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多模玻色二次多项式型系统的特性函数和准概率分布函数

徐秀玮　任廷琦　迟永江　朱友良　刘姝延

(烟台师范学院物理系,烟台　264025)

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运用多模玻色系统广义线性量子变换的普遍理论,对多模玻色二次多项式型系统进行了研究.给出了多模玻色二次多项式型系统的正规、反正规、Wigner 特性函数、P 表示和 Q 表示.举例讨论了该系统的压缩性质.

关键词:多模玻色二次多项式型系统,特性函数,准概率分布函数

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1. 引　　言

多模玻色二次多项式型系统是量子理论诸多领域特别是量子光学中可以严格求解的典型系统之一,由于运用传统的方法求解此类系统将涉及到复杂的、非对易的算符运算,故此类系统的特性函数和准概率分布函数[1,2]的一般表达式尚未见公开发表.本文运用广义线性量子变换的普遍理论,给出了多模玻色二次多项式型系统的特性函数和准概率分布函数中的 P 表示和 Q 表示的普遍表达式,将与特性函数和准概率分布函数有关的、非对易的算符运算转化为容易进行的 c 数运算.作为应用举例具体讨论了该类系统的量子压缩特性.

在 Fock 空间中,n 模玻色二次多项式型系统的 Hamilton 量为

$$H = \frac{1}{2}\boldsymbol{\Lambda N \Sigma_B \tilde{\Lambda}} + \boldsymbol{R\Sigma_B\tilde{\Lambda}}, \tag{1}$$

式中,

$$\boldsymbol{\Lambda} = (\boldsymbol{a}^+, \tilde{\boldsymbol{a}}) = (a_1^+, \cdots, a_n^+, a_1, \cdots, a_n),$$

其中 a_j^+ 和 a_j 分别是第 j 模玻色产生和湮没算符;

$$\boldsymbol{\Sigma}_B = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix},$$

其中 $\boldsymbol{I}$ 是 $n\times n$ 单位矩阵;$\boldsymbol{N}$ 是 $2n\times 2n$ 的负厄米矩阵,即

$$\boldsymbol{N} = \boldsymbol{N}^- = \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix}\boldsymbol{N}^+\begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix};$$

$$\boldsymbol{R} = (\boldsymbol{r}^+, \tilde{\boldsymbol{r}}),$$

其中 $\boldsymbol{r}$ 是任意 n 维复矢量.

2. 演化算符

若 n 模玻色二次多项式型系统(1)从初态 $|\psi(0)\rangle$ 开始演化,则该系统在任意 t 时刻的态函数为

$$|\psi(t)\rangle = U(\boldsymbol{\Lambda}, \boldsymbol{M}, \boldsymbol{L})|\psi(0)\rangle. \tag{2}$$

这里的演化算符为

$$\begin{aligned} U(\boldsymbol{\Lambda}, \boldsymbol{M}, \boldsymbol{L}) &= \exp\left(\frac{t}{\mathrm{i}\hbar}\hat{H}\right) \\ &= \exp\Big[\frac{1}{2}\boldsymbol{\Lambda}(\ln\boldsymbol{M})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}} \\ &\quad + \boldsymbol{L}(\boldsymbol{M}-\boldsymbol{I}')^{-1}(\ln\boldsymbol{M})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}}\Big], \end{aligned} \tag{3}$$

$$\begin{aligned} \boldsymbol{M} &= \exp\left(\frac{t}{\mathrm{i}\hbar}\boldsymbol{N}\right) \equiv \begin{pmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \tilde{\boldsymbol{B}} & \tilde{\boldsymbol{C}} \end{pmatrix}, \\ \tilde{\boldsymbol{M}}\boldsymbol{\Sigma}_B\boldsymbol{M} &= \boldsymbol{\Sigma}_B, \\ \boldsymbol{A} &= \boldsymbol{C}^+, \\ \boldsymbol{B} &= \boldsymbol{D}^+, \end{aligned} \tag{4}$$

$$\boldsymbol{L} = \frac{t}{\mathrm{i}\hbar}\boldsymbol{R}(\ln\boldsymbol{M})^{-1}(\boldsymbol{M}-\boldsymbol{I}') \equiv (\boldsymbol{l}^+, \tilde{\boldsymbol{l}}). \tag{5}$$

其中 $\boldsymbol{M}$ 是复辛且负幺正的 $2n\times 2n$ 矩阵,$\boldsymbol{I}'$ 是 $2n\times 2n$ 单位矩阵.

