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三质点 Toda 晶格微分方程的积分*

何 光[1)†] 梅凤翔[2)]

1)(北京理工大学宇航科学技术学院,北京 100081)

2)(北京理工大学理学院,北京 100081)

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三质点 Toda 晶格的微分方程是一个 Hamilton 系统.研究用 Noether 理论和 Poisson 理论求其积分.

关键词:Toda 晶格, Hamilton 系统, Noether 理论, Poisson 理论

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1. 引 言

Toda 晶格是非线性物理,非线性动力学,几何动力学等的重要研究对象[1—3].文献[2]研究了晶格中的 Toda 孤波,各种 Toda 晶格的动力学.文献[3]研究了 Toda 轨道及其切空间,Toda 晶格括号等.近年,在分析力学中求微分方程的积分已有一些进展[4—8].Toda 晶格的微分方程是一个 Hamilton 系统,我们用 Noether 理论和 Poisson 理论研究三质点 Toda 晶格微分方程的积分.

2. 三质点 Toda 晶格的微分方程

三质点 Toda 晶格的运动由 Hamilton 函数

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+C[\exp(q_1-q_2)+\exp(q_2-q_3)+\exp(q_3-q_1)] \tag{1}$$

来描述[2],其运动微分方程为

$$\begin{aligned}
&\dot q_1=p_1,\dot q_2=p_2,\dot q_3=p_3,\\
&\dot p_1=C[\exp(q_3-q_1)-\exp(q_1-q_2)],\\
&\dot p_2=C[\exp(q_1-q_2)-\exp(q_2-q_3)],\\
&\dot p_3=C[\exp(q_2-q_3)-\exp(q_3-q_1)],
\end{aligned} \tag{2}$$

其中 C 为常数.

3. 三质点 Toda 晶格微分方程的 Noether 理论

对一般的 Hamilton 系统,Noether 等式表为[9—11]

$$p_s\dot\xi_s-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q_s}\xi_s-H\dot\xi_0+\dot G_N=0, \tag{3}$$

其中 $\xi_0=\xi_0(t,\boldsymbol{q},\boldsymbol{p})$,$\xi_s=\xi_s(t,\boldsymbol{q},\boldsymbol{p})$为无限小生成元,$G_N=G_N(t,\boldsymbol{q},\boldsymbol{p})$为规范函数.由 Noether 对称性导致的 Noether 守恒量有形式

$$I_N=p_s\xi_s-H\xi_0+G_N=\text{const}. \tag{4}$$

对系统(1),Noether 等式(3)给出为

$$\begin{aligned}
&p_1\dot\xi_1+p_2\dot\xi_2+p_3\dot\xi_3\\
&-C[\exp(q_1-q_2)-\exp(q_3-q_1)]\xi_1\\
&-C[\exp(q_2-q_3)-\exp(q_1-q_2)]\xi_2\\
&-C[\exp(q_3-q_1)-\exp(q_2-q_3)]\xi_3\\
&-H\dot\xi_0+\dot G_N\\
&=0,
\end{aligned} \tag{5}$$

可找到它的如下 4 个解:

$$\begin{aligned}
&\xi_0=-1,\xi_1=\xi_2=\xi_3=0,\\
&\eta_1=\eta_2=\eta_3=0,G_N=0,
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\xi_0=0,\xi_1=\xi_2=\xi_3=1,\\
&\eta_1=\eta_2=\eta_3=0,G_N=0,
\end{aligned} \tag{7}$$

$$\begin{aligned}
&\xi_0=0,\xi_1=p_2+p_3,\\
&\xi_2=p_3+p_1,\xi_3=p_1+p_2,
\end{aligned}$$

