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弹性矩形板方程在非线性边界条件下整体解的存在唯一性*

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考虑了在非线性边界条件下的弹性矩形板方程.利用 Galerkin 方法,首先证明了该方程在非线性边界(a)及初值 $w^0 \in W, w^1 \in W$ 的条件下初边值问题存在唯一整体弱解 $w(t)$.其次证明了该方程在非线性边界(b)及初值 $w^0 \in W_1, w^1 \in W_1$ 的条件下初边值问题也存在唯一整体弱解 $w(t)$.

关键词:弹性矩形板方程,非线性边界条件,初边值问题,整体解

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1. 引 言

考虑弹性矩形板方程

$$w_{xxxx} + w_{yyyy} + 2w_{xxyy} + \frac{\rho}{D}w_{tt} + \frac{\varepsilon\mu}{D}w_t - \left(\frac{N_1}{D} + \frac{T}{2D}\int_{\Omega} w_x^2 \mathrm{d}x\mathrm{d}y\right)w_{xx} - \left(\frac{N_2}{D} + \frac{T}{2D}\int_{\Omega} w_y^2 \mathrm{d}x\mathrm{d}y\right)w_{yy} = 0,$$

$$(x, y; t) \in \Omega \times (0, \infty), \tag{1}$$

其中 ρ 为板的质量密度,D 是板的弯曲刚度,N_1,N_2 为板中应力,T 为板中面发生单位面积改变所需要的薄膜张力,ε 为正的无量纲的小参数,$\Omega = (0,1) \times (0,1)$ 为变形前板中面所占据的区域,μ 为阻尼系数,在初始条件

$$w(x, y, 0) = w^0, \quad w_t(x, y, 0) = w^1, \tag{2}$$

及非线性边界条件(a)

$$w(0, y, t) = w(x, 0, t) = w_x(1, y, t) = w_y(x, 1, t) = w_{xx}(0, y, t) = w_{yy}(x, 0, t) = 0, \tag{3}$$

$$w_{xxx}(1, y, t) = f(w_t(1, y, t)), \tag{4}$$

$$w_{yyy}(x, 1, t) = g(w_t(x, 1, t)), \tag{5}$$

及非线性边界条件(b)

$$w(0, y, t) = w(x, 0, t) = w_x(0, y, t) = w_y(x, 0, t) = w_{xx}(1, y, t) = w_{yy}(x, 1, t) = 0, \tag{6}$$

$$w_{xxx}(1, y, t) + 2w_{xyy}(1, y, t) - \left(\frac{N_1}{D} + \frac{T}{2D}\| w_x \|^2\right) w_x(1, y, t) = f(w_t(1, y, t)), \tag{7}$$

$$w_{yyy}(x, 1, t) - 2w_{xy}(x, 1, t) - \left(\frac{N_2}{D} + \frac{T}{2D}\| w_y \|^2\right) w_y(x, 1, t) = g(w_t(x, 1, t)) \tag{8}$$

下的整体弱解的存在唯一性.这个问题基于由 To[1] 提出的一维梁方程

$$u_{tt} + u_{xxxx} - M\left(\int_0^l | u_x |^2 \mathrm{d}x\right)u_{xx} = 0$$

$$\text{在}[0, L] \times R^+,$$

To 研究了此方程在非齐次边界条件下整体弱解的存在唯一性及系统能量的指数衰减.事实上,对于一维梁方程,从 1950 年,由 Woinowsky-Krieger[2] 提出来以后,随后一系列的作者研究过它的初边值问题,其中 Ball[3,4] 的方法比较突出,即 Galerkin 方法证明了更一般的梁方程初边值问题的解的存在唯一性.还

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有许多的作者[5—8]研究了上述问题的混沌运动.对于二维板,也有不少作者进行过研究,如 Ciro[9]研究了热弹性板解的稳定态的空间行为,韩强[10]给出了一类非线性板的混沌运动.但却一直很少有人研究过它的初边值问题解的存在性.本文利用 Galerkin 方法,给出了二维板方程(1)在两组非线性边界条件(a)及(b)下初边值问题整体弱解的存在唯一性.

2. 基本假设

对于方程(1),我们有如下假设:设 $\Omega=(0,1)\times(0,1)$,记 $Q=\Omega\times[0,T]$(T 为充分大的正数),D,N_1,N_2,ρ,ε,μ,$T>0$ 为常数.为书写方便,以下讨论中某三元函数 $h(x,y,t)$ 关于 x,y 的偏导数记为 $h_x(x,y,t)$,$h_y(x,y,t)$,$h_{xx}(x,y,t)$,$h_{yy}(x,y,t)$ 等;关于 t 的偏导数记为 $h_t(x,y,t)$,$h_{tt}(x,y,t)$ 等.$w(x,y,t)$ 有时为书写方便简记为 $w(t)$ 或 w.对于非线性函数 $f(\cdot)$,$g(\cdot)$,设 $f,g:R\to R$ 为连续可微函数,满足

$$
\begin{aligned}
f(0)&=0,且[f(r)-f(s)](r-s)\\
&\geqslant l\,|r-s|^2,\forall\, r,s\in R,l>0,\\
g(0)&=0,且[g(r)-g(s)](r-s)\\
&\geqslant k\,|r-s|^2,\forall\, r,s\in R,k>0.
\end{aligned}
$$

我们定义如下 Sobolev 空间.

$$
\begin{aligned}
V&=\{w\in H^2(\Omega)\,|\,w(0,y,t)\\
&=w_x(1,y,t)=w(x,0,t)\\
&=w_y(x,1,t)=0\},\\
W&=\{w\in V\cap H^4(\Omega)\,|\,w_{xx}(0,y,t)\\
&=w_{yy}(x,0,t)=0\},\\
V_1&=\{w\in H^2(\Omega)\,|\,w(0,y,t)\\
&=w_x(0,y,t)=w(x,0,t)\\
&=w_y(x,0,t)=0\},\\
W_1&=\{w\in V\cap H^4(\Omega)\,|\,w_{xx}(1,y,t)\\
&=w_{yy}(x,1,t)=0\}.
\end{aligned}
$$

3. 主要结果及其证明

定理 1 设 $f(s)$,$g(s)$ 满足上述假设条件,则对任意的 w^0,$w^1\in W$ 且满足相容性条件

$$
\begin{aligned}
w^0_{xxx}(1,y)&=f(w^1(1,y)),\\
w^0_{yyy}(x,1)&=g(w^1(x,1)),
\end{aligned}
$$

存在函数

$$
\begin{aligned}
w(t)\in L^2(0,\infty;W)\cap C^0([0,\infty);V)\\
\cap W^{2,\infty}(0,\infty;L^2(\Omega))
\end{aligned}
$$

满足系统(1)—(5).