据文献[3,4]可以给出演化算符 $U(\boldsymbol{\Lambda}, \boldsymbol{M}, \boldsymbol{L})$ 的正、反正规乘积形式如下:

$$\begin{aligned} U^{(\mathrm{n})}(\boldsymbol{\Lambda}, \boldsymbol{M}, \boldsymbol{L}) = \alpha^{(\mathrm{n})}(\boldsymbol{M}, \boldsymbol{L}) :\exp\Big[\frac{1}{2}\boldsymbol{\Lambda}D(\boldsymbol{M})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}} \\ + \boldsymbol{L}(\boldsymbol{M}-\boldsymbol{I}')^{-1}D(\boldsymbol{M})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}}\Big]:, \end{aligned} \tag{6}$$

$$U^{(\mathrm{a})}(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L}) = \alpha^{(\mathrm{a})}(\boldsymbol{M},\boldsymbol{L}) \times {}^{+}_{+}\exp\Big[-\frac{1}{2}\boldsymbol{\Lambda}D(\boldsymbol{M}^{-1})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}} - \boldsymbol{L}(\boldsymbol{M}-\boldsymbol{I}')^{-1}D(\boldsymbol{M}^{-1})\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}}\Big]{}^{+}_{+}, \tag{7}$$

式中，$:\cdots:$为正规乘积符号(如$:a_j^+a_ja_j^+ + a_ja_j^+a_j: = a_j^{+2}a_j + a_j^+a_j^2$)，${}^{+}_{+}\cdots{}^{+}_{+}$为反正规乘积符号(如${}^{+}_{+}a_j^+a_ja_j^+ + a_ja_j^+a_j{}^{+}_{+} = a_ja_j^{+2} + a_j^2a_j^+$)，

$$\alpha^{(\mathrm{n})}(\boldsymbol{M},\boldsymbol{L}) = \frac{1}{\sqrt{\det\boldsymbol{C}}}\exp\Big\{\frac{1}{2}\boldsymbol{L}(\boldsymbol{M}-\boldsymbol{I}')^{-1}[D(\boldsymbol{M}) - \ln\boldsymbol{M}]\boldsymbol{\Sigma}_B(\tilde{\boldsymbol{M}}-\boldsymbol{I}')^{-1}\tilde{\boldsymbol{L}}\Big\}, \tag{8}$$

$$\alpha^{(\mathrm{a})}(\boldsymbol{M},\boldsymbol{L}) = \frac{1}{\sqrt{\det\boldsymbol{A}}}\exp\Big\{-\frac{1}{2}\boldsymbol{L}(\boldsymbol{M}-\boldsymbol{I}')^{-1}[D(\boldsymbol{M}^{-1}) + \ln\boldsymbol{M}]\boldsymbol{\Sigma}_B(\tilde{\boldsymbol{M}}-\boldsymbol{I}')^{-1}\tilde{\boldsymbol{L}}\Big\}, \tag{9}$$

$$D(\boldsymbol{M})\boldsymbol{\Sigma}_B = \begin{pmatrix} -\boldsymbol{D}\tilde{\boldsymbol{C}}^{-1} & \boldsymbol{C}^{-1}-\boldsymbol{I} \\ \tilde{\boldsymbol{C}}^{-1}-\boldsymbol{I} & \boldsymbol{B}\boldsymbol{C}^{-1}\end{pmatrix},$$
$$D(\boldsymbol{M}^{-1})\boldsymbol{\Sigma}_B = \begin{pmatrix} \boldsymbol{A}^{-1}\boldsymbol{D} & \boldsymbol{A}^{-1}-\boldsymbol{I} \\ \tilde{\boldsymbol{A}}^{-1}-\boldsymbol{I} & -\tilde{\boldsymbol{B}}\boldsymbol{A}^{-1}\end{pmatrix}, \tag{10}$$

$$(\boldsymbol{M}-\boldsymbol{I}')^{-1} = \begin{pmatrix} -\boldsymbol{G}_{12}(\tilde{\boldsymbol{C}}-\boldsymbol{I})\boldsymbol{D}^{-1} & \boldsymbol{G}_{12} \\ \boldsymbol{G}_{21} & -\boldsymbol{G}_{21}(\boldsymbol{A}-\boldsymbol{I})\tilde{\boldsymbol{B}}^{-1}\end{pmatrix}, \tag{11}$$

$$\boldsymbol{G}_{12} = [\boldsymbol{D}^{-1}(\boldsymbol{A}-\boldsymbol{I}) + \tilde{\boldsymbol{D}}^{-1}(\boldsymbol{C}-\boldsymbol{I})]^{-1} = \tilde{\boldsymbol{G}}_{12},$$
$$\boldsymbol{G}_{21} = [(\boldsymbol{A}-\boldsymbol{I})\tilde{\boldsymbol{B}}^{-1} + (\boldsymbol{C}-\boldsymbol{I})\boldsymbol{B}^{-1}]^{-1} = \tilde{\boldsymbol{G}}_{21}, \tag{12}$$

