第 58 卷 第 10 期 2009 年 10 月
1000-3290/2009/58(10)/6686-06

物 理 学 报
ACTA PHYSICA SINICA

Vol.58,No.10,October,2009

变系数广义 Gardner 方程的微分不变量及群分类*

郭美玉† 高 洁
(聊城大学数学科学学院,聊城 252059)
(2008 年 6 月 4 日收到;2009 年 2 月 12 日收到修改稿)

应用李无穷小不变规则,得到了变系数广义 Gardner 方程的连续等价变换.从等价代数开始,构造了一阶微分不变量并依据微分不变量对方程作了群分类.最后,通过等价变换将变系数 Gardner 方程映射为常系数 mKdV 方程、KdV-mKdV 方程.同时,也得到了变系数广义 Gardner 方程的一些精确解.

关键词:李无穷小不变规则,微分不变量,群分类,变系数广义 Gardner 方程
PACC:0220

1. 引 言

考虑变系数广义 Gardner 方程

$$u_t + a(x,t)uu_x + b(x,t)u^2u_x + c(x,t)u_{xxx} + d(x,t)u_x + f(x,t)u = 0, \tag{1}$$

其中 $a(x,t)$, $b(x,t)$, $c(x,t)$, $d(x,t)$ 和 $f(x,t)$ 是关于 x 和 t 的实函数.当 $b, d, f = 0$, $c = 1$ 和 $a = q$ 时,方程(1)约化为 mKdV 方程

$$u_t + qu^2u_x + u_{xxx} = 0, \tag{2}$$

其中 q 是任意常数.当 $d, f = 0$, $c = 1$, $a = \mu$ 和 $b = v$ 时,方程(1)约化为 KdV-mKdV 方程

$$u_t + \mu uu_x + vu^2u_x + u_{xxx} = 0, \tag{3}$$

其中 μ, v 是任意常数.方程(2)和(3)[1—3]是众所周知的方程,并且它们在物理学方面都有广泛的应用.关于更一般的 Gardner 方程已经受到了广泛的关注.

近期,一些学者致力于某些重要的非线性动力系统的等价变换的研究.所谓方程(1)的等价变换,是指变量 (t,x,u) 到 $(\hat{t},\hat{x},\hat{u})$ 的非退化的变换,并且将方程(1)映射为另一个相同形式但含有不同函数 $(\hat{a},\hat{b},\hat{c},\hat{d},\hat{f})$ 的方程[4—6].因此,一个方程的解可以转化为等价方程的解.

在微分方程理论中,一个经典研究是寻找李群微分不变量.最初,Laplace 得到了线性双曲方程的两个微分不变量.Cotton 得到了椭圆和双曲方程的微分不变量的一般形式[7,8].最近,关于某些多维线性和非线性偏微分方程的等价代数的微分不变量的研究再次引起了众多学者的兴趣.

本文借助于李无穷小规则[9—11]获得了方程(1)的等价变换代数和若干微分不变量.将 a, b, c, d, f 视为任意参数,在等价变换代数下,可以将方程(1)转变为常系数 KdV 形式的方程.同时借助于常系数方程的解可以得到某些变系数偏微分方程的解.

2. 变系数广义 Gardner 方程的等价变换

在扩张空间 (t,x,u,a,b,c,d,f) 中,在辅助条件

$$a_u = b_u = c_u = d_u = f_u = 0 \tag{4}$$

下,考虑单参数群的如下等价变换 G_ε:

$$\hat{t} = t + \varepsilon\xi^1(t,x,u) + o(\varepsilon^2),$$
$$\hat{x} = x + \varepsilon\xi^2(t,x,u) + o(\varepsilon^2),$$
$$\hat{u} = u + \varepsilon\eta(t,x,u) + o(\varepsilon^2), \tag{5}$$
$$\hat{a} = a + \varepsilon\gamma^1(t,x,u,a,b,d,f,g) + o(\varepsilon^2),$$
$$\hat{b} = b + \varepsilon\gamma^2(t,x,u,a,b,d,f,g) + o(\varepsilon^2), \tag{6}$$
$$\hat{c} = c + \varepsilon\gamma^3(t,x,u,a,b,c,d,f) + o(\varepsilon^2),$$
$$\hat{d} = d + \varepsilon\gamma^3(t,x,u,a,b,d,f,g) + o(\varepsilon^2), \tag{7}$$
$$\hat{f} = f + \varepsilon\gamma^4(t,x,u,a,b,d,f,g) + o(\varepsilon^2), \tag{8}$$

