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两体组合坐标表象的建立、性质及应用*

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为了研究具有相互作用势和运动耦合的两个非全同的量子谐振子体系的动力学问题,利用有序算符乘积内的积分技术,建立了一种两粒子体系的组合坐标新表象$|\eta_1,\eta_2\rangle$,构造了一个双模压缩算符 U 并分析了其压缩特性. 应用组合坐标新表象严格求解了具有相互作用势和运动耦合的两个非全同的量子谐振子体系的动力学问题. 这为研究复杂耦合量子谐振子体系提供了一个有效途径.

关键词:有序算符乘积内的积分技术,组合坐标表象,双模压缩算符,幺正矩阵

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1. 引 言

表象理论及表象变换已成为量子力学、量子光学等领域的一个重要课题. 正如 Dirac 所说,当一个人要解决一个具体的量子力学问题时,他可以通过选择使用一种合适的表象来大大减少他的劳动量,在这种表象中,问题的重要物理量的表示是尽可能的简单[1]. 文献[2]建立了双模纠缠态表象,文献[3]建立了三模纠缠态表象,文献[4]建立了中介坐标-动量表象,文献[5]建立了多模纠缠态表象,文献[6—12]对表象、表象变换及其应用进行了研究与探讨. 本文将利用有序算符乘积内的积分(IWOP)技术[13,14],建立一种两粒子体系的新表象$|\eta_1,\eta_2\rangle$(称之为组合坐标表象),构造一个双模压缩算符并研究其压缩特性. 运用$|\eta_1,\eta_2\rangle$表象来严格求解具有相互作用势和运动耦合的两个非全同的量子谐振子体系的动力学问题.

2. 两粒子体系的组合坐标表象$|\eta_1,\eta_2\rangle$的建立

一个具有运动耦合和相互作用的两非全同谐振子组成的体系,其哈密顿量的一般形式可表示为

$$\hat{H}=\hat{p}_1^2/2m_1+\hat{p}_2^2/2m_2+m_1\omega_1^2\hat{x}_1^2/2+m_2\omega_2^2\hat{x}_2^2/2+\gamma p_1\hat{p}_2+\lambda\hat{x}_1\hat{x}_2.$$

在坐标表象$|x_1x_2\rangle$中求解其本征值及本征波函数是非常困难的. 从代数学的角度观察该哈密顿量,它是二次型哈密顿量. 根据二次型理论实现哈密顿量的正交对角化,需要求出二次型矩阵的特征值和正交归一的特征解,并用这些正交归一的特征解构造一个幺正变换矩阵进行算符变换(包括坐标算符和动量算符). 为此,构造如下两个线性组合坐标算符和两个线性组合动量算符 $\hat{X}_1,\hat{X}_2,\hat{P}_1,\hat{P}_2$:

$$\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\end{pmatrix}=\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}=A\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix},$$
$$\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix}=\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix}\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}=A\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix},\tag{1}$$

其中 $A_{11},A_{12},A_{21},A_{22}$均为实数,$A$ 为幺正矩阵.

$$A=\begin{pmatrix}A_{11}&A_{12}\\ A_{21}&A_{22}\end{pmatrix},$$
$$AA^\dagger=A^\dagger A=I,$$
$$\sum_{k=1}^{2}A_{ki}A_{kj}=\sum_{k=1}^{2}A_{ik}A_{jk}$$

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$$= \delta_{ij} \qquad (i, j = 1,2). \tag{2}$$

显然

$$[\hat{X}_i, \hat{P}_j] = \mathrm{i}\delta_{ij},$$
$$[\hat{X}_1, \hat{X}_2] = 0,$$

即变换后的组合坐标 $\hat{X}_1, \hat{X}_2$ 具有共同本征态，它们构成了新的表象.

Dirac 符号表示的量子力学表象(如坐标表象、动量表象、相干态表象以及两粒子连续变量纠缠态表象等)的完备性为[14]

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle\langle x| = 1,\\
&\int_{-\infty}^{\infty} \mathrm{d}p\,|p\rangle\langle p| = 1,\\
&\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| = 1,\\
&\int \frac{\mathrm{d}^2 \eta}{\pi}|\eta\rangle\langle \eta| = 1.
\end{aligned} \tag{3}$$

这里

$$\begin{aligned}
&|x\rangle = \pi^{-1/4}\exp\left\{-\frac{x^2}{2} + \sqrt{2}x\hat{a}^\dagger - \frac{\hat{a}^{\dagger 2}}{2}\right\}|0\rangle,\\
&\hat{x}|x\rangle = x|x\rangle;\\
&|p\rangle = \pi^{-1/4}\exp\left\{-\frac{p^2}{2} + \mathrm{i}\sqrt{2}p\hat{a}^\dagger + \frac{\hat{a}^{\dagger 2}}{2}\right\}|0\rangle,\\
&\hat{p}|p\rangle = p|p\rangle;\\
&|z\rangle = \exp\left\{-\frac{|z|^2}{2} + z\hat{a}^\dagger\right\}|0\rangle,\\
&\hat{a}|z\rangle = z|z\rangle;\\
&|\eta\rangle = \exp\left\{-\frac{|\eta|^2}{2} + \eta\hat{a}_1^\dagger - \eta^*\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_2^\dagger\right\}|00\rangle,\\
&(\hat{x}_1 - \hat{x}_2)|\eta\rangle = \sqrt{2}\eta_1|\eta\rangle,\\
&(\hat{p}_1 + \hat{p}_2)|\eta\rangle = \sqrt{2}\eta_2|\eta\rangle \qquad (\eta = \eta_1 + \mathrm{i}\eta_2).
\end{aligned} \tag{4}$$