$$(\boldsymbol{M}-\boldsymbol{I}')^{-1}D(\boldsymbol{M})\boldsymbol{\Sigma}_B = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\tilde{\boldsymbol{C}}^{-1} & -\boldsymbol{B}\boldsymbol{C}^{-1}\end{pmatrix},$$
$$(\boldsymbol{M}-\boldsymbol{I}')^{-1}D(\boldsymbol{M}^{-1})\boldsymbol{\Sigma}_B = \begin{pmatrix} -\boldsymbol{A}^{-1}\boldsymbol{D} & -\boldsymbol{A}^{-1} \\ \boldsymbol{I} & 0\end{pmatrix}. \tag{13}$$

显然，演化算符 $U(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L})$的三种表示形式是等价的，即

$$U(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L}) = U^{(\mathrm{n})}(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L}) = U^{(\mathrm{a})}(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L}). \tag{14}$$

3. 特性函数

利用多模玻色系统的广义线性量子变换关系[5,6]

$$U\boldsymbol{\Lambda}U^{-1} = \boldsymbol{\Lambda}\boldsymbol{M} + \boldsymbol{L},$$
$$U^{-1}\boldsymbol{\Lambda}U = (\boldsymbol{\Lambda}-\boldsymbol{L})\boldsymbol{M}^{-1}, \tag{15}$$

可以给出 n 模玻色二次多项式型系统的 Wigner 特性函数如下：

$$\begin{aligned}
\chi^{(\mathrm{W})}(\xi,t) &= \langle\psi(t)|\exp[\eta(\xi^+\tilde{\boldsymbol{a}}^+ + \tilde{\xi}\boldsymbol{a})]|\psi(t)\rangle \\
&= \langle\psi(0)|U^+\exp[\eta(\xi^+\tilde{\boldsymbol{a}}^+ + \tilde{\xi}\boldsymbol{a})]U|\psi(0)\rangle \\
&= \exp\Big\{\frac{\eta^2}{2}[\tilde{\xi}(\boldsymbol{AC}+\boldsymbol{DB})\tilde{\xi}^+ - (\tilde{\xi}\boldsymbol{D}\tilde{\boldsymbol{A}}\xi + \xi^+\boldsymbol{D}^*\tilde{\boldsymbol{A}}^*\tilde{\xi}^+)] \\
&\quad - \eta(\xi^+\boldsymbol{A}^* - \tilde{\xi}\boldsymbol{D},\tilde{\xi}\boldsymbol{A} - \xi^+\boldsymbol{D}^*)\tilde{\boldsymbol{L}}\Big\} \\
&\quad \times\langle\psi(0)|:\exp[\eta(\xi^+\boldsymbol{A}^* - \tilde{\xi}\boldsymbol{D},\tilde{\xi}\boldsymbol{A} - \xi^+\boldsymbol{D}^*)\tilde{\boldsymbol{\Lambda}}]:|\psi(0)\rangle,
\end{aligned} \tag{16}$$

式中，η 是任意常数，ξ 是任意 n 维复矢量.

若初态为 n 模玻色相干态

$$|\psi(0)\rangle = |z_1\rangle\cdots|z_n\rangle \equiv |Z\rangle, \tag{17}$$

则 n 模玻色二次多项式型系统(1)的 Wigner 特性函数成为

$$\begin{aligned}
\chi^{(\mathrm{W})}(\xi,t) = \exp\Big\{&\frac{\eta^2}{2}[\tilde{\xi}(\boldsymbol{AC}+\boldsymbol{DB})\tilde{\xi}^+ - (\tilde{\xi}\boldsymbol{D}\tilde{\boldsymbol{A}}\xi + \xi^+\boldsymbol{D}^*\tilde{\boldsymbol{A}}^*\tilde{\xi}^+)] \\
&+ 2\eta\mathrm{Re}[(\tilde{\xi}\boldsymbol{A} - \xi^+\boldsymbol{D}^*)(\boldsymbol{Z}-\boldsymbol{l})]\Big\},
\end{aligned} \tag{18}$$