其中 ε 是群参数.无穷小等价变换的向量场可以表示为

* 山东省自然科学基金(批准号:2004ZX16,Q2005A01)资助的课题.
† E-mail:skyguomeiyu@163.com

$$Y = \xi^1\partial_t + \xi^2\partial_x + \eta\partial_u + \gamma^1\partial_a + \gamma^2\partial_b + \gamma^3\partial_c + \gamma^4\partial_d + \gamma^5\partial_f. \quad (9)$$

定义

$$(f^1,f^2,f^3,f^4,f^5) \equiv (a,b,c,d,f),(x^1,x^2) \equiv (t,x), \quad (10)$$

$$Y^{(3)} = Y + \zeta_j \frac{\partial}{\partial u_j} + \zeta_{jjj}\frac{\partial}{\partial u_{jjj}} + \hat{w}_j^r \frac{\partial}{\partial f_{x^j}^r} + \bar{w}_u^r \frac{\partial}{\partial f_u^r}, \quad (11)$$

$$\zeta_j = D_j\eta - u_k D_j\xi^k, \zeta_{jj} = D_j\zeta_j - u_{jk}D_j\xi^k,$$

$$\zeta_{jjj} = D_j\zeta_{jj} - u_{jjk}D_j\xi^k, \quad (12)$$

$$D_j = \frac{\partial}{\partial x^j} + u_j\frac{\partial}{\partial u} + u_{jk}\frac{\partial}{\partial u_k}, \quad (13)$$

$$\hat{w}_j^r = \hat{D}_j(\gamma^r) - f_{x^k}^r\hat{D}_j(\xi^k) - f_u^r\hat{D}_j(\eta), \quad (14)$$

$$\hat{D}_j = \frac{\partial}{\partial x^j} + f_{x^j}^r\frac{\partial}{\partial f^r}, \bar{D}_j = \frac{\partial}{\partial y^j} + f_u^r\frac{\partial}{\partial f^r}. \quad (15)$$

$$u_j = \frac{\partial u}{\partial x^j}, u_{jk} = \frac{\partial^2 u}{\partial x^j \partial x^k}$$

$$(j,\ k = 1,2),(r = 1,2,3,4,5).$$

在单参数群的等价变换(5)—(8)式下，方程(1)的不变性要求满足下述条件：

$$Y^{(3)}(u_t + a(x,t)uu_x + b(x,t)u^2u_x + c(x,t)u_{xxx} + d(x,t)u_x + f(x,t)u) = 0, \quad (16)$$

$$Y^{(3)}(a_u) = Y^{(3)}(b_u) = Y^{(3)}(c_u) = Y^{(3)}(d_u) = Y^{(3)}(f_u) = 0. \quad (17)$$

将(12)—(15)式和方程(1)代入(16)，(17)式，可以得

$$\xi^1 = \varphi(t), \xi^2 = s(t)x + q(t), \eta = \beta(t)u,$$

$$\gamma^1 = -a\varphi_t + as(t) - a\beta(t), \quad (18)$$

$$\gamma^2 = -b\varphi_t + bs(t) - 2b\beta(t),$$

$$\gamma^3 = -c\varphi_t + 3cs(t), \quad (19)$$

$$\gamma^4 = s(t)d - d\varphi_t + s_t x + q_t,$$

$$\gamma^5 = -f\varphi_t - \beta_t, \quad (20)$$

其中 $\varphi(t)$，$s(t)$，$q(t)$ 和 $\beta(t)$ 是关于 t 的任意函数.相应的等价代数是无穷维的且是由以下算子生成

$$Y_\varphi = \varphi(t)\partial_t - a\varphi_t\partial_a - b\varphi_t\partial_b - c\varphi_t\partial_c - d\varphi_t\partial_d - f\varphi_t\partial_f, \quad (21)$$

