第 56 卷 第 9 期 2007 年 9 月
1000-3290/2007/56(09)/5045-04
物 理 学 报
ACTA PHYSICA SINICA
Vol.56,No.9,September,2007

Maggi 方程的形式不变性与 Hojman 守恒量*

胡楚勒[1)†] 解加芳[2)]

1)(呼伦贝尔学院物理系，海拉尔 021008)

2)(北京理工大学理学院力学系，北京 100081)

(2006 年 12 月 11 日收到；2006 年 12 月 14 日收到修改稿)

研究 Maggi 方程的形式不变性，给出了 Maggi 方程形式不变性的定义和判据，通过形式不变性间接得到了 Hojman 守恒量. 给出一个算例，说明结果的应用.

关键词：分析力学，Maggi 方程，形式不变性，Hojman 守恒量

PACC：0320

1. 引 言

力学系统的对称性是其本质特征的反映. 力学系统的对称性与守恒量的研究，是现代数理科学，特别是分析力学的一个近代发展方向. 近代利用对称性寻找系统守恒量的方法主要有 Noether 理论[1]、Lie 对称性[2] 和形式不变性[3] (也称 Mei 对称性)等. 相应地有三种主要的守恒量，分别是 Noether 守恒量[1,4,5]、Hojman 守恒量[6] 和新型守恒量[7] (也称 Mei 守恒量). 在以往的研究中通常都是由一种对称性直接或间接寻找一种类型的守恒量. 近年来这些对称性理论的研究已取得了一系列重要成果[4—20].

Maggi[21] 于 1896 年对线性非完整约束系统得到一类动力学方程，后人称为 Maggi 方程. 这些方程被 Przeborski 推广到非线性非完整系统[22]. 文献[23]指出，Maggi 方程对研究非完整系统的运动是很方便的.

本文研究了 Maggi 方程的形式不变性与 Hojman 守恒量. 给出了 Maggi 系统形式不变性的定义和判据，得到了形式不变性间接导致的 Hojman 守恒量. 最后给出一个算例，说明结果的应用.

2. 形式不变性的定义和判据

设力学系统的位形由 n 个广义坐标 q_s ($s=1,\cdots,n$) 来确定，而约束方程为

$$f_\beta=\dot q_{\varepsilon+\beta}-\varphi_\beta(t,q_s,\dot q_\sigma)=0,\quad (s=1,\cdots,n;\ \beta=1,\cdots,g;\ \sigma=1,\cdots,\varepsilon;\ \varepsilon=n-g),\tag{1}$$

则 Maggi 方程表为

$$E_\sigma(L)+E_{\varepsilon+\beta}(L)\frac{\partial\varphi_\beta}{\partial\dot q_\sigma}=Q_\sigma+Q_{\varepsilon+\beta}\frac{\partial\varphi_\beta}{\partial\dot q_\sigma},\tag{2}$$

其中 E_σ 为 Euler 算子.

引入无限小变换

$$t^*=t+\varepsilon\xi_0(t,q_s,\dot q_s),\quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,q_s,\dot q_s).\tag{3}$$

在无限小变换(3)下，动力学函数 L,Q_s,f_β 变为 L^*,Q_s^*,f_β^*，有

$$L^*\left(t^*,q_s^*,\frac{\mathrm dq_s^*}{\mathrm dt^*}\right)=L(t,q_s,\dot q_s)+\varepsilon X^{(1)}(L)+O(\varepsilon^2),$$

$$Q_s^*\left(t^*,q_k^*,\frac{\mathrm dq_k^*}{\mathrm dt^*}\right)=Q_s(t,q_k,\dot q_k)+\varepsilon X^{(1)}(Q_s)+O(\varepsilon^2),$$

$$f_\beta^*\left(t^*,q_s^*,\frac{\mathrm dq_\sigma^*}{\mathrm dt^*}\right)=f_\beta(t,q_s,\dot q_\sigma)+\varepsilon X^{(1)}(f_\beta)+O(\varepsilon^2).\tag{4}$$

定义 对于给定的动力学系统，用变换后的动力学函数代替变换前的动力学函数，方程形式保持不变的对称性称为形式不变性.

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* 国家自然科学基金(批准号：10572021)资助的课题.

† E-mail：huchule@sina.com.cn

根据上述定义有

$$f_\beta^* = \frac{\mathrm{d}q_{\varepsilon+\beta}^*}{\mathrm{d}t^*} - \varphi_\beta^*\left(t^*, q_s^*, \frac{\mathrm{d}q_\sigma^*}{\mathrm{d}t^*}\right) = 0, \tag{5}$$

$$E_\sigma(L^*) + E_{\varepsilon+\beta}(L^*)\frac{\partial\varphi_\beta^*}{\partial\dot{q}_\sigma} = Q_\sigma^* + Q_{\varepsilon+\beta}^*\frac{\partial\varphi_\beta^*}{\partial\dot{q}_\sigma}. \tag{6}$$