$|0\rangle$和$|00\rangle$分别是单模真空态和双模真空态，$\hat{a}_i$ 和 $\hat{a}_i^\dagger$ 分别为湮没算符和产生算符，它们与 $\hat{x}_i$ 和 $\hat{p}_i$ 的关系分别为

$$\begin{aligned}
&\hat{x}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2},\\
&\hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger)/\mathrm{i}\sqrt{2} \qquad (i = 1,2).
\end{aligned} \tag{5}$$

将(4)式代入(3)式，并考虑$:\exp[-\hat{a}^\dagger\hat{a}]: = |0\rangle\langle 0|$和$:\exp[-\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2]: = |00\rangle\langle 00|$，可以得到

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}x\,|x\rangle\langle x| = \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\sqrt{\pi}}:\exp[-(x-\hat{x})^2]: = 1,\\
&\int_{-\infty}^{\infty} \mathrm{d}p\,|p\rangle\langle p| = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{\sqrt{\pi}}:\exp[-(p-\hat{p})^2]: = 1,\\
&\int \frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle z| = \int\frac{\mathrm{d}^2 z}{\pi}:\exp[-(z-\hat{a})(z^*-\hat{a}^\dagger)]:\\
&= 1,\\
&\int \frac{\mathrm{d}^2 \eta}{\pi}|\eta\rangle\langle \eta| = \iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}\\
&\quad\times:\exp\left\{-\left[\left(\eta_1 - \frac{\hat{x}_1 - \hat{x}_2}{\sqrt{2}}\right)^2 + \left(\eta_2 - \frac{\hat{p}_1 + \hat{p}_2}{\sqrt{2}}\right)^2\right]\right\}:\\
&= 1.
\end{aligned} \tag{6}$$

由此可见，由 Dirac 符号表示的量子力学表象的完备性，在 IWOP 理论框架下均可表示为一个含有算符且积分结果为单位‘1’的纯高斯型积分. 受此启发，我们通过构建一个含有算符(或算符的组合)且积分结果为单位‘1’的正规乘积内的纯高斯型积分，选择合适的系数，将该积分化简为由 Dirac 符号表示的量子力学表象完备性的标准形式，这样就可以从结构上把高斯积分中的算符分解为一对相互共轭的态矢，从而构建新表象.

构造如下正规乘积内的高斯积分：

$$\begin{aligned}
1 &= \iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}:\exp\{-(\eta_1 - \hat{X}_1)^2 - (\eta_2 - \hat{X}_2)^2\}:\\
&= \iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}:\exp\{-[\eta_1 - (A_{11}\hat{x}_1 + A_{12}\hat{x}_2)]^2\\
&\quad - [\eta_2 - (A_{21}\hat{x}_1 + A_{22}\hat{x}_2)]^2\}:\\
&= \iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}:\exp\Big\{-(\eta_1^2 + \eta_2^2)\\
&\quad + \sqrt{2}(\hat{a}_1^\dagger \quad \hat{a}_2^\dagger)\tilde{A}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\\
&\quad - \frac{1}{2}(\hat{a}_1^{\dagger 2} + \hat{a}_2^{\dagger 2}) - (\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2)\\
&\quad + \sqrt{2}(\eta_1 \quad \eta_2)A\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} - \frac{1}{2}(\hat{a}_1^2 + \hat{a}_2^2)\Big\}:,
\end{aligned} \tag{7}$$

式中

$$\tilde{A} = \begin{pmatrix} A_{11} & A_{21}\\ A_{12} & A_{22}\end{pmatrix}.$$

利用双模真空投影算符的正规乘积形式

$$|00\rangle\langle 00| = :\exp\{-\hat{a}_1^\dagger\hat{a}_1-\hat{a}_2^\dagger\hat{a}_2\}:,$$

将(7)式变为

$$\begin{aligned}&\iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}:\exp\Big\{-\frac{1}{2}(\eta_1^2+\eta_2^2)\\&+\sqrt{2}(\hat{a}_1^\dagger\quad\hat{a}_2^\dagger)\tilde{A}\begin{pmatrix}\eta_1\\\eta_2\end{pmatrix}-\frac{1}{2}(\hat{a}_1^{\dagger2}+\hat{a}_2^{\dagger2})\Big\}|00\rangle\\&\times\langle 00|\exp\Big\{-\frac{1}{2}(\eta_1^2+\eta_2^2)\\&+\sqrt{2}(\eta_1\quad\eta_2)A\begin{pmatrix}a_1\\a_2\end{pmatrix}-\frac{1}{2}(\hat{a}_1^2+\hat{a}_2^2)\Big\}:\\&=\iint_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1\mathrm{d}\eta_2}{\pi}|\eta_1,\eta_2\rangle\langle\eta_1,\eta_2|\\&=1,\end{aligned}\tag{8}$$

