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Lagrange 系统的共形不变性与 Hojman 守恒量*

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研究了一般完整 Lagrange 系统在无限小变换下的共形不变性,推导出共形不变性的确定方程,并且找到在特殊无限小变换下的共形不变性并且是 Lie 对称性的共形因子,接下来导出 Lagrange 系统的运动微分方程共形不变时的 Hojman 守恒量,并给出应用算例.

关键词:Lagrange 系统,共形不变性,Hojman 守恒量,确定方程

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1. 引 言

动力学系统的对称性和守恒量的研究在现代数理科学中占有重要地位,也是分析力学的一个近代发展方向[1—14]. 1918 年 Noether 研究了力学系统的 Hamilton 作用量范函在时空无限小单参数变换群作用下的不变性,发现作用量的每一种连续对称性都有一个守恒量与之对应,揭示了力学系统守恒量与其内在的动力学对称性之间的潜在关系,建立了 Noether 对称性理论[15],从此被广泛应用于规范场论的研究领域.近年来我国学者深入而广泛地研究了动力学系统的 Noether 对称性,Lie 对称性和形式不变性[16—25].文献[26]对各种约束力学系统的上述三种主要对称性与守恒量进行了系统、深入、全面地研究.共形不变性理论是上个世纪 60—70 年代在规范场论,特别是引力规范场论中的热点课题[27,28],近期在动力学系统中有了新的应用研究[29,30].文献[29]利用几何方法研究了 Hamilton 系统的共形不变性,讨论了 Hamilton 系统共形不变性的几何结构及其与一般对称性的关系.文献[31]研究了 Birkhoff 动力学系统的共形不变性并导出了 Noether 守恒量.本文研究了一般完整 Lagrange 系统的共形不变性,给出 Lagrange 系统共形不变性的定义和确定方程,并且得到了系统的守恒量,最后给出一个算例来验证本文成果的应用.

2. Lagrange 系统的共形不变性

对于具有 n 个自由度的 Lagrange 系统,其运动微分方程有如下形式:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=0 \qquad (s=1,\cdots,n), \tag{1}$$

由(1)式可得

$$A_{sk}\ddot{q}_k+B_s=0 \qquad (s,k=1,\cdots,s), \tag{2}$$

其中

$$A_{sk}=\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k},$$

$$B_s=\frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2 L}{\partial \dot{q}_s\partial t}-\frac{\partial L}{\partial q_s}.$$

假设系统非奇异,即

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0,$$

可解出

$$\ddot{q}_k=\alpha_k(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \tag{3}$$

若令

$$F_s=A_{sk}\ddot{q}_k+B_s, \tag{4}$$

寻求 Lagrange 系统的共形不变性,也就是寻求方程(1)或(2)共形不变所对应的独立或非独立变量的变换集.考虑方程(2)的对称性,为此,取时间 t 和广义

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坐标 q_s 的无限小单参数点变换群

$$t^* = t, \qquad q_s^*(t^*) = q_s + \Delta q_s,$$

其展开式为

$$t^* = t, q_s^*(t^*) = q_s + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{5}$$

引入无限小生成元向量

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s},$$

其一次扩展为

$$X^{(1)} = X^{(0)} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s},$$

二次扩展为

$$X^{(2)} = X^{(1)} + \ddot{\xi}_s \frac{\partial}{\partial \ddot{q}_s}.$$

定义 1　若对于 F_s，在无限小生成元 $\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$，$\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ 的变换下，若满足

$$X^{(2)} F_s = \mathscr{l}_s^k F_k, \tag{6}$$

则称二阶微分方程为共形不变.

3. 共形不变性并且是 Lie 对称性的共形因子

为得到共形不变性的共形因子表达式，要求(3)式在无限小生成元向量的作用下既是共形不变的又要是 Lie 对称的. 因此

需计算差值

$$X^{(1)} F_i - X^{(1)} F_i \big|_{F_i = 0}. \tag{7}$$

因为

$$\begin{aligned} X^{(2)} F_s &= A_{sk}\ddot{\xi}_k + X^{(1)}(B_s) + X^{(0)}(A_{sk})\ddot{q}_k \\ &= A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\ddot{q}_m + \frac{\partial^2 \xi_k}{\partial t \partial q_m}\dot{q}_m + \frac{\partial^2 \xi_k}{\partial t^2} + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\ddot{q}_m \dot{q}_l + \frac{\partial^2 \xi_k}{\partial q_l \partial t}\dot{q}_l + \frac{\partial^2 \xi_k}{\partial q_l \partial q_m}\dot{q}_m \dot{q}_l \right. \\ &\quad \left. + \frac{\partial \xi_k}{\partial q_l}\ddot{q}_l + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_k}{\partial \dot{q}_l}\right)\ddot{q}_l + \frac{\partial \xi_k}{\partial \dot{q}_l}\dddot{q}_l\right] + X^{(0)}(B_s) + \left[\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_l}\dot{q}_l + \frac{\partial \xi_k}{\partial \dot{q}_l}\ddot{q}_l\right]\frac{\partial B_s}{\partial \dot{q}_k} \\ &\quad + X^{(0)}(A_{sk})\ddot{q}_k, \end{aligned} \tag{8}$$