式中 $\tilde{Z} = (z_1,z_2,\cdots,z_n)$是任意 n 维复矢量.正规特性函数为

$$\begin{aligned}
\chi^{(\mathrm{n})}(\xi,t) &= \exp\Big(-\frac{\eta^2}{2}\xi^+\xi\Big)\chi^{(\mathrm{W})}(\xi,t) \\
&= \exp\Big\{\frac{\eta^2}{2}[2\tilde{\xi}\boldsymbol{DB}\tilde{\xi}^+ - (\tilde{\xi}\boldsymbol{D}\tilde{\boldsymbol{A}}\xi + \xi^+\boldsymbol{D}^*\tilde{\boldsymbol{A}}^*\tilde{\xi}^+)] \\
&\quad + 2\eta\mathrm{Re}[(\tilde{\xi}\boldsymbol{A} - \xi^+\boldsymbol{D}^*)(\boldsymbol{Z}-\boldsymbol{l})]\Big\}.
\end{aligned} \tag{19}$$

反正规特性函数为

$$\begin{aligned}
\chi^{(\mathrm{a})}(\xi,t) &= \exp\Big(\frac{\eta^2}{2}\xi^+\xi\Big)\chi^{(\mathrm{W})}(\xi,t) \\
&= \exp\Big\{\frac{\eta^2}{2}[2\tilde{\xi}\boldsymbol{AC}\tilde{\xi}^+ - (\tilde{\xi}\boldsymbol{D}\tilde{\boldsymbol{A}}\xi + \xi^+\boldsymbol{D}^*\tilde{\boldsymbol{A}}^*\tilde{\xi}^+)] \\
&\quad + 2\eta\mathrm{Re}[(\tilde{\xi}\boldsymbol{A} - \xi^+\boldsymbol{D}^*)(\boldsymbol{Z}-\boldsymbol{l})]\Big\}.
\end{aligned} \tag{20}$$

4. Q 表示和 P 表示

4.1. Q 表示

n 模玻色二次多项式型系统(1)的密度算符为

$$\rho = |\psi(t)\rangle\langle\psi(t)|, \tag{21}$$

其 Q 表示为

$$\begin{aligned} Q(\beta,\beta^*,t) &= \langle\beta|\rho|\beta\rangle \\ &= |\langle\beta|U(t)|\psi(0)\rangle|^2, \end{aligned} \tag{22}$$

式中 $|\boldsymbol{\beta}\rangle = |\beta_1\rangle\cdots|\beta_n\rangle$ 是 n 模玻色相干态.

参考文献[7],初态可一般地取作

$$\begin{aligned} |\psi(0)\rangle &= \psi(\boldsymbol{a}^+)|0\rangle \\ &= \prod_{j=1}^{n}(a_j^+)^{n_j}\exp(\sigma\boldsymbol{a}^+\tilde{\boldsymbol{a}}^+ + \boldsymbol{a}^+\boldsymbol{\lambda})|0\rangle \\ &= \prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda_j}\right)^{n_j}\exp(\sigma\boldsymbol{a}^+\tilde{\boldsymbol{a}}^+ + \boldsymbol{a}^+\boldsymbol{\lambda})|0\rangle, \end{aligned} \tag{23}$$

式中,σ 是任意复常数,λ 是任意 n 维复矢量.再利用演化算符的反正规乘积形式(7)式和相干态表象的超完备性关系

$$\int |Z\rangle\langle Z|\prod_{i=1}^{n}\frac{\mathrm{d}^2 z_i}{\pi} = 1, \tag{24}$$

可得 n 模玻色二次多项式型系统演化算符 $U(\boldsymbol{\Lambda},\boldsymbol{M},\boldsymbol{L})$的矩阵元如下

$$\begin{aligned} &\langle\beta|U|\psi(0)\rangle \\ &= \int\langle\beta|U^{(a)}\psi(\boldsymbol{a}^+)|Z\rangle\langle Z||0\rangle\prod_{i=1}^{n}\frac{\mathrm{d}^2 z_i}{\pi} \\ &= \alpha^{(a)}(\boldsymbol{M},\boldsymbol{L})\exp\left(-\frac{\boldsymbol{\beta}^+\boldsymbol{\beta}}{2}\right) \\ &\quad\times\left[(-1)^n\det\begin{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{D}-2\sigma\boldsymbol{I} & \boldsymbol{A}^{-1} \\ \tilde{\boldsymbol{A}}^{-1} & -\tilde{\boldsymbol{B}}\boldsymbol{A}^{-1}\end{pmatrix}\right]^{-1/2} \\ &\quad\times\left[\prod_{j=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}\lambda_j}\right)^{n_j}\right] \\ &\quad\times\exp\left[\frac{1}{2}(\tilde{\boldsymbol{\lambda}}-\tilde{\boldsymbol{l}}+\boldsymbol{k}^+\boldsymbol{A}^{-1}\boldsymbol{D},\boldsymbol{\beta}^+ + \boldsymbol{k}^+\boldsymbol{A}^{-1})\right. \\ &\quad\times\begin{pmatrix}\boldsymbol{A}^{-1}\boldsymbol{D}-2\sigma\boldsymbol{I} & \boldsymbol{A}^{-1} \\ \tilde{\boldsymbol{A}}^{-1} & -\tilde{\boldsymbol{B}}\boldsymbol{A}^{-1}\end{pmatrix}^{-1} \\ &\quad\left.\times\begin{pmatrix}\tilde{\boldsymbol{\lambda}}-\tilde{\boldsymbol{l}}+\boldsymbol{A}^{-1}\boldsymbol{D}\tilde{\boldsymbol{k}}^+ \\ \tilde{\boldsymbol{\beta}}^+ + \tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{k}}^+\end{pmatrix}\right], \end{aligned} \tag{25}$$