式中

$$\begin{aligned}|\eta_1,\eta_2\rangle=\pi^{-1/2}\exp\Big\{&-\frac{1}{2}(\eta_1^2+\eta_2^2)\\&+\sqrt{2}(\hat{a}_1^\dagger\quad\hat{a}_2^\dagger)\tilde{A}\begin{pmatrix}\eta_1\\\eta_2\end{pmatrix}\\&-\frac{1}{2}(\hat{a}_1^{\dagger2}+\hat{a}_2^{\dagger2})\Big\}|00\rangle.\end{aligned}\tag{9}$$

利用有序算符正规乘积的性质[12]

$$:\frac{\partial}{\partial a_i^\dagger}f(\hat{a}_i,\hat{a}_i^\dagger):=[\hat{a}_i,:f(\hat{a}_1,\hat{a}_i^\dagger):],$$

$$\hat{a}_i|00\rangle=0,$$

容易得到

$$\hat{a}_i|\eta_1,\eta_2\rangle=[\sqrt{2}(A_{1i}\eta_1+A_{2i}\eta_2)-\hat{a}_i^\dagger]|\eta_1,\eta_2\rangle\quad(i=1,2).\tag{10}$$

由(1),(5)及(10)式可得

$$\hat{X}_i|\eta_1,\eta_2\rangle=\eta_i|\eta_1,\eta_2\rangle\quad(i=1,2).\tag{11}$$

这表明$|\eta_1,\eta_2\rangle$是$\hat{X}_1$,$\hat{X}_2$的共同本征态,相应的本征值分别为η_1和η_2. 由(8)式可见,$|\eta_1,\eta_2\rangle$的完备性是显然的. 再由(11)式得

$$\begin{aligned}\langle\eta_1',\eta_2'|\hat{X}_i|\eta_1,\eta_2\rangle&=\eta_i'\langle\eta_1',\eta_2'|\eta_1,\eta_2\rangle\\&=\eta_i\langle\eta_1',\eta_2'|\eta_1,\eta_2\rangle\\&\quad(i=1,2).\end{aligned}\tag{12}$$

即

$$\langle\eta_1',\eta_2'|\eta_1,\eta_2\rangle=\delta(\eta_1'-\eta_1)\delta(\eta_2'-\eta_2).\tag{13}$$

可见$|\eta_1,\eta_2\rangle$既是完备的又是正交的,完全可以构成一种表象. 因为算符$\hat{X}_1$,$\hat{X}_2$是坐标算符$\hat{x}_1$,$\hat{x}_2$的线性组合,所以称$|\eta_1,\eta_2\rangle$为组合坐标表象. 容易证明$\hat{P}_1$和$\hat{P}_2$在$|\eta_1,\eta_2\rangle$表象中的表示如下:

$$\langle\eta_1,\eta_2|\hat{P}_i=-\mathrm{i}\frac{\partial}{\partial\eta_i}\langle\eta_1,\eta_2|\quad(i=1,2).\tag{14}$$

如果把(1)式中的算符$\hat{X}_1$,$\hat{X}_2$和$\hat{P}_1$,$\hat{P}_2$再次进行组合,即

$$\begin{aligned}\begin{pmatrix}\hat{X}_1'\\\hat{X}_2'\end{pmatrix}&=\begin{pmatrix}A_{11}'&A_{12}'\\A_{21}'&A_{22}'\end{pmatrix}\begin{pmatrix}\hat{X}_1\\\hat{X}_2\end{pmatrix}=A'\begin{pmatrix}\hat{X}_1\\\hat{X}_2\end{pmatrix},\\\begin{pmatrix}\hat{P}_1'\\\hat{P}_2'\end{pmatrix}&=\begin{pmatrix}A_{11}'&A_{12}'\\A_{21}'&A_{22}'\end{pmatrix}\begin{pmatrix}\hat{P}_1\\\hat{P}_2\end{pmatrix}=A'\begin{pmatrix}\hat{P}_1\\\hat{P}_2\end{pmatrix},\end{aligned}\tag{15}$$

式中A'为一个幺正矩阵. 利用IWOP技术,同样可以证明$\hat{X}_1'$,$\hat{X}_2'$的共同本征态可以表示为

$$\begin{aligned}|\chi_1,\chi_2\rangle=\pi^{-1/2}\exp\Big\{&-\frac{1}{2}(\chi_1^2+\chi_2^2)\\&+\sqrt{2}(\hat{A}_1^\dagger\quad\hat{A}_2^\dagger)\tilde{A}'\begin{pmatrix}\chi_1\\\chi_2\end{pmatrix}\\&-\frac{1}{2}(\hat{A}_1^{\dagger2}+\hat{A}_2^{\dagger2})\Big\}\|00\rangle,\end{aligned}\tag{16}$$