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† E-mail:heguang@bit.edu.cn

$$
\begin{aligned}
\eta_1 &= C[\exp(q_1-q_2)-\exp(q_3-q_1)],\\
\eta_2 &= C[\exp(q_2-q_3)-\exp(q_1-q_2)],\\
\eta_3 &= C[\exp(q_3-q_1)-\exp(q_2-q_3)],\\
G_N &= -(p_1p_2+p_2p_3+p_3p_1)\\
&\quad -C[\exp(q_1-q_2)+\exp(q_2-q_3)\\
&\quad +\exp(q_3-q_1)],
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
\xi_0 &= 0,\ \xi_1 = p_2p_3 - C\exp(q_2-q_3),\\
\xi_2 &= p_3p_1 - C\exp(q_3-q_1),\\
\xi_3 &= p_1p_2 - C\exp(q_1-q_2),\\
\eta_1 &= C[p_3\exp(q_1-q_2)-p_2\exp(q_3-q_1)],\\
\eta_2 &= C[p_1\exp(q_2-q_3)-p_3\exp(q_1-q_2)],\\
\eta_3 &= C[p_2\exp(q_3-q_1)-p_1\exp(q_2-q_3)],\\
G_N &= -2p_1p_2p_3.
\end{aligned}
\tag{9}
$$

守恒量(4)式分别给出

$$
\begin{aligned}
I_1 = H &= \frac{1}{2}(p_1^2+p_2^2+p_3^2)\\
&\quad +C[\exp(q_1-q_2)+\exp(q_2-q_3)\\
&\quad +\exp(q_3-q_1)],
\end{aligned}
\tag{10}
$$

$$
I_2 = p_1+p_2+p_3,
\tag{11}
$$

$$
\begin{aligned}
I_3 &= p_1p_2+p_2p_3+p_3p_1\\
&\quad -C[\exp(q_1-q_2)+\exp(q_2-q_3)\\
&\quad +\exp(q_3-q_1)],
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
I_4 &= p_1p_2p_3 - C[p_1\exp(q_2-q_3)\\
&\quad +p_2\exp(q_3-q_1)+p_3\exp(q_1-q_2)].
\end{aligned}
\tag{13}
$$

这 4 个积分是不完全独立的，因为有

$$
I_3 = \frac{1}{2}I_2^2 - I_1.
\tag{14}
$$

4. 三质点 Toda 晶格微分方程的 Poisson 理论

$I=I(t,\boldsymbol{q},\boldsymbol{p})$为 Hamilton 系统积分的充要条件为

$$
\frac{\partial I}{\partial t}+(I,H)=0,
\tag{15}
$$

其中

$$
(I,H)=\frac{\partial I}{\partial q_s}\frac{\partial H}{\partial p_s}-\frac{\partial I}{\partial p_s}\frac{\partial H}{\partial q_s}
\tag{16}
$$

为 Poisson 括号.

对系统(1)，因

$$
\frac{\partial H}{\partial t}=0,
\tag{17}
$$

故由(15)式得到积分

$$
I_1 = H.
\tag{18}
$$

展开(15)式，有

$$
\begin{aligned}
&\frac{\partial I}{\partial t}+\frac{\partial I}{\partial q_1}p_1+\frac{\partial I}{\partial q_2}p_2+\frac{\partial I}{\partial q_3}p_3\\
&-\frac{\partial I}{\partial p_1}C[\exp(q_1-q_2)-\exp(q_3-q_1)]\\
&-\frac{\partial I}{\partial p_2}C[\exp(q_2-q_3)-\exp(q_1-q_2)]\\
&-\frac{\partial I}{\partial p_3}C[\exp(q_3-q_1)-\exp(q_2-q_3)]\\
&=0,
\end{aligned}
\tag{19}
$$

它等价于方程

$$
\begin{aligned}
\frac{\mathrm{d}t}{1}&=\frac{\mathrm{d}q_1}{p_1}=\frac{\mathrm{d}q_2}{p_2}=\frac{\mathrm{d}q_3}{p_3}\\
&=\frac{\mathrm{d}p_1}{C[\exp(q_3-q_1)-\exp(q_1-q_2)]}\\
&=\frac{\mathrm{d}p_2}{C[\exp(q_1-q_2)-\exp(q_2-q_3)]}\\
&=\frac{\mathrm{d}p_3}{C[\exp(q_2-q_3)-\exp(q_3-q_1)]}.
\end{aligned}
\tag{20}
$$