$$\begin{aligned} X^{(2)} F_s \big|_{F_s = 0} &= A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m}\alpha_m + \frac{\partial^2 \xi_k}{\partial t \partial q_m}\dot{q}_m + \frac{\partial^2 \xi_k}{\partial t^2} + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\alpha_m \dot{q}_l + \frac{\partial^2 \xi_k}{\partial q_l \partial t}\dot{q}_l \right. \\ &\quad \left. + \frac{\partial^2 \xi_k}{\partial q_l \partial q_m}\dot{q}_m \dot{q}_l + \frac{\partial \xi_k}{\partial q_l}\alpha_l + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_k}{\partial \dot{q}_l}\right)\alpha_l + \frac{\partial \xi_k}{\partial \dot{q}_l}\dddot{q}\right] + X^{(0)}(B_s) \\ &\quad + \left[\frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_l}\dot{q}_l + \frac{\partial \xi_k}{\partial \dot{q}_l}\alpha_l\right]\frac{\partial B_s}{\partial \dot{q}_k} + X^{(0)}(A_{sk})\alpha_k, \end{aligned} \tag{9}$$

因此(8)式减(9)式，并且利用

$$\ddot{q}_m - \alpha_m = \ddot{q}_m + A^{mr}B_r = A^{mr}(A_{rm}\ddot{q}_m + B_r) = A^{mr}F_r, \tag{10}$$

令

$$B_s^r = A_{sk}\left[\frac{\partial^2 \xi_k}{\partial t \partial \dot{q}_m} + \frac{\partial^2 \xi_k}{\partial q_l \partial \dot{q}_m}\dot{q}_l + \frac{\partial \xi_k}{\partial q_m} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \xi_k}{\partial \dot{q}_m}\right)\right]A^{mr} + \frac{\partial \xi_k}{\partial \dot{q}_m}A^{mr}\frac{\partial B_s}{\partial \dot{q}_k} + X^{(0)}(A_{sk})A^{kr}, \tag{11}$$

则

$$X^{(2)} F_s - X^{(2)} F_s \big|_{F_s = 0} = B_s^r F_r. \tag{12}$$

若 Lagrange 系统具有共形不变性且是 Lie 对称性则可得

$$\mathscr{l}_s^r F_r - B_s^r F_r = X^{(2)} F_s \big|_{F_s = 0} = 0,$$

即

$$(\mathscr{l}_s^r - B_s^r) F_r = 0, \tag{13}$$

于是可得

$$\mathscr{l}_s^r = B_s^r, \tag{14}$$

于是有以下命题：

命题 1 对于 Lagrange 系统，其 Lie 对称性同时又是共形不变性的充分必要条件是生成元满足

$$\ell_s^r = B_s^r = A_{sk}\left[\frac{\partial^2\xi_k}{\partial t\partial\dot{q}_m} + \frac{\partial^2\xi_k}{\partial q_l\partial\dot{q}_m}\dot{q}_l + \frac{\partial\xi_k}{\partial q_m} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial\dot{q}_m}\right)\right]A^{mr} + \frac{\partial\xi_k}{\partial\dot{q}_m}A^{mr}\frac{\partial B_s}{\partial\dot{q}_k} + X^{(0)}(A_{sk})A^{kr}$$

$$(s,k,m,l,r = 1,\cdots,n). \tag{15}$$

推论　若 Lagrange 方程(1)可规范为标准形式

$$F_s \equiv \ddot{q}_s - \alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0 \quad (s = 1,\cdots,n), \tag{16}$$

在无限小单参数点变换群(5)作用下,方程(16)共形不变性同时是 Lie 对称性的充分与必要条件是共形因子

$$\ell_m^k = \frac{\partial^2\xi_k}{\partial t\partial\dot{q}_m} + \frac{\partial^2\xi_k}{\partial q_l\partial\dot{q}_m}\dot{q}_l + \frac{\partial\xi_k}{\partial q_m} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial\dot{q}_m}\right) + \frac{\partial\xi_l}{\partial\dot{q}_m}\frac{\partial B_k}{\partial\dot{q}_l}. \tag{17}$$

事实上,此时(16)式相当于(2)式中 $A_{sk} = \delta_{sk}$,$B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = -\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$,容易由(15)式得到(17)式.

