$

式中

$$\tilde{\Omega}_{\mu\nu}^* = \frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}$$

称为变质量 Birkhoff 张量.设系统非奇异,即有

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$$\det(\tilde{\Omega}^*_{\mu\nu}) \neq 0.$$

如果 $\tilde{R}^*_\mu$ 和 $\tilde{B}^*$ 都不显含时间 t,则系统(4)是自治的.如果 $\tilde{R}^*_\mu$ 不显含时间 t,则系统(4)是半自治的.

方程(4)展开后可得

$$\dot{a}^\mu = \tilde{h}^*_\mu(t,\boldsymbol{a}) = \tilde{h}^*_\mu(m_i(t,\boldsymbol{a}),t,\boldsymbol{a}) \quad (\mu = 1,\cdots,2n). \tag{5}$$

于是有

$$\frac{\partial \tilde{h}^*_\mu}{\partial t} = \frac{\partial h^*_\mu}{\partial t} + \frac{\partial h^*_\mu}{\partial m_i}\frac{\partial m_i}{\partial t},$$

$$\frac{\partial \tilde{h}^*_\mu}{\partial a^\mu} = \frac{\partial h^*_\mu}{\partial a^\mu} + \frac{\partial h^*_\mu}{\partial m_i}\frac{\partial m_i}{\partial a^\mu}.$$

3.变质量 Birkhoff 系统的 Lie 对称性

引入无限小群变换

$$t^* = t + \varepsilon\xi_0(t,\boldsymbol{a}),\quad a^{\mu *} = a^\mu + \varepsilon\xi_\mu(t,\boldsymbol{a}), \tag{6}$$

式中 ξ_0,ξ_μ 为无限小群变换的生成元.

无限小群变换(6)式的生成元向量为

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_\mu\frac{\partial}{\partial a^\mu}.$$

其一次扩展为

$$X^{(1)} = X^{(0)} + (\dot{\xi}_\mu - \dot{a}^\mu\dot{\xi}_0)\frac{\partial}{\partial \dot{a}^\mu}.$$

根据 Lie 对称性理论,变质量 Birkhoff 系统的不变性归结为如下确定方程:

$$X^{(1)}(\dot{a}^\mu - \tilde{h}^*_\mu(t,\boldsymbol{a})) = 0 \quad (\mu = 1,\cdots,2n). \tag{7}$$

展开(7)式,得

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\mu - \tilde{h}^*_\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \xi_0\frac{\partial \tilde{h}^*_\mu}{\partial t} + \xi_\nu\frac{\partial \tilde{h}^*_\mu}{\partial a^\nu} \quad (\nu,\mu = 1,\cdots,2n). \tag{8}$$

式中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \tilde{h}^*_\nu\frac{\partial}{\partial a^\nu}.$$

命题 1 如果无限小变换(6)式下的生成元 ξ_0, ξ_μ 满足确定方程(8),则相应的对称性为变质量 Birkhoff 系统(4)的 Lie 对称性.

4.变质量 Birkhoff 系统的非 Noether 守恒量

在一定的条件,根据 Hojman 守恒律可直接找到变质量 Birkhoff 系统 Lie 对称性的非 Noether 守恒量.我们有命题 2.

命题 2 对于满足确定方程(8)的无限小生成元 ξ_0,ξ_μ,如果存在函数

$$\tilde{\lambda}^* = \tilde{\lambda}^*(t,\boldsymbol{a}) = \lambda^*(m_i(t,\boldsymbol{a}),t,\boldsymbol{a})$$

满足条件

$$\frac{\partial \tilde{h}^*_\mu}{\partial a^\mu} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\lambda}^* = 0, \tag{9}$$

则变质量 Birkhoff 系统的 Lie 对称性导致如下形式的非 Noether 守恒量:

$$I = \frac{1}{\tilde{\lambda}^*}\frac{\partial(\tilde{\lambda}^*\xi_0)}{\partial t} + \frac{1}{\tilde{\lambda}^*}\frac{\partial(\tilde{\lambda}^*\xi_\mu)}{\partial a^\mu} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \mathrm{const}. \tag{10}$$

证明 我们有

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\ln\tilde{\lambda}^*}{\partial t}\xi_0 + \frac{\partial\ln\tilde{\lambda}^*}{\partial a^\mu}\xi_\mu\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\mu}{\partial a^\mu} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0; \tag{11}$$

$$\begin{gathered}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_0}{\partial t} = \frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \frac{\partial\tilde{h}^*_\mu}{\partial t}\frac{\partial\xi_0}{\partial a^\mu},\\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\xi_\nu}{\partial a^\nu} = \frac{\partial}{\partial a^\nu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_\nu - \frac{\partial\tilde{h}^*_\mu}{\partial a^\nu}\frac{\partial\xi_\nu}{\partial a^\mu}.\end{gathered} \tag{12}$$