式中χ_1,χ_2分别为算符$\hat{X}_1'$,$\hat{X}_2'$的本征值,$\|00\rangle$是$\hat{A}_i^\dagger\hat{A}_i$的基态态矢,$\hat{A}_1$,$\hat{A}_2$及$\hat{A}_1^\dagger$,$\hat{A}_2^\dagger$为双模湮没算符及产生算符,它们与$\hat{X}_1$,$\hat{X}_2$和$\hat{P}_1$,$\hat{P}_2$的关系为

$$\begin{aligned}\hat{X}_i&=(\hat{A}_i+\hat{A}_i^\dagger)/\sqrt{2},\\\hat{P}_i&=(\hat{A}_i-\hat{A}_i^\dagger)/\mathrm{i}\sqrt{2}\quad(i=1,2).\end{aligned}\tag{17}$$

$|\chi_1,\chi_2\rangle$也是完备正交的,可以构成表象以及

$$\langle\chi_1,\chi_2|\hat{P}_i'=-\mathrm{i}\frac{\partial}{\partial\chi_i}\langle\chi_1,\chi_2|\quad(i=1,2).\tag{18}$$

3. $|\eta_1,\eta_2\rangle$的不对称积分诱导双模压缩算符

构造如下积分型算符:

$$U=\iint_{-\infty}^{\infty}\mathrm{d}\eta_1\mathrm{d}\eta_2\sqrt{\kappa_1\kappa_2}|\kappa_1\eta_1,\kappa_2\eta_2\rangle\langle\eta_1,\eta_2|.\tag{19}$$

这里κ_1,κ_2是两个独立的正数. 由(19)及(13)式和IWOP技术可得

$$UU^{\dagger}=\kappa_1\kappa_2\iint_{-\infty}^{\infty}\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_1'\mathrm{d}\eta_2'|\kappa_1\eta_1,\kappa_2\eta_2\rangle\times\langle\eta_1,\eta_2||\eta_1',\eta_2'\rangle\langle\kappa_1\eta_1',\kappa_2\eta_2'|=\kappa_1\kappa_2\iint_{-\infty}^{\infty}\mathrm{d}\eta_1\mathrm{d}\eta_2|\kappa_1\eta_1,\kappa_2\eta_2\rangle\langle\kappa_1\eta_1,\kappa_2\eta_2|=1=U^{\dagger}U. \tag{20}$$

可见 U 是幺正的.利用(11)式和 IWOP 技术,我们有

$$U\hat{X}_1U^{\dagger}=\kappa_1\kappa_2\iint_{-\infty}^{\infty}\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_1'\mathrm{d}\eta_2'|\kappa_1\eta_1,\kappa_2\eta_2\rangle\times\langle\eta_1,\eta_2|\hat{X}_1|\eta_1',\eta_2'\rangle\langle\kappa_1\eta_1',\kappa_2\eta_2'|=\kappa_1\kappa_2\iint_{-\infty}^{\infty}\eta_1\mathrm{d}\eta_1\mathrm{d}\eta_2|\kappa_1\eta_1,\kappa_2\eta_2\rangle\langle\kappa_1\eta_1,\kappa_2\eta_2|=\kappa_1^{-1}\hat{X}_1. \tag{21}$$

在计算中使用了下列积分公式：

$$\int_{-\infty}^{\infty}x\exp(-\alpha x^2+\beta x)\mathrm{d}x=\frac{\beta}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2}{4\alpha}\right)\quad(\mathrm{Re}(\alpha)>0).$$

同样地,我们有

$$U\hat{X}_2U^{\dagger}=\kappa_2^{-1}\hat{X}_2,\quad U\hat{P}_1U^{\dagger}=\kappa_1\hat{P}_1,\quad U\hat{P}_2U^{\dagger}=\kappa_2\hat{P}_2. \tag{22}$$

从(21)和(22)式可知, U 的确是一个双模压缩算符.

4. $\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle$和$\langle x_1,x_2|\eta_1,\eta_2\rangle$的计算

在$\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle$中插入相干态的超完备性,利用(9),(16)式以及

$$|z_1z_2\rangle=\exp\left[-\frac{|z_1|^2+|z_2|^2}{2}+(z_1A_1^{\dagger}+z_2A_2^{\dagger})\right]|00\rangle,$$

可得

$$\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle=\pi^{-1}\exp\left\{-\frac{\chi_1^2}{2}-\frac{\chi_2^2}{2}-\frac{\eta_1^2}{2}-\frac{\eta_2^2}{2}\right\}\times\iint_{-\infty}^{\infty}\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}\exp\left\{-|z_1|^2+\sqrt{2}\eta_1z_1+\sqrt{2}(A_{11}'\chi_1+A_{21}'\chi_2)z_1^*-\frac{z_1^2}{2}-\frac{z_1^{*2}}{2}-|z_2|^2+\sqrt{2}\eta_2z_2+\sqrt{2}(A_{12}'\chi_1+A_{22}'\chi_2)z_2^*-\frac{z_2^2}{2}-\frac{z_2^{*2}}{2}\right\}. \tag{23}$$