4. 共形不变性与 Hojman 守恒量

命题 2　在特殊无限小变换(5)作用下,如果生成元 ξ_s 满足方程

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial\alpha_s}{\partial q_k}\xi_k + \frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k, \tag{18}$$

其中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial\dot{q}_s},$$

且存在某函数 $\mu = \mu(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 使得

$$\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{19}$$

则共形不变性同时是 Lie 对称的,并可以直接导致 Hojman 守恒量

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \mathrm{const}. \tag{20}$$

命题 2 的证明参考文献[26].

5. 算　　例

二自由度系统为

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2),$$

$$Q_1 = -\frac{t}{1+t^2}\dot{q}_1,\quad Q_2 = \frac{1}{1+t^2}\dot{q}_1, \tag{21}$$

系统的运动微分方程为

$$\ddot{q}_1 = -\frac{t}{1+t^2}\dot{q}_1,\quad \ddot{q}_2 = \frac{1}{1+t^2}\dot{q}_1, \tag{22}$$

无限小生成元为

$$\xi_0 = \xi_1 = 0,\quad \xi_2 = (\dot{q}_1 + t\dot{q}_2 - q_2)^2, \tag{23}$$

所以

$$X^{(2)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\xi_0)\frac{\partial}{\partial\dot{q}_s} + (\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial\ddot{q}_s} = \xi_2\frac{\partial}{\partial q} + \dot{\xi}_2\frac{\partial}{\partial\dot{q}} + \ddot{\xi}_2\frac{\partial}{\partial\ddot{q}},$$

因此

$$X^{(2)}F = \xi_2\frac{\partial}{\partial q_2} + \dot{\xi}_2\frac{\partial}{\partial\dot{q}_2} + \ddot{\xi}_2\frac{\partial}{\partial\ddot{q}_2} \times \begin{pmatrix} \ddot{q}_1 + \frac{t}{1+t^2}\dot{q}_1 \\ \ddot{q}_2 - \frac{1}{1+t^2}\dot{q}_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2(\dot{q}_1 + t\dot{q}_2 - q_2) \end{pmatrix} \times \begin{pmatrix} \ddot{q}_1 + \frac{t}{1+t^2}\dot{q}_1 \\ \ddot{q}_2 - \frac{1}{1+t^2}\dot{q}_1 \end{pmatrix}. \tag{24}$$

也可以利用推论中的(17)式计算出共形因子

$$\ell_s^r = \begin{pmatrix} 0 & 0 \\ 0 & 2(\dot{q}_1 + t\dot{q}_2 - q_2) \end{pmatrix}, \tag{25}$$

显然其结果与(24)式相同.因此共形不变性的确定方程为

$$X^{(2)}F = \begin{pmatrix} 0 & 0 \\ 0 & 2(\dot{q}_1 + t\dot{q}_2 - q_2) \end{pmatrix}$$

$$\times\begin{pmatrix}\ddot{q}_1+\dfrac{t}{1+t^2}\dot{q}_1\\ \ddot{q}_2-\dfrac{1}{1+t^2}\dot{q}_1\end{pmatrix}. \tag{26}$$

此时微分方程(21)在特殊无限小变换(22)变换下是共形不变的同时还是 Lie 对称的．利用命题 2，当 $\mu=(1+t^2)^{1/2}$ 时，可求得此系统的 Hojman 守恒量

$$I_{\mathrm{H}}=\frac{\partial\xi_2}{\partial q_2}=-2(\dot{q}_1+t\dot{q}_2-q_2)=\text{const}. \tag{27}$$

6. 结　　论

本文研究了一般完整 Lagrange 系统在特殊无限小变换下的共形不变性，给出了这类动力学系统共形不变性的定义和确定方程，得到共形因子，最后找到系统在特殊无限小变换下的 Hojman 守恒量．

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Conformal symmetry and Hojman conserved quantity of Lagrange system*

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Abstract

In this paper the conformal invariance under special infinitesimal transformations of Lagrange systems is studied. The necessary and sufficient conditions for the conformal invariance under infinitesimal transformations which is Lie symmetric at the same time are given. Finally we get the Hojman conserved quantities of the conformal invariance.

Keywords: Lagrange systems, conformal invariance, Hojman conserved quantities, determining equations

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