证明 首先考虑系统(1)—(5)的变形问题,即求 $w(t)\in W$,对一切 $v\in V$ 有

$$
\begin{aligned}
&\int_\Omega w_{xx}v_{xx}\,\mathrm{d}x\mathrm{d}y+\int_\Omega w_{yy}v_{yy}\,\mathrm{d}x\mathrm{d}y\\
&+2\int_\Omega w_{xy}v_{xy}\,\mathrm{d}x\mathrm{d}y+\frac{\rho}{D}\int_\Omega w_{tt}v\,\mathrm{d}x\mathrm{d}y+\frac{\varepsilon\mu}{D}\int_\Omega w_tv\,\mathrm{d}x\mathrm{d}y\\
&+\left(\frac{N_1}{D}+\frac{T}{2D}\int_\Omega(w_x)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w_xv_x\,\mathrm{d}x\mathrm{d}y\\
&+\left(\frac{N_2}{D}+\frac{T}{2D}\int_\Omega(w_y)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w_yv_y\,\mathrm{d}x\mathrm{d}y\\
&+\int_0^1 f(w_t(1,y,t))v(1,y,t)\,\mathrm{d}y\\
&+\int_0^1 g(w_t(x,1,t))v(x,1,t)\,\mathrm{d}x=0.
\end{aligned}\tag{9}
$$

下面我们用 Faedo-Galerkin 逼近方法得到解的存在性.

步骤 1 近似解.设 $\{\omega_j\}$ 是 W 的一个解,并且 W^m 是由向量 ω_1,ω_2,…,ω_m 所生成的空间,寻求一个函数

$$
w^m(t)=\sum_{j=1}^m a_j^m(t)\,\omega_j,
$$

其中 $a_j^m(t)$ 为未知函数,使得对于任意的 $\omega\in W^m$ 满足逼近方程

$$
\begin{aligned}
&\int_\Omega w^m_{xx}\omega_{xx}\,\mathrm{d}x\mathrm{d}y+\int_\Omega w^m_{yy}\omega_{yy}\,\mathrm{d}x\mathrm{d}y\\
&+2\int_\Omega w^m_{xy}\omega_{xy}\,\mathrm{d}x\mathrm{d}y+\frac{\rho}{D}\int_\Omega w^m_{tt}\omega\,\mathrm{d}x\mathrm{d}y+\frac{\varepsilon\mu}{D}\int_\Omega w^m_t\omega\,\mathrm{d}x\mathrm{d}y\\
&+\left(\frac{N_1}{D}+\frac{T}{2D}\int_\Omega(w^m_x)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w^m_x\omega_x\,\mathrm{d}x\mathrm{d}y\\
&+\left(\frac{N_2}{D}+\frac{T}{2D}\int_\Omega(w^m_y)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w^m_y\omega_y\,\mathrm{d}x\mathrm{d}y\\
&+\int_0^1 f(w^m_t(1,y,t))\,\omega(1,y)\,\mathrm{d}y\\
&+\int_0^1 g(w^m_t(x,1,t))\,\omega(x,1)\,\mathrm{d}x=0,
\end{aligned}\tag{10}
$$

及

$$
w^m(x,y,0)=w^{0m}\to w^0\quad(在\ W\ 中强收敛),\tag{11}
$$

$$
w^m_t(x,y,0)=w^{1m}\to w^1\quad(在\ W\ 中强收敛).\tag{12}
$$

则(10)—(12)式等价于一个常微分方程组的柯西问题,则逼近方程(10)—(12)在$[0,t_m]$中存在解 $w^m(t)$.

以下讨论中,用 C 表示与 m,T 均无关的常数,且它们在不同的表达式中可能有不同的值.

步骤 2　作先验估计.

估计 1　在(10)式中取 $\omega = w_t^m(t)$，再利用分部积分得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w_{xx}^m\|^2+\|w_{yy}^m\|^2+2\|w_{xy}^m\|^2+\frac{\rho}{D}\|w_t^m\|^2+\frac{N_1}{D}\|w_x^m\|^2+\frac{T}{4D}\|w_x^m\|^4+\frac{N_2}{D}\|w_y^m\|^2+\frac{T}{4D}\|w_y^m\|^4\Big]+\frac{\varepsilon\mu}{D}\|w_t^m\|^2+\int_0^1 f(w_t^m(1,y,t))w_t^m(1,y,t)\mathrm{d}y+\int_0^1 g(w_t^m(x,1,t))w_t^m(x,1,t)\mathrm{d}x=0, \tag{13}$$

考虑到 f,g 的假设，然后再从 0 到 $t(<t_m)$ 积分(13)式得

$$\begin{aligned}&\|w_{xx}^m\|^2+\|w_{yy}^m\|^2+2\|w_{xy}^m\|^2+\frac{N_1}{D}\|w_x^m\|^2+\frac{N_2}{D}\|w_y^m\|^2+\frac{\rho}{D}\|w_t^m\|^2+\frac{T}{4D}\|w_x^m\|^4+\frac{T}{4D}\|w_y^m\|^4\\ \leqslant\ &\|w_{xx}^m(0)\|^2+\|w_{yy}^m(0)\|^2+2\|w_{xy}^m(0)\|^2+\frac{N_1}{D}\|w_x^m(0)\|^2+\frac{N_2}{D}\|w_y^m(0)\|^2+\frac{\rho}{D}\|w_t^m(0)\|^2+\frac{T}{4D}\|w_x^m(0)\|^4+\frac{T}{4D}\|w_y^m(0)\|^4.\end{aligned} \tag{14}$$