利用积分公式[14]

$$\int\frac{\mathrm{d}^2z}{\pi}\exp\{\zeta|z|^2+\xi z+\eta z^*+fz^2+gz^{*2}\}=\frac{1}{\sqrt{\zeta^2-4fg}}\exp\left\{\frac{-\zeta\xi\eta+\xi^2g+\eta^2f}{\zeta^2-4fg}\right\},$$

对(23)式积分得

$$\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle=\pi^{-1}\exp\left\{-\frac{\chi_1^2}{2}-\frac{\chi_2^2}{2}-\frac{\eta_1^2}{2}-\frac{\eta_2^2}{2}\right\}\times\lim_{t\to1}\left\{\left(\frac{1}{\sqrt{1-t^2}}\right)^2\times\exp[(2\chi_1(A_{11}'\eta_1+A_{12}'\eta_2)+2\chi_2(A_{21}'\eta_1-A_{22}'\eta_2)-t\eta_1^2-t\eta_2^2-t\chi_1^2-t\chi_2^2)/(1-t^2)]\right\}. \tag{24}$$

利用 δ 函数的表示式[13]

$$\delta(q'-q)=\pi^{-1/2}\exp\left[-\frac{q'^2+q^2}{2}\right]\lim_{t\to1}\frac{1}{\sqrt{1-t^2}}\times\exp\left\{\frac{2q'qt-(q^2+q'^2)t^2}{1-t^2}\right\}.$$

注意到 A' 的幺正性及积分公式,可得

$$\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle=\delta[\chi_1-(A_{11}'\eta_1+A_{12}'\eta_2)]\times\delta[\chi_2-(A_{21}'\eta_1-A_{22}'\eta_2)]. \tag{25}$$

同理可得

$$\langle\eta_1,\eta_2|x_1,x_2\rangle=\delta[x_1-(A_{11}x_1-A_{12}x_2)]\times\delta[\eta_2-(A_{21}\eta_1-A_{22}\eta_2)]. \tag{26}$$

由(25)和(26)式可知,表象变换后,$\langle\eta_1,\eta_2|\chi_1,\chi_2\rangle$和$\langle x_1,x_2|\eta_1,\eta_2\rangle$恰好为 δ 函数,这为下面精确求解体系的波函数带来了极大的方便.

5. 利用$|\eta_1,\eta_2\rangle$严格求解耦合谐振子动力学问题

作为两体组合坐标表象$|\eta_1,\eta_2\rangle$的应用,研究两个非全同耦合谐振子体系的动力学问题.其哈密顿量为

$$\hat{H}=\frac{\hat{p}_1^2}{2m_1}+\frac{\hat{p}_2^2}{2m_2}+\frac{1}{2}m_1\omega_1^2\hat{x}_1^2+\frac{1}{2}m_2\omega_2^2\hat{x}_2^2+\gamma\hat{p}_1\hat{p}_2+\lambda\hat{x}_1\hat{x}_2$$

$$=\frac{\hat{p}_1^2}{2m_1}+\frac{\hat{p}_2^2}{2m_2}+\gamma\hat{p}_1\hat{p}_2+(\hat{x}_1\quad\hat{x}_2)D\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix},\qquad(27)$$

式中 D 为二次型的矩阵，

$$D=\begin{pmatrix}\frac{1}{2}m_1\omega_1^2 & \frac{\lambda}{2}\\ \frac{\lambda}{2} & \frac{1}{2}m_2\omega_2^2\end{pmatrix}.$$

根据二次型理论可求得 D 的特征值及相应的正交归一化的特征解分别为

$$\Lambda_1=\frac{1}{4}\left[m_1\omega_1^2+m_2\omega_2^2+\sqrt{(m_1\omega_1^2-m_2\omega_2^2)^2+4\lambda^2}\right],$$

$$\xi_1=\begin{pmatrix}\frac{d_1}{\sqrt{2}}\\ \frac{d_2}{\sqrt{2}}\end{pmatrix};$$

$$\Lambda_2=\frac{1}{4}\left[m_1\omega_1^2+m_2\omega_2^2-\sqrt{(m_1\omega_1^2-m_2\omega_2^2)^2+4\lambda^2}\right],$$

$$\xi_2=\begin{pmatrix}\frac{d_2}{\sqrt{2}}\\ \frac{-d_1}{\sqrt{2}}\end{pmatrix}.$$

这里

$$d_1=\sqrt{1+\frac{m_1\omega_1^2-m_2\omega_2^2}{\sqrt{(m_1\omega_1^2-m_2\omega_2^2)^2+4\lambda^2}}},$$

$$d_2=\sqrt{1-\frac{m_1\omega_1^2-m_2\omega_2^2}{\sqrt{(m_1\omega_1^2-m_2\omega_2^2)^2+4\lambda^2}}}.$$