式中 $\boldsymbol{\beta}$ 是与 Z 类似的 n 维复矢量.在(25)式的推导中,还用到了复变数的多维高斯积分公式

$$\begin{aligned} &\int\exp\left[-\frac{1}{2}(\boldsymbol{Z}^+,\tilde{\boldsymbol{Z}})\boldsymbol{W}\begin{pmatrix}\tilde{\boldsymbol{Z}}^+ \\ \boldsymbol{Z}\end{pmatrix}+(\boldsymbol{u},\boldsymbol{v})\begin{pmatrix}\tilde{\boldsymbol{Z}}^+ \\ \boldsymbol{Z}\end{pmatrix}\right]\prod_{i=1}^{n}\frac{\mathrm{d}^2 z_i}{\pi} \\ &= [(-1)^n\det\boldsymbol{W}]^{-1/2}\exp\left[\frac{1}{2}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{W}^{-1}\begin{pmatrix}\tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}}\end{pmatrix}\right], \end{aligned} \tag{26}$$

式中 $\boldsymbol{u},\boldsymbol{v}$ 是任意 n 维复矢量,W 是 $\det W\neq 0$ 的 $2n\times 2n$ 的复矩阵.将(25)式代入(22)式即可给出 n 模玻色二次多项式型系统的 Q 表示.

若初态为相干态

$$\begin{aligned} |Z'\rangle &= \exp(\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{a}}^+ - \boldsymbol{Z}'^+\boldsymbol{a})|0\rangle \\ &= \exp\left(-\frac{|\boldsymbol{Z}'|^2}{2}+\tilde{\boldsymbol{Z}}'\tilde{\boldsymbol{a}}^+\right)|0\rangle, \end{aligned} \tag{27}$$

则 n 模玻色二次多项式型系统的 Q 表示为

$$\begin{aligned} Q &= \frac{\det\boldsymbol{A}}{\det\boldsymbol{C}}\exp[-(|\boldsymbol{\beta}|^2+|\boldsymbol{Z}'|^2)] \\ &\quad\times\left|\alpha^{(a)}(\boldsymbol{M},\boldsymbol{L})\right. \\ &\quad\times\exp\left[\frac{1}{2}(\tilde{\boldsymbol{Z}}'-\tilde{\boldsymbol{l}}+\boldsymbol{l}^+\boldsymbol{A}^{-1}\boldsymbol{D},\boldsymbol{\beta}^+ + \boldsymbol{l}^+\boldsymbol{A}^{-1})\right. \\ &\quad\left.\left.\times\begin{pmatrix}\boldsymbol{B}\boldsymbol{C}^{-1} & \tilde{\boldsymbol{C}}^{-1} \\ \boldsymbol{C}^{-1} & -\boldsymbol{D}\tilde{\boldsymbol{C}}^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{Z}'-\boldsymbol{l}+\boldsymbol{A}^{-1}\boldsymbol{D}\tilde{\boldsymbol{l}}^+ \\ \tilde{\boldsymbol{\beta}}^+ + \tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{l}}^+\end{pmatrix}\right]\right|^2. \end{aligned} \tag{28}$$

在(28)式推导中,用到了分块矩阵的行列式

$$\det\begin{pmatrix}\boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{G} & \boldsymbol{H}\end{pmatrix} = \det\boldsymbol{E}\cdot\det(\boldsymbol{H}-\boldsymbol{G}\boldsymbol{E}^{-1}\boldsymbol{F}). \tag{29}$$

4.2. P 表示

考虑到算符

$$\begin{aligned} |0\rangle\langle 0| &= :\exp(-\boldsymbol{a}^+\boldsymbol{a}): \\ &= \lim_{\boldsymbol{A}\to\varepsilon\boldsymbol{I}}:\exp[\boldsymbol{a}^+(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{a}]: \\ &= \lim_{\varepsilon\to 0}\varepsilon^{-n/2}\exp\left[\frac{1}{2}\boldsymbol{\Lambda}\ln\boldsymbol{M}_0\boldsymbol{\Sigma}_B\tilde{\boldsymbol{\Lambda}}\right], \end{aligned} \tag{30}$$