又由于初始条件

$w^m(x,y,0)=w^{0m}\to w^0$　**(在 W 中强收敛)，**

$w_t^m(x,y,0)=w^{1m}\to w^1$　**(在 W 中强收敛)，**

则存在一常数 C 仅仅依赖于 T 使得

$$\|w_{xx}^m\|^2+\|w_{yy}^m\|^2+2\|w_{xy}^m\|^2+\frac{N_1}{D}\|w_x^m\|^2+\frac{N_2}{D}\|w_y^m\|^2+\frac{\rho}{D}\|w_t^m\|^2+\frac{T}{4D}\|w_x^m\|^4+\frac{T}{4D}\|w_y^m\|^4\leqslant C, \tag{15}$$

对一切 $t\in[0,T]$ 和对一切 $m\in N$ 成立. 再由庞加莱不等式有

$$\|w^m\|\leqslant\frac{1}{\sqrt{2}}\|w_x^m\|\leqslant C.$$

估计 2　在(10)式中取 $\omega = w_{tt}^m(0)$ 及 $t=0$，利用分部积分，并考虑到相容性条件得

$$\frac{\rho}{D}\|w_{tt}^m(0)\|^2\leqslant\Big[\|w_{xxxx}^{0m}\|+2\|w_{xxyy}^{0m}\|+\|w_{yyyy}^{0m}\|+\frac{\varepsilon\mu}{D}\|w^{1m}\|+\frac{N_1}{D}\|w_{xx}^{0m}\|+\frac{T}{2D}\|w_x^{0m}\|^2\|w_{xx}^{0m}\|+\frac{N_2}{D}\|w_{yy}^{0m}\|+\frac{T}{2D}\|w_y^{0m}\|^2\|w_{yy}^{0m}\|\Big]\|w_{tt}^m(0)\|. \tag{16}$$

故存在常数 $C>0$，使得

$$\|w_{tt}^m(0)\|\leqslant C,\ \forall m\in N.$$

估计 3　在(10)式中分别取 $t=t+\xi$ 和 $t=t$ 后，两式相减，再取 $\omega=w_t^m(t+\xi)-w_t^m(t)$，并利用分部积分得

$$\begin{aligned}&\frac{\rho}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t^m(t+\xi)-w_t^m(t)\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{xx}^m(t+\xi)-w_{xx}^m(t)\|^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{yy}^m(t+\xi)-w_{yy}^m(t)\|^2\\&+\frac{\mathrm{d}}{\mathrm{d}t}\|w_{xy}^m(t+\xi)-w_{xy}^m(t)\|^2+\frac{N_1}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x^m(t+\xi)-w_x^m(t)\|^2+\frac{N_2}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_y^m(t+\xi)-w_y^m(t)\|^2\\&+\frac{\varepsilon\mu}{D}\|w_t^m(t+\xi)-w_t^m(t)\|^2\\&+\frac{T}{2D}\int_\Omega(w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi)\times(w_{xt}^m(t+\xi)-w_{xt}^m(t))\mathrm{d}x\mathrm{d}y\\&-\frac{T}{2D}\int_\Omega(w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t)\times(w_{xt}^m(t+\xi)-w_{xt}^m(t))\mathrm{d}x\mathrm{d}y\\&+\frac{T}{2D}\int_\Omega(w_y^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t+\xi)\times(w_{yt}^m(t+\xi)-w_{yt}^m(t))\mathrm{d}x\mathrm{d}y\\&-\frac{T}{2D}\int_\Omega(w_y^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t)\times(w_{yt}^m(t+\xi)-w_{yt}^m(t))\mathrm{d}x\mathrm{d}y\\&+\int_0^1[g(w_t^m(x,1,t+\xi))-g(w_t^m(x,1,t))]\times(w_t^m(x,1,t+\xi)-w_t^m(x,1,t))\mathrm{d}x\\&+\int_0^1[f(w_t^m(1,y,t+\xi))-f(w_t^m(1,y,t))]\times(w_t^m(1,y,t+\xi)-w_t^m(1,y,t))\mathrm{d}y=0,\end{aligned} \tag{17}$$

记

$$I_1=\frac{T}{2D}\int_\Omega(w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi)$$

$$
\begin{aligned}
&\times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y \\
&- \frac{T}{2D}\int_\Omega (w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t) \\
&\times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

$$
\begin{aligned}
I_2 = &\frac{T}{2D}\int_\Omega (w_y^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t+\xi) \\
&\times (w_{yt}^m(t+\xi) - w_{yt}^m(t))\mathrm{d}x\mathrm{d}y \\
&- \frac{T}{2D}\int_\Omega (w_y^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t) \\
&\times (w_{yt}^m(t+\xi) - w_{yt}^m(t))\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

现在开始估计 I_1，

$$
\begin{aligned}
|I_1| = &\Big|\frac{T}{2D}\int_\Omega (w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi) \\
&\times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y \\
&- \frac{T}{2D}\int_\Omega (w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t) \\
&\times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y\Big| \\
= &\Big|\frac{T}{2D}\|(w_x^m(t+\xi))\|^2\int_\Omega (w_{xx}^m(tt\xi) \\
&- w_{x^2}^m(t))(w_t^m(t+\xi) - w_t^m(t))\mathrm{d}x\mathrm{d}y \\
&+ \frac{T}{2D}\int_\Omega ((w_x^m(t+\xi))^2 \\
&- (w_x^m(t))^2)\mathrm{d}x\mathrm{d}y\int_\Omega w_{xx}^m(t) \\
&\times (w_t^m(t+\xi) - w_t^m(t))\mathrm{d}x\mathrm{d}y\Big| \\
\leqslant &C\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 \\
&+ C\frac{\rho}{2D}\|w_t^m(t+\xi) - w_t^m(t)\|^2.
\end{aligned}
$$

同理

$$
\begin{aligned}
|I_2| \leqslant &C\|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 \\
&+ C\frac{\rho}{2D}\|w_t^m(t+\xi) - w_t^m(t)\|^2.
\end{aligned}
$$

故从(17)式有

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 \\
&+ \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 \\
&+ 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&+ \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 \\
&+ \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 \\
&+ \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big] \\
\leqslant &C\Big(\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 \\
&+ \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 \\
&+ 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&+ \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 \\
&+ \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 \\
&+ \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big) \\
&- \int_0^1 [g(w_t^m(x,1,t+\xi)) - g(w_t^m(x,1,t))] \\
&\times (w_t^m(x,1,t+\xi) - w_t^m(x,1,t))\mathrm{d}x \\
&- \int_0^1 [f(w_t^m(1,y,t+\xi)) - f(w_t^m(1,y,t))] \\
&\times (w_t^m(1,y,t+\xi) - w_t^m(1,y,t))\mathrm{d}y. \qquad (18)
\end{aligned}
$$

