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# Birkhoff 系统约化的 Routh 方法

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研究 Birkhoff 系统的约化.首先,列出系统的运动微分方程及其循环积分;其次,构造 Birkhoff 系统的 Routh 函数组,利用循环积分约化 Birkhoff 系统的运动微分方程,并使约化后的动力学方程仍保持 Birkhoff 方程的形式;最后,举例说明结果的应用.

关键词:Birkhoff 系统,约化,循环积分
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## 1. 引 言

对于完整力学系统,利用循环积分通过构造 Routh 函数消去循环坐标,将 Lagrange 方程约化,并使得约化后的方程仍能保持 Lagrange 方程的形式,这就是分析力学中著名的 Routh 方法[1,2].近 20 年来,Routh 方法得到了数学、力学和物理学家的重视,取得了一系列重要研究成果[3—8].

1927 年 Birkhoff 在文献[9]中给出了比 Hamilton 方程更普遍的一类新型运动微分方程.1978 年 Santilli 将其推广并建议方程命名为 Birkhoff 方程[10].1989 年 Галиуллин 指出[11],对 Birkhoff 方程的研究是近代分析力学的一个重要发展方向.1996 年梅凤翔[12]构建了 Birkhoff 系统动力学的基本理论框架.近年来,对 Birkhoff 系统动力学的研究取得了重要进展[13—24].本文进一步将 Routh 方法推广到 Birkhoff 系统,通过构造一组新的动力学函数,或称之为 Routh 函数组,研究了 Birkhoff 系统的约化问题.

## 2. Birkhoff 系统的循环积分

Birkhoff 系统的运动微分方程为[12]

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0,\quad (\mu=1,\cdots,2n),\tag{1}$$

其中 $B=B(t,\boldsymbol{a})$ 称为 Birkhoff 函数,$R_\mu=R_\mu(t,\boldsymbol{a})$ 称为 Birkhoff 函数组,$\boldsymbol{a}=(a^1,a^2,\cdots,a^{2n})^{\mathrm{T}}$.

如果函数 $B$ 和 $R_\mu$ 不显含某个 Birkhoff 变量 $a^k$ $(1\leqslant k\leqslant 2n)$,即有

$$\frac{\partial B}{\partial a^k}=0,\frac{\partial R_\mu}{\partial a^k}=0,(\mu=1,\cdots,2n;1\leqslant k\leqslant 2n),\tag{2}$$

则称 $a^k$ 为 Birkhoff 系统的一个循环坐标[7].

当 $a^k$ 是 Birkhoff 系统的一个循环坐标时,由方程(1),有

$$-\frac{\partial R_k}{\partial a^\nu}\dot{a}^\nu-\frac{\partial R_k}{\partial t}=0,\tag{3}$$

即

$$-\frac{\mathrm{d}R_k}{\mathrm{d}t}=0.\tag{4}$$

于是

$$R_k(t,a^1,a^2,\cdots,a^{k-1},a^k,a^{k+1},\cdots,a^{2n})=\beta_k,\tag{5}$$

其中 $\beta_k$ 为任意常数.显然,积分(5)是方程(1)的一个第一积分,称之为 Birkhoff 系统与循环坐标 $a^k$ 相应的循环积分[7].

## 3. Birkhoff 系统约化的 Routh 方法

Routh 于 1877 年提出应用循环积分消去循环变量,将 Lagrange 方程约化的方法[1,2],既达到了约化的目的,又能使动力学方程仍保持 Lagrange 方程的形式.下面将这一思想应用于 Birkhoff 系统的约化.

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假设 $a^k$ 是循环坐标，系统有循环积分(5). 设由(5)式可解出某个 $a^j$ 作为 $t, a^\nu (\nu \neq k, j), \beta_k$ 的函数，有

$$a^j = f_j(t, a^\nu, \beta_k), (\nu \neq k, j). \tag{6}$$

现在定义

$$B^* = B + \frac{\partial}{\partial t}(f_j R_j), \tag{7}$$

$$R_\mu^* = R_\mu - \frac{\partial}{\partial a^\mu}(f_j R_j), (\mu = 1, \cdots, 2n), \tag{8}$$

称 $B^*, R_\mu^*$ 为 Birkhoff 系统的 Routh 函数组. 应用(6)式可消去 $B^*, R_\mu^*$ 中的 $a^j$，特记为

$$B^* = \hat{B} + \frac{\partial}{\partial t}(\hat{R}_j f_j) = B^*(t, a^\nu, \beta_k), \quad (\nu \neq k, j), \tag{9}$$

$$R_\mu^* = \hat{R}_\mu - \frac{\partial}{\partial a^\mu}(\hat{R}_j f_j) = R_\mu^*(t, a^\nu, \beta_k), \quad (\mu = 1, \cdots, 2n; \nu \neq k, j). \tag{10}$$