令

$$\tilde{A}=(\xi_1\quad\xi_2),$$

显然 $\tilde{A}$ 是一个幺正矩阵. 至此，我们已经得到了(1)式中的幺正矩阵 A. 利用(1)式的逆变换

$$\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}=\tilde{A}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{d_1}{\sqrt{2}} & \frac{d_2}{\sqrt{2}}\\ \frac{d_2}{\sqrt{2}} & \frac{-d_1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{X}_1\\ \hat{X}_2\end{pmatrix},$$

$$\begin{pmatrix}\hat{p}_1\\ \hat{p}_2\end{pmatrix}=\tilde{A}\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix}\qquad(28)$$

$$=\begin{pmatrix}\frac{d_1}{\sqrt{2}} & \frac{d_2}{\sqrt{2}}\\ \frac{d_2}{\sqrt{2}} & \frac{-d_1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix},$$

改写 (27) 式所示的哈密顿算符，可得

$$\begin{aligned}\hat{H}=&\left(\frac{d_1^2}{4m_1}+\frac{d_2^2}{4m_2}+\frac{\gamma d_1d_2}{2}\right)\hat{P}_1^2\\&+\left(\frac{d_2^2}{4m_1}+\frac{d_1^2}{4m_2}-\frac{\gamma d_1d_2}{2}\right)\hat{P}_2^2\\&+\left(\frac{d_1d_2}{2m_1}-\frac{d_1d_2}{2m_2}-\frac{\gamma(d_1^2-d_2^2)}{2}\right)\hat{P}_1\hat{P}_2\\&+\Lambda_1\hat{X}_1^2+\Lambda_2\hat{X}_2^2.\end{aligned}\qquad(29)$$

根据(21)，(22)式，选择压缩系数 $\kappa_1=1$，$\kappa_2=\sqrt{\Lambda_2/\Lambda_1}$，于是有

$$\begin{aligned}U\hat{H}U^{\dagger}=&\left(\frac{d_1^2}{4m_1}+\frac{d_2^2}{4m_2}+\frac{\gamma d_1d_2}{2}\right)\hat{P}_1^2\\&+\frac{\Lambda_2}{\Lambda_1}\left(\frac{d_2^2}{4m_1}+\frac{d_1^2}{4m_2}-\frac{\gamma d_1d_2}{2}\right)\hat{P}_2^2\\&+\sqrt{\frac{\Lambda_2}{\Lambda_1}}\left(\frac{d_1d_2}{2m_1}-\frac{d_1d_2}{2m_2}-\frac{\gamma(d_1^2-d_2^2)}{2}\right)\hat{P}_1\hat{P}_2\\&+\Lambda_1\hat{X}_1^2+\Lambda_1\hat{X}_2^2\\=&(\hat{P}_1\quad\hat{P}_2)D'\begin{pmatrix}\hat{P}_1\\ \hat{P}_2\end{pmatrix}+\Lambda_1\hat{X}_1^2+\Lambda_1\hat{X}_2^2,\end{aligned}\qquad(30)$$

式中

$$D'=\begin{pmatrix}\frac{d_1^2}{4m_1}+\frac{d_2^2}{4m_2}+\frac{\gamma d_1d_2}{2} & \sqrt{\frac{\Lambda_2}{\Lambda_1}}\left(\frac{d_1d_2}{4m_1}-\frac{d_1d_2}{4m_2}-\frac{\gamma(d_1^2-d_2^2)}{4}\right)\\ \sqrt{\frac{\Lambda_2}{\Lambda_1}}\left(\frac{d_1d_2}{4m_1}-\frac{d_1d_2}{4m_2}-\frac{\gamma(d_1^2-d_2^2)}{4}\right) & \frac{\Lambda_2}{\Lambda_1}\left(\frac{d_2^2}{4m_1}+\frac{d_1^2}{4m_2}-\frac{\gamma d_1d_2}{2}\right)\end{pmatrix}$$

$$\equiv\begin{pmatrix}D'_{11} & D'_{12}\\ D'_{12} & D'_{22}\end{pmatrix}.$$

这里的 D' 也是一个二次型的矩阵，它的两个特征根及相应的两个正交归一化特征解分别为

$$\Lambda'_1=\frac{1}{2}\left[D'_{11}+D'_{22}+\sqrt{(D'_{11}-D'_{22})^2+4D'^2_{12}}\right],$$

$$\zeta_1 = \begin{pmatrix} \frac{e_1}{\sqrt{2}} \\ \frac{e_2}{\sqrt{2}} \end{pmatrix};$$

$$A_2' = \frac{1}{2}\left[D_{11}' + D_{22}' - \sqrt{(D_{11}' - D_{22}')^2 + 4D_{12}'^2}\right],$$

$$\zeta_2 = \begin{pmatrix} \frac{e_2}{\sqrt{2}} \\ \frac{-e_1}{\sqrt{2}} \end{pmatrix},$$