考虑到 f,g 的条件故(18)式变为

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 \\
&+ \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 \\
&+ 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&+ \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 \\
&+ \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 \\
&+ \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big] \\
\leqslant &C\Big(\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 \\
&+ \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 \\
&+ 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&+ \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 \\
&+ \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 \\
&+ \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big).
\end{aligned}
$$

然后上式两边同除以 ξ^2，再令 $\xi\to 0$，则有

$$
\begin{aligned}
&\|w_{xxt}^m(t)\|^2 + \|w_{yyt}^m(t)\|^2 \\
&+ 2\|w_{xyt}^m(t)\|^2 + \frac{N_1}{D}\|w_{xt}^m(t)\|^2 \\
&+ \frac{N_2}{D}\|w_{yt}^m(t)\|^2 + \frac{\rho}{D}\|w_{tt}^m(t)\|^2
\end{aligned}
$$

$$\leqslant\Big(\|w_{xx}^{1m}\|^2+\|w_{yy}^{1m}\|^2+2\|w_{xy}^{1m}\|^2+\frac{N_1}{D}\|w_x^{1m}\|^2+\frac{N_2}{D}\|w_y^{1m}\|^2+\frac{\rho}{D}\|w_{tt}^m(0)\|^2\Big)\exp(CT),$$

则我们可找到一仅仅依赖于 T 的常数 $C>0$，使得

$$\|w_{xxt}^m(t)\|^2+\|w_{yyt}^m(t)\|^2+2\|w_{xyt}^m(t)\|^2+\frac{N_1}{D}\|w_{xt}^m(t)\|^2+\frac{N_2}{D}\|w_{yt}^m(t)\|^2+\frac{\rho}{D}\|w_{tt}^m(t)\|^2\leqslant C.$$

步骤 3　收敛性. 综上可得 $\|w^m\|$，$\|w_x^m\|$，$\|w_y^m\|$，$\|w_{xx}^m\|$，$\|w_{yy}^m\|$，$\|w_{xy}^m\|$，$\|w_{tt}^m\|$，$\|w_{xt}^m\|$，$\|w_{yt}^m\|$，$\|w_{xxt}^m\|$，$\|w_{yyt}^m\|$，$\|w_{xyt}^m\|<C$，由 $\|w^m\|$，$\|w_x^m\|$，$\|w_{xx}^m\|$，$\|w_y^m\|$，$\|w_{yy}^m\|$ 有界可得 w^m 在 $L^\infty(0,T;V)$ 中有界，由 $\|w_t^m\|$，$\|w_{tx}^m\|$，$\|w_{ty}^m\|$，$\|w_{txx}^m\|$，$\|w_{tyy}^m\|$ 有界可得 w_t^m 在 $L^\infty(0,T;V)$ 中有界，故存在子序列 $\{w^\mu\}$ 满足 $w^\mu\to w$ 在 $L^\infty(0,T;V)$ 中弱星收敛；$w_t^\mu\to w_t$ 在 $L^\infty(0,T;V)$ 中弱星收敛. 从而说明存在整体解

$$w(t)\in L^2(0,\infty;W)\cap C^0([0,\infty);V)\cap W^{2,\infty}(0,\infty;L^2(\Omega)).$$

证毕.

定理 2　定理 1 中的解是唯一的.

证明　假设 w,u 是系统的两个解，记 $p=w-u$，则由于(9)式，对一切 $v\in V$，并取 $v=p_t$，则有

$$\int_\Omega p_{xx}p_{xxt}\mathrm{d}x\mathrm{d}y+\int_\Omega p_{yy}p_{yyt}\mathrm{d}x\mathrm{d}y+2\int_\Omega p_{xy}p_{xyt}\mathrm{d}x\mathrm{d}y+\frac{\rho}{D}\int_\Omega p_{tt}p_t\mathrm{d}x\mathrm{d}y$$
$$+\frac{\varepsilon\mu}{D}\int_\Omega p_tp_t\mathrm{d}x\mathrm{d}y+\frac{N_1}{D}\int_\Omega p_xp_{xt}\mathrm{d}x\mathrm{d}y+\frac{T}{2D}\int_\Omega w_x^2\mathrm{d}x\mathrm{d}y\int_\Omega w_xp_{xt}\mathrm{d}x\mathrm{d}y$$
$$-\frac{T}{2D}\int_\Omega u_x^2\mathrm{d}x\mathrm{d}y\int_\Omega u_xp_{xt}\mathrm{d}x\mathrm{d}y+\frac{N_2}{D}\int_\Omega p_yp_{yt}\mathrm{d}x\mathrm{d}y+\frac{T}{2D}\int_\Omega w_y^2\mathrm{d}x\mathrm{d}y\int_\Omega w_yp_{yt}\mathrm{d}x\mathrm{d}y$$
$$-\frac{T}{2D}\int_\Omega u_y^2\mathrm{d}x\mathrm{d}y\int_\Omega u_yp_{yt}\mathrm{d}x\mathrm{d}y+\int_0^1f(w_t(1,y,t))p_t(1,y,t)\mathrm{d}y-\int_0^1f(u_t(1,y,t))p_t(1,y,t)\mathrm{d}y$$
$$+\int_0^1g(w_t(x,1,t))p_t(x,1,t)\mathrm{d}x-\int_0^1g(u_t(x,1,t))p_t(x,1,t)\mathrm{d}x=0.\quad(19)$$