将(4)式代入方程(5)和(6)有

$$f_\beta + \varepsilon X^{(1)}(f_\beta) + O(\varepsilon^2) = 0,$$

$$\begin{aligned}&E_\sigma(L) + E_{\varepsilon+\beta}(L)\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} + \varepsilon\Big\{E_\sigma[X^{(1)}(L)] \\ &+ E_{\varepsilon+\beta}[X^{(1)}(L)]\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} + E_{\varepsilon+\beta}(L)\frac{\partial[X^{(1)}(\varphi_\beta)]}{\partial\dot{q}_\sigma}\Big\} \\ &= Q_\sigma + \varepsilon X^{(1)}(Q_\sigma) + \varepsilon X^{(1)}(Q_{\varepsilon+\beta})\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} \\ &+ \varepsilon Q_{\varepsilon+\beta}\frac{\partial[X^{(1)}(\varphi_\beta)]}{\partial\dot{q}_\sigma} + O(\varepsilon^2) = 0.\end{aligned}$$

考虑到方程(1),(2),舍去 ε^2 及更高阶小项,则有

$$X^{(1)}(\dot{q}_{\varepsilon+\beta} - \varphi_\beta) = 0, \tag{7}$$

$$\begin{aligned}&E_\sigma[X^{(1)}(L)] + E_{\varepsilon+\beta}[X^{(1)}(L)]\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} \\ &+ E_{\varepsilon+\beta}(L)\frac{\partial[X^{(1)}(q_{\varepsilon+\beta})]}{\partial\dot{q}_\sigma} \\ &= X^{(1)}(Q_\sigma) + X^{(1)}(Q_{\varepsilon+\beta})\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} \\ &+ Q_{\varepsilon+\beta}\frac{\partial[X^{(1)}(\dot{q}_{\varepsilon+\beta})]}{\partial\dot{q}_\sigma}.\end{aligned} \tag{8}$$

方程(7),(8)称为 Maggi 方程形式不变性的判据方程.

3. 由形式不变性导出 Hojman 守恒量

将方程(1),(2)表为

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s), \tag{9}$$

在 t 不变的特殊无限小变换下,即(3)式中当 $\xi_0 = 0$ 时,Lie 对称性确定方程为

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial\alpha_s}{\partial q_k}\xi_k + \frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k, \tag{10}$$

其中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial\dot{q}_s}.$$

如果生成元 ξ_s 满足方程(10),且存在某函数 $\mu = \mu(t, q_s, \dot{q}_s)$ 使得

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{11}$$

则形式不变性导致方程(9)的 Hojman 守恒量

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \mathrm{const}. \tag{12}$$

考虑到约束不变性,还应满足式(7).

4. 算 例

二自由度系统的 Lagrange 函数为

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2),$$

约束方程为

$$f = \dot{q}_2 - t\dot{q}_1 = 0,$$

试由 Maggi 方程的形式不变性导出 Hojman 守恒量.

方程(7)给出

$$\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0 - \dot{q}_1\dot{\xi}_0 - t(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) = 0, \tag{13}$$

方程(8)给出

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_1}[\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)] \\ &- \frac{\partial}{\partial q_1}[\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)] \\ &+ \Big\{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_2}[\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)] \\ &- \frac{\partial}{\partial q_2}[\dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)]\Big\}t, \\ &+ \ddot{q}_2\frac{\partial}{\partial\dot{q}_1}(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) = 0.\end{aligned} \tag{14}$$

方程(13),(14)有解

$$\xi_0 = 0, \quad \xi_1 = \xi_2 = 1. \tag{15}$$

由系统的 Maggi 方程和约束方程,得

$$\ddot{q}_1 = -\frac{t\dot{q}_1}{1+t^2}, \quad \ddot{q}_2 = \frac{\dot{q}_1}{1+t^2}, \tag{16}$$

方程(10)给出

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1 = -\frac{t}{1+t^2}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1, \\ &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_2 = \frac{1}{1+t^2}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_1,\end{aligned} \tag{17}$$

显然(15)式满足方程(17).

(11)式给出

$$-\frac{t}{1+t^2} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0,$$

它有解

$$\mu = (1 + t^2)^{1/2}, \tag{18}$$

$$\mu = (1 + t^2)^{1/2}(\dot{q}_1 + t\dot{q}_2 - q_2). \tag{19}$$

由(15),(19)式得 Hojman 守恒量

$$\begin{aligned} I_{\mathrm{H}} &= \frac{1}{\mu}\frac{\partial}{\partial q_1}(\mu) + \frac{1}{\mu}\frac{\partial}{\partial q_2}(\mu) \\ &= -(\dot{q}_1 + t\dot{q}_2 - q_2)^{-1} \\ &= \text{const}. \end{aligned}$$

5. 结　　论

本文研究了 Maggi 方程的形式不变性与 Hojman 守恒量. 给出了 Maggi 系统形式不变性的定义和判据,得到了形式不变性间接导致的 Hojman 守恒量. 对于利用对称性理论寻找动力学系统的守恒量提供了一种方便、简捷的新途径.

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Form invariance and Hojman conserved quantity of Maggi equation*

Hu Chu-Le[1)†] Xie Jia-Fang[2)]
1)(Department of Physics, Hulun Buir College, Hailaer 021008, China)
2)(Department of Mechanics, Faculty of Science, Beijing Institute of Technology, Beijing 100081, China)
(Received 11 December 2006; revised manuscript received 14 December 2006)

Abstract

This paper studies the form invariance of Maggi equation. Its definition and criterion are presented. A Hojman conserved quantity can be deduced using the form invariance. An example is given to illustrate the application of the result.

Keywords: analytical mechanics, Maggi equation, form invariance, Hojman conserved quantity
PACC: 0320

* Project supported by the National Natural Science Foundation of China (Grant No. 10572021).
† E-mail: huchule@sina.com.cn