$$Y_s = s(t)x\partial_x + as(t)\partial_a + bs(t)\partial_b + 3cs(t)\partial_c + (s(t)d + s_t x)\partial_d, \quad (22)$$

$$Y_q = q(t)\partial_x + q_t\partial_d,$$

$$Y_\beta = \beta(t)u\partial_u - a\beta(t)\partial_a - 2b\beta(t)\partial_b - \beta_t\partial_f. \quad (23)$$

3. 等价代数的微分不变量

3.1. 零阶微分不变量

假定零阶微分不变量的形式为

$$J = J(t,x,a,b,c,d,f). \quad (24)$$

应用微分不变检验 $Y(J) = 0$ 作用到算子 $Y_\varphi, Y_s, Y_q, Y_\beta$ 上.因为任意函数 φ, s, q, β 及其导数被视为独立的，等式 $Y_\varphi(J) = 0$，$Y_s(J) = 0$，$Y_q(J) = 0$ 及 $Y_\beta(J) = 0$ 可以被简化为如下条件：

$$\frac{\partial J}{\partial t} = 0, \frac{\partial J}{\partial x} = 0, \frac{\partial J}{\partial d} = 0, \frac{\partial J}{\partial f} = 0, \frac{\partial J}{\partial c} = 0, \frac{\partial J}{\partial a} = 0, \frac{\partial J}{\partial b} = 0. \quad (25)$$

由方程(25)可知方程(1)不含有零阶微分不变量.

3.2. 一阶微分不变量

如果一阶微分不变量的形式为

$$J = J(t,x,a,b,c,d,f,a_x,a_t,b_x,b_t,c_x,c_t,d_x,d_t,f_x,f_t), \quad (26)$$

我们考虑算子 Y 的一阶延拓为

$$Y^{(1)} = Y + \hat{w}_j^r \frac{\partial}{\partial f_{x^j}^r}, \quad (27)$$

一阶延拓算子 $Y_\varphi^{(1)}$，$Y_s^{(1)}$，$Y_q^{(1)}$ 和 $Y_\beta^{(1)}$ 的表达形式如下：

$$\begin{aligned} Y_\varphi^{(1)} = {} & \varphi(t)\partial_t - a\varphi_t\partial_a - b\varphi_t\partial_b - c\varphi_t\partial_c - d\varphi_t\partial_d \\ & - f\varphi_t\partial_f - (2a_t\varphi_t + a\varphi_{tt})\frac{\partial}{\partial a_t} - a_x\varphi_t\frac{\partial}{\partial a_x} \\ & - (2b_t\varphi_t + b\varphi_{tt})\frac{\partial}{\partial b_t} - b_x\varphi_t\frac{\partial}{\partial b_x} \\ & - (2c_t\varphi_t + c\varphi_{tt})\frac{\partial}{\partial c_t} - c_x\varphi_t\frac{\partial}{\partial c_x} \\ & - (2d_t\varphi_t + d\varphi_{tt})\frac{\partial}{\partial d_t} - d_x\varphi_t\frac{\partial}{\partial d_x} \\ & - (2f_t\varphi_t + f\varphi_{tt})\frac{\partial}{\partial f_t} - f_x\varphi_t\frac{\partial}{\partial f_x}, \end{aligned} \quad (28)$$