由(9)和(10)式，有

$$\frac{\partial B^*}{\partial a^\nu} = \frac{\partial \hat{B}}{\partial a^\nu} + \frac{\partial^2 (\hat{R}_j f_j)}{\partial t \partial a^\nu}, \tag{11}$$

$$\frac{\partial R_\mu^*}{\partial a^\nu} = \frac{\partial \hat{R}_\mu}{\partial a^\nu} - \frac{\partial^2 (\hat{R}_j f_j)}{\partial a^\mu \partial a^\nu}, \tag{12}$$

$$\frac{\partial R_\mu^*}{\partial t} = \frac{\partial \hat{R}_\mu}{\partial t} - \frac{\partial^2 (\hat{R}_j f_j)}{\partial a^\mu \partial t}. \tag{13}$$

由于

$$\frac{\partial \hat{B}}{\partial a^\nu} = \frac{\partial B}{\partial a^\nu} + \frac{\partial B}{\partial a^j}\frac{\partial f_j}{\partial a^\nu}, \quad (\nu = 1, \cdots, 2n; \nu \neq k, j), \tag{14}$$

$$\frac{\partial \hat{R}_\mu}{\partial a^\nu} = \frac{\partial R_\mu}{\partial a^\nu} + \frac{\partial R_\mu}{\partial a^j}\frac{\partial f_j}{\partial a^\nu}, \quad (\mu, \nu = 1, \cdots, 2n; \nu \neq k, j), \tag{15}$$

$$\frac{\partial \hat{R}_\mu}{\partial t} = \frac{\partial R_\mu}{\partial t} + \frac{\partial R_\mu}{\partial a^j}\frac{\partial f_j}{\partial t}, \quad (\mu = 1, \cdots, 2n). \tag{16}$$

将(14)—(16)式分别代入(11)—(13)式，有

$$\frac{\partial B}{\partial a^\nu} = \frac{\partial B^*}{\partial a^\nu} - \frac{\partial B}{\partial a^j}\frac{\partial f_j}{\partial a^\nu} - \frac{\partial^2 (\hat{R}_j f_j)}{\partial t \partial a^\nu}, \quad (\nu = 1, \cdots, 2n; \nu \neq k, j), \tag{17}$$

$$\frac{\partial R_\mu}{\partial a^\nu} = \frac{\partial R_\mu^*}{\partial a^\nu} - \frac{\partial R_\mu}{\partial a^j}\frac{\partial f_j}{\partial a^\nu} + \frac{\partial^2 (\hat{R}_j f_j)}{\partial a^\mu \partial a^\nu}, \quad (\mu, \nu = 1, \cdots, 2n; \nu \neq k, j), \tag{18}$$

$$\frac{\partial R_\mu}{\partial t} = \frac{\partial R_\mu^*}{\partial t} - \frac{\partial R_\mu}{\partial a^j}\frac{\partial f_j}{\partial t} + \frac{\partial^2 (\hat{R}_j f_j)}{\partial a^\mu \partial t}, \quad (\mu = 1, \cdots, 2n). \tag{19}$$

由(17)—(19)式，并考虑到关系式(6)，可得到

$$\begin{aligned}
&\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} \\
&= \left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} \\
&\quad + \frac{\partial R_\mu}{\partial a^j}\left(\frac{\partial f_j}{\partial a^\nu}\dot{a}^\nu + \frac{\partial f_j}{\partial t} - \dot{a}^j\right) \\
&\quad + \frac{\partial f_j}{\partial a^\mu}\left(\frac{\partial B}{\partial a^j} - \frac{\partial R_\nu}{\partial a^j}\dot{a}^\nu\right) \\
&= \left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} \\
&\quad + \frac{\partial f_j}{\partial a^\mu}\left(\frac{\partial B}{\partial a^j} - \frac{\partial R_\nu}{\partial a^j}\dot{a}^\nu\right), \\
&\qquad (\mu, \nu = 1, \cdots, 2n; \mu \neq k, j).
\end{aligned} \tag{20}$$

而由(7)式和(8)式，有

$$\frac{\partial B^*}{\partial \beta_k} = R_j \frac{\partial^2 f_j}{\partial t \partial \beta_k} + \frac{\partial R_j}{\partial t}\frac{\partial f_j}{\partial \beta_k}, \tag{21}$$

$$\frac{\partial R_\mu^*}{\partial \beta_k} = -R_j \frac{\partial^2 f_j}{\partial a^\mu \partial \beta_k} - \frac{\partial R_j}{\partial a^\mu}\frac{\partial f_j}{\partial \beta_k}. \tag{22}$$

将(22)式的两边乘以 $\dot{a}^\mu$，并对 $\mu$ 求和，有

$$\frac{\partial R_\mu^*}{\partial \beta_k}\dot{a}^\mu = -R_j \frac{\partial^2 f_j}{\partial a^\mu \partial \beta_k}\dot{a}^\mu - \frac{\partial R_j}{\partial a^\mu}\frac{\partial f_j}{\partial \beta_k}\dot{a}^\mu. \tag{23}$$