由此得

$$
\begin{aligned}
\mathrm{d}p_1 &= C[\exp(q_3-q_1)-\exp(q_1-q_2)]\mathrm{d}t,\\
\mathrm{d}p_2 &= C[\exp(q_1-q_2)-\exp(q_2-q_3)]\mathrm{d}t,\\
\mathrm{d}p_3 &= C[\exp(q_2-q_3)-\exp(q_3-q_1)]\mathrm{d}t,
\end{aligned}
$$

因此有

$$
\mathrm{d}(p_1+p_2+p_3)=0.
$$

积分得

$$
I_2 = p_1+p_2+p_3 = \mathrm{const}.
\tag{21}
$$

由(20)式，得

$$
\begin{aligned}
&(p_2+p_3)\mathrm{d}p_1+(p_3+p_1)\mathrm{d}p_2\\
&+(p_1+p_2)\mathrm{d}p_3\\
&=C[\exp(q_3-q_1)-\exp(q_1-q_2)]\\
&\quad\times(\mathrm{d}q_2+\mathrm{d}q_3)\\
&\quad+C[\exp(q_1-q_2)-\exp(q_2-q_3)]\\
&\quad\times(\mathrm{d}q_3+\mathrm{d}q_1)\\
&\quad+C[\exp(q_2-q_3)-\exp(q_3-q_1)]\\
&\quad\times(\mathrm{d}q_1+\mathrm{d}q_2)\\
&=C\mathrm{d}[\exp(q_3-q_1)+\exp(q_1-q_2)\\
&\quad+\exp(q_2-q_3)],
\end{aligned}
$$

积分得

$$
\begin{aligned}
I_3 &= p_1p_2+p_2p_3+p_3p_1-C[\exp(q_3-q_1)\\
&\quad+\exp(q_1-q_2)+\exp(q_2-q_3)]\\
&=\mathrm{const}.
\end{aligned}
\tag{22}
$$

由(20)式，还有

$$
\begin{aligned}
& p_2 p_3 \mathrm{d} p_1 + p_3 p_1 \mathrm{d} p_2 + p_1 p_2 \mathrm{d} p_3 \\
&= C p_1 \exp(q_2 - q_3)(p_2 - p_3)\mathrm{d}t \\
&\quad + C p_2 \exp(q_3 - q_1)(p_3 - p_1)\mathrm{d}t \\
&\quad + C p_3 \exp(q_1 - q_2)(p_1 - p_2)\mathrm{d}t,
\end{aligned}
$$

积分得

$$
\begin{aligned}
I_4 &= p_1 p_2 p_3 - C[p_1 \exp(q_2 - q_3) \\
&\quad + p_2 \exp(q_3 - q_1) + p_3 \exp(q_1 - q_2)] \\
&= \text{const}. \qquad (23)
\end{aligned}
$$

所得积分(18),(21),(22)和(23)分别与积分(10),(11),(12)和(13)一致.

5. 结 论

本文用 Poisson 方法和 Noether 方法得到三质点 Toda 晶格运动的 4 个积分,其中 3 个是彼此独立的. 对于一般的微分方程,只要可以 Hamilton 化,都可用上述两种方法求积分.

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Integral of differential equations of three-particle Toda lattice*

He Guang[1)†] Mei Feng-Xiang[2)]

1) (*School of Aerospace Science and Engineering*, *Beijing Institute of Technology*, *Beijing* 100081, *China*)

2) (*School of Science*, *Beijing Institute of Technology*, *Beijing* 100081, *China*)

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Abstract

The differential equations of three-particle Toda lattice are a Hamiltonian system. The integrals of the equations can be obtained by using the Noether theory and the Poisson theory.

Keywords: Toda lattice, Hamiltonian system, Noether theory, Poisson theory

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† Corresponding author. E-mail: heguang@bit.edu.cn