记

$$I_3=\frac{T}{2D}\int_\Omega w_x^2\mathrm{d}x\mathrm{d}y\int_\Omega w_xp_{xt}\mathrm{d}x\mathrm{d}y-\frac{T}{2D}\int_\Omega u_x^2\mathrm{d}x\mathrm{d}y\int_\Omega u_xp_{xt}\mathrm{d}x\mathrm{d}y,$$
$$I_4=\frac{T}{2D}\int_\Omega w_y^2\mathrm{d}x\mathrm{d}y\int_\Omega w_yp_{yt}\mathrm{d}x\mathrm{d}y-\frac{T}{2D}\int_\Omega u_y^2\mathrm{d}x\mathrm{d}y\int_\Omega u_yp_{yt}\mathrm{d}x\mathrm{d}y,$$
$$I_5=\int_0^1f(w_t(1,y,t))p_t(1,y,t)\mathrm{d}y-\int_0^1f(u_t(1,y,t))p_t(1,y,t)\mathrm{d}y$$
$$I_6=\int_0^1g(w_t(x,1,t))p_t(x,1,t)\mathrm{d}x-\int_0^1g(u_t(x,1,t))p_t(x,1,t)\mathrm{d}x,$$

则(19)式变形为

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_{xx}\|^2+\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_{yy}\|^2+2\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_{xy}\|^2+\frac{N_1}{D}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_x\|^2+\frac{N_2}{D}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_y\|^2+\frac{\rho}{D}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|p_t\|^2$$
$$+\frac{\varepsilon\mu}{D}\|p_t\|^2+I_3+I_4+I_5+I_6=0,\quad(20)$$

其中

$$|I_3|=\Big|\frac{T}{2D}\int_\Omega w_x^2\mathrm{d}x\mathrm{d}y\int_\Omega p_xp_{xt}\mathrm{d}x\mathrm{d}y+\frac{T}{2D}\int_\Omega w_x^2\mathrm{d}x\mathrm{d}y\int_\Omega u_xp_{xt}\mathrm{d}x\mathrm{d}y-\frac{T}{2D}\int_\Omega u_x^2\mathrm{d}x\mathrm{d}y\int_\Omega u_xp_{xt}\mathrm{d}x\mathrm{d}y\Big|$$
$$=\Big|-\frac{T}{2D}\|w_x\|^2\int_\Omega p_{xx}p_t\mathrm{d}x\mathrm{d}y+\frac{T}{2D}\int_\Omega p_x(w_x+u_x)\mathrm{d}x\mathrm{d}y\int_\Omega u_xp_{xt}\mathrm{d}x\mathrm{d}y\Big|$$
$$=\Big|-\frac{T}{2D}\|w_x\|^2\int_\Omega p_{xx}p_t\mathrm{d}x\mathrm{d}y+\frac{T}{2D}\int_\Omega p_{xx}(w+u)\mathrm{d}x\mathrm{d}y\int_\Omega u_{xx}p_t\mathrm{d}x\mathrm{d}y\Big|$$
$$\leqslant C\|p_{xx}\|\|p_t\|+C\|p_{xx}\|\|p_t\|$$
$$\leqslant C\|p_{xx}\|^2+\frac{\rho}{2D}\|p_t\|^2).$$

同理

$$|I_4| \leqslant C\left(\|p_{yy}\|^2+\frac{\rho}{2D}\|p_t\|^2\right),$$

又

$$\begin{aligned}I_5&=\int_0^1 f(w_t(1,y,t))p_t(1,y,t)\mathrm{d}y\\&\quad-\int_0^1 f(u_t(1,y,t))p_t(1,y,t)\mathrm{d}y\\&\geqslant\int_0^1 l|p_t(1,y,t)|^2\mathrm{d}y\\&\geqslant 0.\end{aligned}$$

同理

$$\begin{aligned}I_6&=\int_0^1 g(w_t(x,1,t))p_t(x,1,t)\mathrm{d}x\\&\quad-\int_0^1 g(u_t(x,1,t))p_t(x,1,t)\mathrm{d}x\\&\geqslant\int_0^1 k|p_t(x,1,t)|^2\mathrm{d}x\\&\geqslant 0.\end{aligned}$$

则(20)式变形为

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big((\|p_{xx}\|^2+\|p_{yy}\|^2+2\|p_{xy}\|^2\\&\quad+\frac{N_1}{D}\|p_x\|^2+\frac{N_2}{D}\|p_y\|^2+\frac{\rho}{D}\|p_t\|^2\Big)\\&\leqslant|I_1|+|I_2|\\&\leqslant C\left(\|p_{xx}\|^2+\|p_{yy}\|^2+\frac{\rho}{D}\|p_t\|^2\right),\end{aligned}$$

则由 Gronwall 不等式和 $w^0=u^0, w^1=u^1$,则有

$$p=0,$$

从而唯一性得证.

定理 3 设 $f(s), g(s)$ 满足上述假设条件,则对任意的 $w^0\in W_1, w^1\in W_1$ 且满足相容性条件

$$\begin{aligned}&w^0_{xxx}(1,y)+2w^0_{xyy}(1,y)\\&-\left(\frac{N_1}{D}+\frac{T}{2D}\int_\Omega w_x^2\mathrm{d}x\mathrm{d}y\right)w_x^0(1,y)\\&=f(w^1(1,y)),\\&w^0_{yyy}(x,1)-2w^0_{xy}(x,1)\\&-\left(\frac{N_2}{D}+\frac{T}{2D}\int_\Omega w_y^2\mathrm{d}x\mathrm{d}y\right)w_x^0(x,1)\\&=g(w^1(x,1)),\end{aligned}$$

存在函数

$$\begin{aligned}w(t)&\in L^2(0,\infty,W_1)\cap C^0([0,\infty);V_1)\\&\cap W^{2,\infty}(0,\infty;L^2(\Omega)).\end{aligned}$$

满足系统(1),(2),(6)—(8).

证明 首先考虑系统(1),(2),(6)—(8)的变形问题,即求 $w(t)\in W_1$,对一切 $v\in V_1$ 有

$$\begin{aligned}&\int_\Omega w_{xx}v_{xx}\mathrm{d}x\mathrm{d}y+\int_\Omega w_{yy}v_{yy}\mathrm{d}x\mathrm{d}y\\&+2\int_\Omega w_{xy}v_{xy}\mathrm{d}x\mathrm{d}y+\frac{\rho}{D}\int_\Omega w_{tt}v\mathrm{d}x\mathrm{d}y\\&+\frac{\varepsilon\mu}{D}\int_\Omega w_t v\mathrm{d}x\mathrm{d}y\\&+\left(\frac{N_1}{D}+\frac{T}{2D}\int_\Omega(w_x)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w_x v_x\mathrm{d}x\mathrm{d}y\\&+\left(\frac{N_2}{D}+\frac{T}{2D}\int_\Omega(w_y)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w_y v_y\mathrm{d}x\mathrm{d}y\\&+\int_0^1 f(w_t(1,y,t))v(1,y,t)\mathrm{d}y\\&+\int_0^1 g(w_t(x,1,t))v(x,1,t)\mathrm{d}x=0.\end{aligned}\tag{21}$$

下面我们也利用 Faedo-Galerkin 逼近方法得到解的存在性.