将(12)式代入(11)式并利用(8)式,得

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\ln\tilde{\lambda}^*}{\partial t}\xi_0 + \frac{\partial\ln\tilde{\lambda}^*}{\partial a^\mu}\xi_\mu\right) + \frac{\partial\tilde{h}^*_\mu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \xi_0\frac{\partial^2\tilde{h}^*_\mu}{\partial a^\mu\partial t} - \xi_\mu\frac{\partial^2\tilde{h}^*_\mu}{\partial a^\mu\partial a^\nu}. \tag{13}$$

将条件(9)式分别对 t,a^ν 求偏导数,并将其代入(13)式,可得

$$\begin{aligned}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial\ln\tilde{\lambda}^*}{\partial t}\xi_0 + \frac{\partial\ln\tilde{\lambda}^*}{\partial a^\mu}\xi_\mu\right) - \xi_0\frac{\partial}{\partial t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\lambda}^* - \xi_\mu\frac{\partial}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\tilde{\lambda}^* + \frac{\partial\tilde{h}^*_\mu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\\ &= \frac{\partial\tilde{h}^*_\mu}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \frac{\partial\ln\tilde{\lambda}^*}{\partial t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{h}^*_\mu\frac{\partial\ln\tilde{\lambda}^*}{\partial a^\mu}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = 0.\end{aligned} \tag{14}$$

证毕.

根据命题 2,我们得到两个推论.

推论 1　对于满足确定方程(8)的无限小生成元 ξ_0,ξ_μ,如果有条件

$$\frac{\partial \tilde{h}_\mu^*}{\partial a^\mu} = 0, \tag{15}$$

那么变质量 Birkhoff 系统(4)的 Lie 对称性导致如下形式的非 Noether 守恒量:

$$I = \frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_\mu}{\partial a^\mu} - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \mathrm{const}. \tag{16}$$

推论 2　如果取无限小生成元 $\xi_0 = 0$,那么变质量 Birkhoff 系统(4)的 Lie 对称性导致 Hojman 守恒量

$$I = \frac{1}{\tilde{\lambda}^*}\frac{\partial(\tilde{\lambda}^*\xi_\mu)}{\partial a^\mu} = \mathrm{const}. \tag{17}$$

5. 算　　例

已知四阶变质量 Birkhoff 系统为

$$\begin{aligned}
R_1 &= 0,\\
R_2 &= ra^3 + \frac{1}{m},\\
R_3 &= 0,\\
R_4 &= -\frac{1}{\gamma m} - \frac{t}{\gamma} + ma^1\\
&\quad - (1 + tm)a^3 + \frac{1}{\gamma^2},
\end{aligned} \tag{18}$$

$$B = \left(-t + \gamma m a^1 - \gamma t m a^3 + \frac{1}{\gamma}\right)a^4, \tag{19}$$

式中 $m = m_0\exp(-\gamma t)$,试研究系统的 Lie 对称性及其导致的非 Noether 守恒量.

变质量系统的 Birkhoff 方程(4)给出

$$\begin{aligned}
\dot{a}^1 &= a^3,\\
\dot{a}^2 &= a^4,\\
\dot{a}^3 &= -\frac{1}{m},\\
\dot{a}^4 &= \gamma a^4.
\end{aligned} \tag{20}$$

确定方程(8)给出

$$\begin{aligned}
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 - a^3\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \xi_3,\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 - a^4\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \xi_4,\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_3 + \frac{1}{m}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = -\frac{\gamma}{m}\xi_0,\\
&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_4 - \gamma a^4\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \gamma\xi_4.
\end{aligned} \tag{21}$$

(21)式有如下一组解:

$$\begin{aligned}
\xi_0 &= 0,\\
\xi_1 &= 1,\\
\xi_2 &= \frac{1}{\gamma}\exp(\gamma t),\\
\xi_3 &= 0,\\
\xi_4 &= \exp(\gamma t).
\end{aligned} \tag{22}$$

由条件(9)式可得

$$\lambda = \exp(-\gamma t); \tag{23}$$

$$\lambda = \left(a^1 + ta^3 - \frac{t}{\gamma m} + \frac{1}{\gamma^2 m}\right)\exp(-\gamma t). \tag{24}$$

对(22)和(23)式,由(10)式可得守恒量

$$I_1 = 0. \tag{25}$$

对(22)和(24)式,由(10)式可得非 Noether 守恒量

$$I_2 = \lambda^{-1}\frac{\partial(\lambda)}{\partial a^1} = \left(a^1 + ta^3 - \frac{t}{\gamma m} + \frac{1}{\gamma^2 m}\right)^{-1} = \mathrm{const}. \tag{26}$$

守恒量 I_1 是平凡的,I_2 是非平凡的.

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Lie symmetry and non-Noether conserved quantities of variable mass Birkhoffian system

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(Received 30 May 2005; revised manuscript received 25 April 2006)

Abstract

In this paper, the Lie symmetrical non-Noether conserved quantity of variable mass Birkhoffian system under the infinitesimal transformations is studied. First, variable mass Birkhoffian equations are established; second, Lie symmetry determining equations are given. Then the non-Noether conserved quantity is obtained. An example is given to illustrate the application of the result.

Keywords: variable mass, Birkhoffian system, Lie symmetry, non-Nonther conserved quantity
PACC: 0320

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