将(21)式与上式相减，整理后可得

$$\begin{aligned}
\dot{a}^k &= -\frac{\mathrm{d}}{\mathrm{d}t}\left(R_j \frac{\partial f_j}{\partial \beta_k}\right) + \frac{\partial B^*}{\partial \beta_k} \\
&\quad - \frac{\partial R_\mu^*}{\partial \beta_k}\dot{a}^\mu, \\
&\qquad (\mu = 1, \cdots, 2n; \mu \neq k),
\end{aligned} \tag{24}$$

积分可得

$$a^k = -R_j \frac{\partial f_j}{\partial \beta_k} + \int\left(\frac{\partial B^*}{\partial \beta_k} - \frac{\partial R_\mu^*}{\partial \beta_k}\dot{a}^\mu\right)\mathrm{d}t, \quad (\mu = 1, \cdots, 2n; \mu \neq k). \tag{25}$$

综合上述分析，我们有

**命题 1**　对于 Birkhoff 系统(1)，如果 $a^k$ 是循环坐标，且满足条件

$$\frac{\partial f_j}{\partial a^\mu}\left(\frac{\partial B}{\partial a^j} - \frac{\partial R_\nu}{\partial a^j}\dot{a}^\nu\right) = 0, (\mu = 1, \cdots, 2n), \tag{26}$$

则此系统的 $2n$ 个 Birkhoff 方程(1)可约化为($2n - 2$)个方程

$$\left(\frac{\partial R_\nu^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial B^*}{\partial a^\mu} - \frac{\partial R_\mu^*}{\partial t} = 0, \quad (\mu = 1, \cdots, 2n; \mu \neq k, j), \tag{27}$$

其中 $B^*, R_\mu^*$ 由(9)和(10)式确定. 方程(27)可称为 Birkhoff 系统的 Routh 方程，而 $a^j, a^k$ 分别由(6)式，(25)式给出.

由命题 1,可得到以下重要推论:

**命题 2** 对于 Birkhoff 系统(1),如果 $a^k$ 是循环坐标,且由循环积分 $R_k = \beta_k$ 可解出某个 $a^j$ 为

$$a^j = f_j(t, \beta_k), \tag{28}$$

则此系统的 $2n$ 个 Birkhoff 方程(1)可约化为($2n-2$)个 Routh 方程(27),而 $a^k$ 由(25)式给出.

## 4. 算 例

已知 4 阶 Birkhoff 系统的 Birkhoff 函数和 Birkhoff 函数组分别为[12]

$$B = \frac{1}{2}[(a^2)^2 + (a^3)^2], \tag{29}$$

$$R_1 = R_2 = 0, R_3 = a^1 a^2, R_4 = \frac{1}{2}(a^2)^2. \tag{30}$$

系统的 Birkhoff 方程给出

$$\begin{aligned} & a^2 \dot{a}^3 = 0, a^1 \dot{a}^3 + a^2 \dot{a}^4 - a^2 = 0, \\ & -a^2 \dot{a}^1 - a^1 \dot{a}^2 - a^3 = 0, \\ & -a^2 \dot{a}^2 = 0. \end{aligned} \tag{31}$$

显然,$a^4$ 是系统的循环坐标,于是,系统有循环积分

$$\frac{1}{2}(a^2)^2 = \beta_4 = \text{const}. \tag{32}$$

由(32)式解得

$$a^2 = (2\beta_4)^{1/2}. \tag{33}$$

(9)式和(10)式分别给出

$$B^* = \beta_4 + \frac{1}{2}(a^3)^2, \tag{34}$$

$$R_1^* = 0, R_2^* = 0, R_3^* = (2\beta_4)^{1/2} a^1, R_4^* = \beta_4, \tag{35}$$

于是方程(27)给出

$$(2\beta_4)^{1/2} \dot{a}^3 = 0, -(2\beta_4)^{1/2} \dot{a}^1 - a^3 = 0. \tag{36}$$

方程(36)是 2 个一阶微分方程,比较(31)式,方程数减少了 2 个.

由(25)式得

$$a^4 = \int \{1 - (2\beta_4)^{-1/2} a^1 \dot{a}^3\} \mathrm{d}t + c_4, \tag{37}$$

其中 $c_4$ 为任意常数.方程(36)和(33)式,(37)式给出系统的运动.


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# Routh method of reduction of Birkhoffian systems

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Abstract

Reduction of Birkhoffian systems is studied. First，the differential equations of motion for the Birkhoffian systems are established and their cyclic integrals are given. Second，the Routh functions of Birkhoffian systems are constructed，and the order of the systems is reduced by using the cyclic integrals and the Birkhoffian form made to hold. Finally，an example is given to illustrate the application of the results.

**Keywords**：Birkhoffian system，reduction，cyclic integral

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