式中

$$\boldsymbol{M}_0 = \begin{pmatrix}\varepsilon\boldsymbol{I} & 0 \\ 0 & \varepsilon^{-1}\boldsymbol{I}\end{pmatrix},$$

其中 ε 为任意小的正实数.运用广义线性量子变换的乘法性质[5,6]和(2),(21),(30)式,n 模玻色二次多项式型系统的密度算符可表示为

$$\begin{aligned} \rho &= \lim_{\varepsilon\to 0}\varepsilon^{-n/2}U(\boldsymbol{M},\boldsymbol{L})U(\boldsymbol{M}_0,0)U^+(\boldsymbol{M},\boldsymbol{L}) \\ &= \lim_{\varepsilon\to 0}\varepsilon^{-n/2}U(\boldsymbol{M}',\boldsymbol{L}'), \end{aligned} \tag{31}$$

式中,

$$\begin{aligned} M' &= MM_0M^{-1}, \\ L' &= L(M_0 - I')M^{-1}. \end{aligned} \tag{32}$$

再运用指数二次多项式型算符的反正规乘积形式(7)式,可给出玻色二次多项式型系统(1)的 P 表示如下：

$$P(\boldsymbol{\alpha}) = \lim_{\varepsilon\to 0}\frac{1}{\pi^n\sqrt{\varepsilon^n}}U^{(a)}(M',L')\Big|_{a=\alpha,a^+=\alpha^+}, \tag{33}$$

式中 α 为任意 n 维复矢量.

最后利用辛条件($\widetilde{M}\Sigma_B M = \Sigma_B$)进行矩阵运算,将(33)式简化为

$$\begin{aligned} P(\boldsymbol{\alpha}) = \lim_{\varepsilon\to 0}\frac{1}{\pi^n\sqrt{\det\boldsymbol{\Omega}}}\mathrm{Re}\Big\{\exp\Big[& \boldsymbol{\alpha}^+(\boldsymbol{I} - \varepsilon\boldsymbol{\Omega}^{-1})\boldsymbol{\alpha} \\ & - \frac{1}{\varepsilon}\boldsymbol{l}^+(\boldsymbol{I} + \boldsymbol{\Theta}^{-1}\widetilde{\boldsymbol{D}}\boldsymbol{D}^*)\boldsymbol{l} - \boldsymbol{l}^+\boldsymbol{C}\boldsymbol{\Omega}^{-1}\boldsymbol{D}\tilde{\boldsymbol{l}}^+ \\ & - \tilde{\boldsymbol{\alpha}}\widetilde{\boldsymbol{C}}\boldsymbol{D}^+\boldsymbol{\Omega}^{-1}\boldsymbol{\alpha} - 2\tilde{\boldsymbol{l}}\boldsymbol{D}^+\boldsymbol{\Omega}^{-1}\boldsymbol{\alpha} \\ & - 2\varepsilon\boldsymbol{l}^+\boldsymbol{C}\boldsymbol{\Omega}^{-1}\boldsymbol{\alpha}\Big]\Big\}, \end{aligned} \tag{34}$$

式中,

$$\begin{aligned} \boldsymbol{\Omega} &= \varepsilon^2\boldsymbol{C}^+\boldsymbol{C} - \boldsymbol{D}\boldsymbol{D}^+, \\ \boldsymbol{\Theta} &= \varepsilon^2\boldsymbol{C}\boldsymbol{C}^+ - \widetilde{\boldsymbol{D}}\boldsymbol{D}^*. \end{aligned} \tag{35}$$

5. 应用举例——系统的压缩特性

据文献[8]引入 n 模正交相位振幅算符

$$\begin{aligned} X(\varphi) &= \frac{1}{\sqrt{2n}}\sum_{j=1}^{n}[a_j\exp(\mathrm{i}\varphi) + a_j^+\exp(-\mathrm{i}\varphi)] \\ &= (\boldsymbol{\xi}^+, \tilde{\boldsymbol{\xi}})\tilde{\boldsymbol{\Lambda}}. \end{aligned} \tag{36}$$

这里的 n 维复矢量 ξ 的各分量全为相同的 $\frac{1}{\sqrt{2n}}\times\exp(\mathrm{i}\varphi)$. 将(36)式代入 Wigner 特性函数(18)式,有

$$\begin{aligned} \chi^{(\mathrm{w})}(\varphi,t) &\equiv \langle\psi(t)|\exp[\eta X(\varphi)]|\psi(t)\rangle \\ &= \exp\Big(\frac{\eta^2}{2}f + \eta g\Big). \end{aligned} \tag{37}$$

$$\begin{aligned} f = \frac{1}{2} + \frac{1}{n}\big[&\mathrm{Sum}\boldsymbol{DB} \\ &- |\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|\cos(\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}} + 2\varphi)\big], \end{aligned} \tag{38}$$