步骤 1 近似解.设 $\{\omega_j\}$ 是 W_1 的一个解,并且 W_1^m 是由向量 $\omega_1,\omega_2,\cdots,\omega_m$ 所生成的空间,寻求一个函数

$$w^m(t)=\sum_{j=1}^m a_j^m(t)\omega_j,$$

其中 $a_j^m(t)$ 为未知函数,使得对于任意的 $\omega\in W_1^m$ 满足逼近方程

$$\begin{aligned}&\int_\Omega w^m_{xx}\omega_{xx}\mathrm{d}x\mathrm{d}y+\int_\Omega w^m_{yy}\omega_{yy}\mathrm{d}x\mathrm{d}y\\&+2\int_\Omega w^m_{xy}\omega_{xy}\mathrm{d}x\mathrm{d}y+\frac{\rho}{D}\int_\Omega w^m_{tt}\omega\mathrm{d}x\mathrm{d}y+\frac{\varepsilon\mu}{D}\int_\Omega w^m_t\omega\mathrm{d}x\mathrm{d}y\\&+\left(\frac{N_1}{D}+\frac{T}{2D}\int_\Omega(w^m_x)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w^m_x\omega_x\mathrm{d}x\mathrm{d}y\\&+\left(\frac{N_2}{D}+\frac{T}{2D}\int_\Omega(w^m_y)^2\mathrm{d}x\mathrm{d}y\right)\int_\Omega w^m_y\omega_y\mathrm{d}x\mathrm{d}y\\&+\int_0^1 f(w^m_t(1,y,t))\omega(1,y)\mathrm{d}y\\&+\int_0^1 g(w^m_t(x,1,t))\omega(x,1)\mathrm{d}x=0,\end{aligned}\tag{22}$$

及

$$w^m(x,y,0)=w^{0m}\to w^0\quad(\text{在 } W_1 \text{ 中强收敛}),\tag{23}$$

$$w^m_t(x,y,0)=w^{1m}\to w^1\quad(\text{在 } W_1 \text{ 中强收敛}),\tag{24}$$

则(22)—(24)等价于一个常微分方程组的柯西问题,则逼近方程(22)—(24)在 $[0,t_m]$ 中存在解 $w^m(t)$.

步骤 2 作先验估计.

估计 1 在(22)式中取 $\omega=w_t^m(t)$,再利用分部积分得

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w^m_{xx}\|^2+\|w^m_{yy}\|^2+2\|w^m_{xy}\|^2$$

$$+\frac{\rho}{D}\|w_t^m\|^2+\frac{N_1}{D}\|w_x^m\|^2+\frac{T}{4D}\|w_x^m\|^4$$
$$+\frac{N_2}{D}\|w_y^m\|^2+\frac{T}{4D}\|w_y^m\|^4\Big]+\frac{\varepsilon\mu}{D}\|w_t^m\|^2$$
$$+\int_0^1 f(w_t^m(1,y,t))w_t^m(1,y,t)\mathrm{d}y$$
$$+\int_0^1 g(w_t^m(x,1,t))w_t^m(x,1,t)\mathrm{d}x=0. \tag{25}$$

考虑到 f,g 的假设,然后再从 0 到 $t(<t_m)$ 积分(25)式得

$$\|w_{xx}^m\|^2+\|w_{yy}^m\|^2+2\|w_{xy}^m\|^2$$
$$+\frac{N_1}{D}\|w_x^m\|^2+\frac{N_2}{D}\|w_y^m\|^2$$
$$+\frac{\rho}{D}\|w_t^m\|^2+\frac{T}{4D}\|w_x^m\|^4+\frac{T}{4D}\|w_y^m\|^4$$
$$\leqslant\|w_{xx}^m(0)\|^2+\|w_{yy}^m(0)\|^2+2\|w_{xy}^m(0)\|^2$$
$$+\frac{N_1}{D}\|w_x^m(0)\|^2+\frac{N_2}{D}\|w_y^m(0)\|^2$$
$$+\frac{\rho}{D}\|w_t^m(0)\|^2+\frac{T}{4D}\|w_x^m(0)\|^4$$
$$+\frac{T}{4D}\|w_y^m(0)\|^4. \tag{26}$$

又由于初始条件

$w^m(x,y,0)=w^{0m}\to w^0$ (在 W_1 中强收敛),

$w_t^m(x,y,0)=w^{1m}\to w^1$ (在 W_1 中强收敛),

则存在一常数 C 仅仅依赖于 T 使得

$$\|w_{xx}^m\|^2+\|w_{yy}^m\|^2+2\|w_{xy}^m\|^2$$
$$+\frac{N_1}{D}\|w_x^m\|^2+\frac{N_2}{D}\|w_y^m\|^2+\frac{\rho}{D}\|w_t^m\|^2$$
$$+\frac{T}{4D}\|w_x^m\|^4+\frac{T}{4D}\|w_y^m\|^4\leqslant C, \tag{27}$$

对一切 $t\in[0,T]$ 和对一切 $m\in N$ 成立.再由庞加莱不等式有

$$\|w^m\|\leqslant\frac{1}{\sqrt{2}}\|w_x^m\|\leqslant C.$$

估计 2 在(22)式中取 $\omega=w_{tt}^m(0)$ 及 $t=0$,利用分部积分,并考虑到相容性条件得

$$\frac{\rho}{D}\|w_{tt}^m(0)\|^2$$
$$\leqslant\Big[\|w_{xxxx}^{0m}\|+2\|w_{xxyy}^{0m}\|$$
$$+\|w_{yyyy}^{0m}\|+\frac{\varepsilon\mu}{D}\|w^{1m}\|+\frac{N_1}{D}\|w_{xx}^{0m}\|$$
$$+\frac{T}{2D}\|w_x^{0m}\|^2\|w_{xx}^{0m}\|+\frac{N_2}{D}\|w_{yy}^{0m}\|$$
$$+\frac{T}{2D}\|w_y^{0m}\|^2\|w_{yy}^{0m}\|\Big]\|w_{tt}^m(0)\|. \tag{28}$$