式中

$$e_1 = \sqrt{1 + \frac{D_{11}' - D_{22}'}{\sqrt{(D_{11}' - D_{22}')^2 + 4D_{12}'^2}}},$$

$$e_2 = \sqrt{1 - \frac{D_{11}' - D_{22}'}{\sqrt{(D_{11}' - D_{22}')^2 + 4D_{12}'^2}}}.$$

令

$$\tilde{A}' = (\zeta_1 \quad \zeta_2),$$

显然也是一个幺正矩阵，即能使(30)式所示哈密顿量对角化的变换矩阵，亦即(16)式中的 $\tilde{A}'$. 再次作变换

$$\begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix} = \tilde{A}' \begin{pmatrix} \hat{X}_1' \\ \hat{X}_2' \end{pmatrix}, \quad \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix} = \tilde{A}' \begin{pmatrix} \hat{P}_1' \\ \hat{P}_2' \end{pmatrix}, \tag{31}$$

其逆变换为

$$\begin{pmatrix} \hat{X}_1' \\ \hat{X}_2' \end{pmatrix} = A' \begin{pmatrix} \hat{X}_1 \\ \hat{X}_2 \end{pmatrix}, \quad \begin{pmatrix} \hat{P}_1' \\ \hat{P}_2' \end{pmatrix} = A' \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}.$$

利用(31)式改写(30)式所示的哈密顿算符，得

$$\begin{aligned} U\hat{H}U^\dagger &= \Lambda_1' \hat{P}_1'^2 + \Lambda_2' \hat{P}_2'^2 + \Lambda_1 \hat{X}_1'^2 + \Lambda_1 \hat{X}_2'^2 \\ &= \frac{\hat{P}_1'^2}{2\mu_1} + \frac{\hat{P}_2'^2}{2\mu_2} + \frac{1}{2}\mu_1 \Omega_1^2 \hat{X}_1'^2 + \frac{1}{2}\mu_2 \Omega_2^2 \hat{X}_2'^2, \end{aligned} \tag{32}$$

式中

$$\mu_i = \frac{1}{2\Lambda_i'},$$

$$\Omega_i = 2\sqrt{\Lambda_1 \Lambda_i'} \qquad (i = 1, 2).$$

设 $\hat{H}$ 的本征态为 $|E\rangle$，用 $\langle \chi_1, \chi_2|$ 和 $U|E\rangle$ 来乘(32)式，并结合(18)式便得到

$$\begin{aligned} E\langle \chi_1, \chi_2 | U | E\rangle = &- \frac{1}{2\mu_1}\frac{\partial^2}{\partial^2 \chi_1^2}\langle \chi_1, \chi_2 | U | E\rangle \\ &- \frac{1}{2\mu_2}\frac{\partial^2}{\partial \chi_2^2}\langle \chi_1, \chi_2 | U | E\rangle \\ &+ \frac{1}{2}\mu_1 \Omega_1^2 \chi_1^2 \langle \chi_1, \chi_2 | U | E\rangle \\ &+ \frac{1}{2}\mu_2 \Omega_2^2 \chi_2^2 \langle \chi_1, \chi_2 | U | E\rangle. \end{aligned} \tag{33}$$

(33)式就是在表象 $\langle \chi_1, \chi_2|$ 中的 Schrödinger 方程，它的解为

$$E_{n_1 n_2} = \sum_{i=1}^{2}\left\{\left(n_i + \frac{1}{2}\right)\hbar \Omega_i\right\},$$

$$\langle \chi_1, \chi_2 | U | E_{n_1 n_2}\rangle = \prod_{i=1}^{2}\left\{N_{n_i} \exp\left(-\frac{\alpha_i'^2}{2}\chi_i^2\right) H_{n_i}(\alpha_i' \chi_i)\right\}. \tag{34}$$

这里已经恢复了普朗克常数 $\hbar$，其中归一化常数

$$\alpha_i' = \sqrt{\frac{\mu_i \Omega_i}{\hbar}},$$

$$N_i = \sqrt{\frac{\alpha_i'}{\sqrt{\pi} 2^{n_i} n_i!}} \qquad (n_1, n_2 = 0, 1, 2, \cdots).$$

由(34)和(25)式可以得到在 $\langle \eta_1, \eta_2|$ 表象中压缩后的本征波函数为

$$\begin{aligned} &\langle \eta_1, \eta_2 | U | E_{n_1 n_2}\rangle \\ &= \iint_{-\infty}^{\infty} d\chi_1 d\chi_2 \langle \eta_1, \eta_2 \| \chi_1, \chi_2\rangle\langle \chi_1, \chi_2 | U | E_{n_1 n_2}\rangle \\ &= N_{n_1} \exp\left\{-\frac{\alpha_1'^2}{2}\left[\frac{1}{\sqrt{2}}(e_1 \eta_1 + e_2 \eta_2)\right]^2\right\} \\ &\quad \times H_{n_1}\left[\alpha_1' \frac{1}{\sqrt{2}}(e_1 \eta_1 + e_2 \eta_2)\right] \\ &\quad \times N_{n_2} \exp\left\{-\frac{\alpha_2'^2}{2}\left[\frac{1}{\sqrt{2}}(e_2 \eta_1 + e_1 \eta_2)\right]^2\right\} \\ &\quad \times H_{n_2}\left[\alpha_2' \frac{1}{\sqrt{2}}(e_2 \eta_1 - e_1 \eta_2)\right]. \end{aligned} \tag{35}$$