式中 Sum 是对所有矩阵元求和的记号$\Big($如 $\mathrm{Sum}\boldsymbol{Y} = \sum_{i,j=1}^{n}Y_{ij}\Big)$,

$$g = 2\mathrm{Re}[(\tilde{\boldsymbol{\xi}}\boldsymbol{A} - \boldsymbol{\xi}^+\boldsymbol{D}^*)(\boldsymbol{Z} - \boldsymbol{l})]. \tag{39}$$

类似地,由(19)和(36)式,可给出正规特性函数

$$\begin{aligned} \chi^{(n)}(\varphi,t) &\equiv \langle\psi(t)|:\exp[\eta X(\varphi)]:|\psi(t)\rangle \\ &= \exp\Big(\frac{\eta^2}{2}f' + \eta g\Big), \end{aligned} \tag{40}$$

$$f' = f - \frac{1}{2}. \tag{41}$$

由(40)式可得

$$\begin{aligned} \bar{X} &= \langle\psi(t)|X|\psi(t)\rangle = g \\ &= \langle\psi(t)|:X:|\psi(t)\rangle = :\bar{X}:. \end{aligned} \tag{42}$$

因此,

$$\begin{aligned} \langle\psi(t)|\exp[\eta\Delta X(\varphi)]|\psi(t)\rangle &= \exp\Big(\frac{\eta^2}{2}f\Big), \\ \langle\psi(t)|:\exp[\eta\Delta X(\varphi)]:|\psi(t)\rangle &= \exp\Big(\frac{\eta^2}{2}f'\Big). \end{aligned} \tag{43}$$

将(43)式两边展开成 η 的幂级数并比较 η 的同幂项系数,得

$$\begin{aligned} &\langle\psi(t)|(\Delta X)^{2n}|\psi(t)\rangle \\ &= \frac{(2n)!}{n!}\Big(\frac{f}{2}\Big)^n \\ &= \begin{cases} (2n-1)!!f^n & (n = 1,2,\cdots), \\ 1 & (n = 0). \end{cases} \end{aligned} \tag{44}$$

$$\begin{aligned} &\langle\psi(t)|(\Delta X)^{2n+1}|\psi(t)\rangle \\ &= 0 \quad (n = 0,1,2,\cdots); \end{aligned} \tag{45}$$

$$\begin{aligned} &\langle\psi(t)|:(\Delta X)^{2n}:|\psi(t)\rangle \\ &= \frac{(2n)!}{n!}\Big(\frac{f'}{2}\Big)^n \\ &= \begin{cases} (2n-1)!!f'^n & (n = 1,2,\cdots), \\ 1 & (n = 0). \end{cases} \end{aligned} \tag{46}$$

$$\begin{aligned} &\langle\psi(t)|:(\Delta X)^{2n+1}:|\psi(t)\rangle \\ &= 0 \quad (n = 0,1,2,\cdots). \end{aligned} \tag{47}$$

当 $M = I'$ 时,$f = \frac{1}{2}$,$f' = 0$,由(44)和(46)式给出正交相位振幅算符的相干态起伏为

$$\langle Z|(\Delta X)^{2n}|Z\rangle = \frac{1}{4^n}\frac{(2n)!}{n!} \quad (n = 1,2,\cdots), \tag{48}$$

$$\langle Z|:(\Delta X)^{2n}:|Z\rangle = 0 \quad (n = 1,2,\cdots). \tag{49}$$

将(44)与(48)式、(46)与(49)式分别加以比较可见,当 $f < \frac{1}{2}$ 时,

$$\langle\psi(t)|(\Delta X)^{2n}|\psi(t)\rangle < \langle Z|(\Delta X)^{2n}|Z\rangle,$$

则 $X(\varphi)$ 可以被压缩到任意 $2n$ 阶;当 $f' < 0$ 时,

$$\begin{aligned} &\langle\psi(t)|:(\Delta X)^{2(2n+1)}:|\psi(t)\rangle \\ &< \langle Z|:(\Delta X)^{2(2n+1)}:|Z\rangle, \end{aligned}$$

则 $X(\varphi)$ 可以被本征压缩到任意 $4n+2$ 阶. 演化算

符是幺正的，即 $\boldsymbol{C}=\boldsymbol{A}^{+}$，$\boldsymbol{D}=\boldsymbol{B}^{+}$，则

$$f'=\frac{1}{n}\Big\{\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}-|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|\times\cos\big[\arg(\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}})+2\varphi\big]\Big\}. \tag{50}$$