故存在常数 $C>0$,使得

$$\|w_{tt}^m(0)\|\leqslant C,\forall m\in N.$$

估计 3 在(22)式中分别取 $t=t+\xi$ 和 $t=t$ 后,两式相减,再取 $\omega=w_t^m(t+\xi)-w_t^m(t)$,并利用分部积分得

$$\frac{\rho}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t^m(t+\xi)-w_t^m(t)\|^2$$
$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{xx}^m(t+\xi)-w_{xx}^m(t)\|^2$$
$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_{yy}^m(t+\xi)-w_{yy}^m(t)\|^2$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\|w_{xy}^m(t+\xi)-w_{xy}^m(t)\|^2$$
$$+\frac{N_1}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x^m(t+\xi)-w_x^m(t)\|^2$$
$$+\frac{N_2}{2D}\frac{\mathrm{d}}{\mathrm{d}t}\|w_y^m(t+\xi)-w_y^m(t)\|^2$$
$$+\frac{\varepsilon\mu}{D}\|w_t^m(t+\xi)-w_t^m(t)\|^2$$
$$+\frac{T}{2D}\int_\Omega(w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi)$$
$$\times(w_{xt}^m(t+\xi)-w_{xt}^m(t))\mathrm{d}x\mathrm{d}y$$
$$-\frac{T}{2D}\int_\Omega(w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t)$$
$$\times(w_{xt}^m(t+\xi)-w_{xt}^m(t))\mathrm{d}x\mathrm{d}y$$
$$+\frac{T}{2D}\int_\Omega(w_y^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t+\xi)$$
$$\times(w_{yt}^m(t+\xi)-w_{yt}^m(t))\mathrm{d}x\mathrm{d}y$$
$$-\frac{T}{2D}\int_\Omega(w_y^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t)$$
$$\times(w_{yt}^m(t+\xi)-w_{yt}^m(t))\mathrm{d}x\mathrm{d}y$$
$$+\int_0^1[g(w_t^m(x,1,t+\xi))-g(w_t^m(x,1,t))]$$
$$\times(w_t^m(x,1,t+\xi)-w_t^m(x,1,t))\mathrm{d}x$$
$$+\int_0^1[f(w_t^m(1,y,t+\xi))-f(w_t^m(1,y,t))]$$
$$\times(w_t^m(1,y,t+\xi)-w_t^m(1,y,t))\mathrm{d}y=0. \tag{29}$$

记

$$I_7=\frac{T}{2D}\int_\Omega(w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi)$$
$$\times(w_{xt}^m(t+\xi)-w_{xt}^m(t))\mathrm{d}x\mathrm{d}y$$
$$-\frac{T}{2D}\int_\Omega(w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t)$$

$$\times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y,$$

$$I_8 = \frac{T}{2D}\int_\Omega (w_y^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t+\xi) \times (w_{yt}^m(t+\xi) - w_{yt}^m(t))\mathrm{d}x\mathrm{d}y - \frac{T}{2D}\int_\Omega (w_y^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_y^m(t) \times (w_{yt}^m(t+\xi) - w_{yt}^m(t))\mathrm{d}x\mathrm{d}y.$$

现在开始估计 I_7，

$$\begin{aligned}
|I_7| &= \Big|\frac{T}{2D}\int_\Omega (w_x^m(t+\xi))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t+\xi) \times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y \\
&\quad - \frac{T}{2D}\int_\Omega (w_x^m(t))^2\mathrm{d}x\mathrm{d}y\int_\Omega w_x^m(t) \times (w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y\Big| \\
&= \Big|\frac{T}{2D}\|(w_x^m(t+\xi))\|^2\int_\Omega w_x^m(t+\xi) - w_x^m(t))(w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{T}{2D}\int_\Omega ((w_x^m(t+\xi))^2 - (w_x^m(t))^2)\mathrm{d}x\mathrm{d}y \times \int_\Omega w_x^m(t)(w_{xt}^m(t+\xi) - w_{xt}^m(t))\mathrm{d}x\mathrm{d}y\Big| \\
&= \Big|\frac{T}{2D}\|w_x^m(t+\xi)\|^2\int_0^1 [w_x^m(1,y,t+\xi) - w_x^m(1,y,t)][w_t(1,y,t+\xi) - w_t(1,y,t)]\mathrm{d}y \\
&\quad - \frac{T}{2D}\|w_x^m(t+\xi))\|^2\int_\Omega (w_{xx}^m(t+\xi) - w_{xx}^m(t))(w_t^m(t+\xi) - w_t^m(t))\mathrm{d}x\mathrm{d}y \\
&\quad + \frac{T}{2D}\int_\Omega [w_x^m(t+\xi) + w_x^m(t)] \times [w_x^m(t+\xi) - w_x^m(t)]\mathrm{d}x\mathrm{d}y \times \int_0^1 w_x^m(1,y,t)[w_t^m(1,y,t+\xi) - w_t^m(1,y,t)]\mathrm{d}y \\
&\quad - \frac{T}{2D}\int_\Omega [w_x^m(t+\xi) + w_x^m(t)] \times [w_x^m(t+\xi) - w_x^m(t)]\mathrm{d}x\mathrm{d}y \times \int_0^1 w_{xx}^m(t)[w_t^m(t+\xi) - w_t^m(t)]\,dy\Big| \\
&\leqslant C\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 + C\frac{\rho}{2D}\|w_t^m(t+\xi) - w_t^m(t)\|^2 \\
&\quad + l\int_0^1 |w_t^m(1,y,t+\xi) - w_t^m(1,y,t)|^2\mathrm{d}y.
\end{aligned}$$

同理

$$|I_8| \leqslant C\|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 + C\frac{\rho}{2D}\|w_t^m(t+\xi) - w_t^m(t)\|^2 + k\int_0^1 |w_t^m(x,1,t+\xi) - w_t^m(x,1,t)|^2\mathrm{d}x.$$