结合(31)式可以得到在 $\langle \eta_1, \eta_2|$ 表象中压缩前的本征波函数为

$$\begin{aligned} \langle \eta_1, \eta_2 | E_{n_1 n_2}\rangle &= \langle \eta_1, \eta_2 | U^\dagger U | E_{n_1 n_2}\rangle \\ &= \sqrt{\kappa_1 \kappa_2}\langle \kappa_1 \eta_1, \kappa_2 \eta_2 | U | E_{n_1 n_2}\rangle \\ &= \left(\frac{\Lambda_2}{\Lambda_1}\right)^{1/4} N_{n_1} \exp\left\{-\frac{\alpha_1'^2}{2}\left[\frac{1}{\sqrt{2}}\left(e_1 \eta_1 \right.\right.\right. \\ &\quad \left.\left.\left. + e_2 \sqrt{\frac{\Lambda_2}{\Lambda_1}} \eta_2\right)\right]^2\right\} H_{n_1}\left[\alpha_1' \frac{1}{\sqrt{2}}\left(e_1 \eta_1 \right.\right. \\ &\quad \left.\left. + e_2 \sqrt{\frac{\Lambda_2}{\Lambda_1}} \eta_2\right)\right] \end{aligned}$$

$$\times N_{n_2}\exp\left\{-\frac{\alpha_2'^2}{2}\left[\frac{1}{\sqrt{2}}\left(e_2\eta_1-e_1\sqrt{\frac{\lambda_2}{\lambda_1}}\eta_2\right)\right]^2\right\}H_{n_2}\left[\alpha_2'\frac{1}{\sqrt{2}}\left(e_2\eta_1-e_1\sqrt{\frac{\Lambda_2}{\Lambda_1}}\eta_2\right)\right]. \qquad (36)$$

将(36)式代入

$$\langle x_1,x_2|E_{n_1n_2}\rangle=\iint_{-\infty}^{\infty}\mathrm{d}\eta_1\mathrm{d}\eta_2\langle x_1,x_2\|\eta_1,\eta_2\rangle\times\langle\eta_1,\eta_2|E_{n_1n_2}\rangle,$$

并结合(26)式可得在坐标表象$\langle x_1,x_2|$中的波函数为

$$\langle x_1,x_2|E_{n_1n_2}\rangle=\left(\frac{\Lambda_2}{\Lambda_1}\right)^{1/4}N_{n_1}\exp\left[-\frac{\alpha'^2}{2}(\sigma_1x_1+\sigma_2x_2)^2\right]\times H_{n_1}[\alpha_1'(\sigma_1x_1+\sigma_2x_2)]\times N_{n_2}\exp\left[-\frac{\alpha'^2}{2}(\tau_1x_1+\tau_2x_2)^2\right]\times H_{n_2}[\alpha_2'(\tau_1x_1+\tau_2x_2)].$$

这里

$$\sigma_1=\frac{1}{2}\left(e_1d_1+e_2d_2\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right),$$

$$\sigma_2=\frac{1}{2}\left(e_1d_2-e_2d_1\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right);$$

$$\tau_1=\frac{1}{2}\left(e_2d_1-e_1d_2\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right),$$

$$\tau_2=\frac{1}{2}\left(e_2d_2+e_1d_1\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right).$$

6. 结 论

本文建立了一种称之为组合坐标表象的新表象,讨论了这种表象的性质.借助这种组合坐标表象和 IWOP 技术可以诱导双模压缩算符 U,实现不对称积分及压缩变换. 应用组合坐标新表象,灵活选取变换参数,研究了具有相互作用势和运动耦合的两个非全同的量子谐振子体系的动力学问题. 这为研究复杂耦合量子谐振子体系提供了一个有效途径.

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The characteristics of a new two mode composite coordinate representation and its application *

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Abstract

In order to solve dynamic problems of double non-identical quantum harmonic oscillators with interaction potential and movement coupling, the new two mode composite coordinate representation $|\eta_1, \eta_2\rangle$ is proposed by the technique of integration within an ordered product of operators. The two mode squeezing operator U is constructed, and its characteristics are analyzed. Moreover, its application to double non-identical quantum harmonic oscillators with interaction potential and movement coupling is presented for solving some the dynamic problems. Approach of solving some the dynamic problems involving complicated quantum harmonic oscillators is proposed.

Keywords: integration with an ordered product of operators technique, composite coordinate representation, double mode squeezing operator, unitary matrix
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