由(50)式可见，若 $|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|>\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}\geqslant0$，可以产生 $2n$ 阶压缩或 $4n+2$ 阶本征压缩效应，其压缩参数角 φ 的取值范围为

$$k\pi-\frac{1}{2}\left[\arccos\frac{\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}}{|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|}+\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}\right]<\varphi<k\pi+\frac{1}{2}\left[\arccos\frac{\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}}{|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|}-\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}\right], \tag{51}$$

式中 $k=0,\pm1,\pm2,\cdots$，

$$0<\arccos\frac{\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}}{|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|}<\frac{\pi}{2}.$$

最大压缩时的参数角为

$$\varphi_{\max}=k\pi-\frac{1}{2}\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}. \tag{52}$$

若 $|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|>-\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}>0$ 时，亦可产生压缩效应，压缩参数角 φ 的取值范围与(51)式形式相同，而 $\frac{\pi}{2}<\arccos\frac{\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}}{|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|}<\frac{3\pi}{2}$，最大压缩时的参数角为

$$\varphi_{\max}=\left(k+\frac{1}{2}\right)\pi-\frac{1}{2}\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}. \tag{53}$$

若 $-\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}>|\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}|$ 时，总可产生压缩效应，最大压缩时的参数角为

$$\varphi_{\max}=k\pi-\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}, \tag{54}$$

最小压缩时的参数角为

$$\varphi_{\min}=\left(k+\frac{1}{2}\right)\pi-\arg\mathrm{Sum}\boldsymbol{D}\tilde{\boldsymbol{A}}. \tag{55}$$

由以上所述可知，多模玻色二次多项式型系统的压缩特性仅与 M 有关而与 L 无关，即仅与(1)式中产生和湮没算符的二次项有关而与线性项无关．注意到(41)式，不难发现 n 模玻色二次多项式型系统的 $2n$ 阶压缩和 $4n+2$ 阶本征压缩的条件相同．

6. 结 论

利用上述关于特性函数、P 表示和 Q 表示的结论，可将与多模玻色二次多项式型系统有关的力学量期望值、量子密度算符主方程求解等涉及到非对易量子算符的运算转化为 c 数的运算[1,2]，从而可降低求解难度、简化求解过程．

关于多模玻色二次多项式型系统压缩特性，文献[8]只给出了二阶压缩的条件，而本文则给出了 $2n$ 阶压缩和 $4n+2$ 阶本征压缩的条件．二者对于 $\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}\geqslant0$ 的情况，得到的压缩条件是一致的，区别在于两文采用的 $2n\times2n$ 复辛且负幺正矩阵 $\boldsymbol{M}$ 是互逆的．文献[8]认为"对于非零复矩阵 $\boldsymbol{Y}$ 容易证明 $\mathrm{Sum}\boldsymbol{Y}^{+}\boldsymbol{Y}\geqslant0$"，我们认为这一结论有欠妥之处，如 $\boldsymbol{M}=\begin{pmatrix}0 & \boldsymbol{D}\\ \tilde{\boldsymbol{D}}^{+} & 0\end{pmatrix}$ 为 $2n\times2n$ 复辛且负幺正矩阵，由于 $\boldsymbol{D}\boldsymbol{D}^{+}=-\boldsymbol{I}$，所以 $\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}=-n<0$．文献[8]据此而未讨论 $\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}<0$ 时的压缩特性．然而，由上述不难发现，$\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}<0$ 时的压缩特性比 $\mathrm{Sum}\boldsymbol{D}\boldsymbol{D}^{+}>0$ 时的压缩特性更显著．

从本文的推导过程和给出的结论易见，运用多模玻色系统线性量子变换普遍理论研究多模玻色二次多项式型系统具有推导过程简单、结论普遍且应用方便等特点．

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Characteristic functions and quasi-probability distribution functions of multi-mode Bose quadratic polynomial system

Xu Xiu-Wei Ren Ting-Qi Chi Yong-Jiang Zhu You-Liang Liu Shu-Yan

(*Department of Physics*, *Yantai Teachers' University*, *Yantai* 264025, *China*)

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Abstract

The multi-mode Bose quadratic polynomial system (MBQPS) is studied. The normal, anti-normal, Wigner characteristic functions, P-representation and Q-representation of MBQPS are presented by utilizing the general theory of linear quantum transformation of multi-mode Bose system. And as a example, its quantum squeezed characteristics are studied.

Keywords: multi-mode Bose quadratic polynomial system, characteristic functions, quasi-probability distribution functions

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