故从(29)式有

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 + \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 + 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&\quad + \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 + \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 + \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big] \\
&\leqslant C\Big(\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 + \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 + 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 \\
&\quad + \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 + \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2 + \frac{\rho}{D}\|w_t^m(t+\xi) - w_t^m(t)\|^2\Big) \\
&\quad + l\int_0^1 |w_t^m(1,y,t+\xi) - w_t^m(1,y,t)|^2\mathrm{d}y + k\int_0^1 |w_t^m(x,1,t+\xi) - w_t^m(x,1,t)|^2\mathrm{d}x \\
&\quad - \int_0^1 [f(w_t^m(1,y,t+\xi)) - f(w_t^m(1,y,t))] \times (w_t^m(1,y,t+\xi) - w_t^m(1,y,t))\mathrm{d}y \\
&\quad - \int_0^1 [g(w_t^m(x,1,t+\xi)) - g(w_t^m(x,1,t))] \times (w_t^m(x,1,t+\xi) - w_t^m(x,1,t))\mathrm{d}x. \qquad (30)
\end{aligned}$$

考虑到 f,g 的条件故(30)式变为

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|w_{xx}^m(t+\xi) - w_{xx}^m(t)\|^2 + \|w_{yy}^m(t+\xi) - w_{yy}^m(t)\|^2 + 2\|w_{xy}^m(t+\xi) - w_{xy}^m(t)\|^2 + \frac{N_1}{D}\|w_x^m(t+\xi) - w_x^m(t)\|^2 + \frac{N_2}{D}\|w_y^m(t+\xi) - w_y^m(t)\|^2$$

$$
\begin{aligned}
&+\frac{\rho}{D}\|w_t^m(t+\xi)-w_t^m(t)\|^2\Big] \\
\leqslant C\Big(&\|w_{xx}^m(t+\xi)-w_{xx}^m(t)\|^2 \\
&+\|w_{yy}^m(t+\xi)-w_{yy}^m(t)\|^2 \\
&+2\|w_{xy}^m(t+\xi)-w_{xy}^m(t)\|^2 \\
&+\frac{N_1}{D}\|w_x^m(t+\xi)-w_x^m(t)\|^2 \\
&+\frac{N_2}{D}\|w_y^m(t+\xi)-w_y^m(t)\|^2 \\
&+\frac{\rho}{D}\|w_t^m(t+\xi)-w_t^m(t)\|^2\Big).
\end{aligned}
$$

然后上式两边同除以 ξ^2，再令 $\xi\to 0$，则有

$$
\begin{aligned}
&\|w_{xxt}^m(t)\|^2+\|w_{yyt}^m(t)\|^2 \\
&+2\|w_{xyt}^m(t)\|^2+\frac{N_1}{D}\|w_{xt}^m(t)\|^2 \\
&+\frac{N_2}{D}\|w_{yt}^m(t)\|^2+\frac{\rho}{D}\|w_{tt}^m(t)\|^2 \\
\leqslant\Big(&\|w_{xx}^{1m}\|^2+\|w_{yy}^{1m}\|^2 \\
&+2\|w_{xy}^{1m}\|^2+\frac{N_1}{D}\|w_x^{1m}\|^2 \\
&+\frac{N_2}{D}\|w_y^{1m}\|^2+\frac{\rho}{D}\|w_{tt}^m(0)\|^2\Big)\exp(CT),
\end{aligned}
$$

则我们可找到一仅仅依赖于 T 的常数 $C>0$，使得

$$
\begin{aligned}
&\|w_{xxt}^m(t)\|^2+\|w_{yyt}^m(t)\|^2 \\
&+2\|w_{xyt}^m(t)\|^2+\frac{N_1}{D}\|w_{xt}^m(t)\|^2 \\
&+\frac{N_2}{D}\|w_{yt}^m(t)\|^2+\frac{\rho}{D}\|w_{tt}^m(t)\|^2\leqslant C.
\end{aligned}
$$

步骤 3　讨论收敛性. 综上可得 $\|w^m\|$，$\|w_x^m\|$，$\|w_y^m\|$，$\|w_{xx}^m\|$，$\|w_{yy}^m\|$，$\|w_{xy}^m\|$，$\|w_{tt}^m\|$，$\|w_{xt}^m\|$，$\|w_{yt}^m\|$，$\|w_{xxt}^m\|$，$\|w_{yyt}^m\|$，$\|w_{xyt}^m\|<C$，由 $\|w^m\|$，$\|w_x^m\|$，$\|w_{xx}^m\|$，$\|w_y^m\|$，$\|w_{yy}^m\|$ 有界可得 w^m 在 $L^\infty(0,T;V_1)$ 中有界，由 $\|w_t^m\|$，$\|w_{tx}^m\|$，$\|w_{ty}^m\|$，$\|w_{txx}^m\|$，$\|w_{tyy}^m\|$ 有界可得 w_t^m 在 $L^\infty(0,T;V_1)$ 中有界，故存在子序列 $\{w^\mu\}$ 满足 $w^\mu\to w$ 在 $L^\infty(0,T;V_1)$ 中弱星收敛；$w_t^\mu\to w_t$ 在 $L^\infty(0,T;V_1)$ 中弱星收敛. 从而说明存在整体解

$$
w(t)\in L^2(0,\infty;W)\cap C^0([0,\infty);V_1)\cap W^{2,\infty}(0,\infty;L^2(\Omega)).
$$

证毕.

定理 4　定理 3 中的解是唯一的.

证明　如同定理 2 的证明.

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The existence and uniqueness of the global solution for the viscoelastic-plate equation under nonlinear boundary conditions*

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Abstract

In this paper, we consider the viscoelastic-plate equation under non-linear boundary conditions. Firstly, by the aid of Galerkin method, under non-linear boundary conditions (a) and the initial values $w^0 \in W$, and $w^1 \in W$, we prove the existence and uniqueness of a global weak solution $w(t)$ for the initial boundary value problems. Secondly, under non-linear boundary conditions (b) and the initial values $w^0 \in W$, and $w^1 \in W_1$, the existence and uniqueness of a global weak solution $w(t)$ is also proved by using Galerkin method.

Keywords: viscoelastic-plate equation, initial boundary value problems, Galerkin method, global solution

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