$$\begin{aligned} Y_s^{(1)} = {} & s(t)x\partial_x + as(t)\partial_a + bs(t)\partial_b + 3cs(t)\partial_c \\ & + (s(t)d + s_t x)\partial_d + (a_t s + as_t - a_x s_t x)\frac{\partial}{\partial a_t} \\ & + (b_t s + bs_t - b_x s_t x)\frac{\partial}{\partial b_t} + (3c_t s + 3cs_t - c_x s_t x) \\ & \times \frac{\partial}{\partial c_t} + 2c_x s\frac{\partial}{\partial c_x} + (d_t s + ds_t + s_{tt}x - d_x s_t x) \end{aligned}$$

$$\times \frac{\partial}{\partial a_t} + s_t \frac{\partial}{\partial d_x} - f_x s \frac{\partial}{\partial f_x} - f_x s_t x \frac{\partial}{\partial f_t}, \tag{29}$$

$$Y_q^{(1)} = q(t)\partial_x + q_t\partial_d + (q_{tt} - d_x q_t)\frac{\partial}{\partial d_t} - a_x q_t \times \frac{\partial}{\partial a_t} - b_x q_t \frac{\partial}{\partial b_t} - c_x q_t \frac{\partial}{\partial c_t} - f_x q_t \frac{\partial}{\partial f_t}, \tag{30}$$

$$Y_\beta^{(1)} = \beta(t) u\partial_u - a\beta(t)\partial_a - 2b\beta(t)\partial_b - \beta_t\partial_f - (a_t\beta + a\beta_t)\frac{\partial}{\partial a_t} - a_x\beta\frac{\partial}{\partial a_x} - (2b_t\beta + 2b\beta_t) \times \frac{\partial}{\partial b_t} - 2b_x\beta\frac{\partial}{\partial b_x} - \beta_{tt}\frac{\partial}{\partial f_t}. \tag{31}$$

现在应用不变量检验 $Y^{(1)}(J) = 0$ 作用到算子 $Y_\varphi^{(1)}$，$Y_s^{(1)}$，$Y_q^{(1)}$，$Y_\beta^{(1)}$ 上．因为任意函数 φ, s, q, β 及其导数被视为独立的，等式 $Y_\varphi^{(1)}(J) = 0$，$Y_s^{(1)}(J) = 0$，$Y_q^{(1)}(J) = 0$和 $Y_\beta^{(1)}(J) = 0$ 可以被约化为如下条件：

$$\frac{\partial J}{\partial t} = \frac{\partial J}{\partial x} = \frac{\partial J}{\partial d_t} = \frac{\partial J}{\partial f_t} = 0, \tag{32}$$

$$\frac{\partial J}{\partial d} - a_x\frac{\partial J}{\partial a_t} - b_x\frac{\partial J}{\partial b_t} - c_x\frac{\partial J}{\partial c_t} = 0, \tag{33}$$

$$a\frac{\partial J}{\partial a} + 2b\frac{\partial J}{\partial b} + a_t\frac{\partial J}{\partial a_t} + a_x\frac{\partial J}{\partial a_x} + 2b_t\frac{\partial J}{\partial b_t} + 2b_x\frac{\partial J}{\partial b_x} = 0, \tag{34}$$

$$\frac{\partial J}{\partial f} + a\frac{\partial J}{\partial a_t} + 2b\frac{\partial J}{\partial b_t} = 0, \tag{35}$$

$$a\frac{\partial J}{\partial a} + b\frac{\partial J}{\partial b} + 3c\frac{\partial J}{\partial c} + d\frac{\partial J}{\partial d} + a_t\frac{\partial J}{\partial a_t} + b_t\frac{\partial J}{\partial b_t} + 3c_t\frac{\partial J}{\partial c_t} + 2c_x\frac{\partial J}{\partial c_x} - f_x\frac{\partial J}{\partial f_x} = 0, \tag{36}$$

$$x\frac{\partial J}{\partial d} + (a - xa_x)\frac{\partial J}{\partial a_t} + (b - xb_x)\frac{\partial J}{\partial b_t} + (3c - xc_x)\frac{\partial J}{\partial c_t} + \frac{\partial J}{\partial d_x} = 0, \tag{37}$$

$$a\frac{\partial J}{\partial a} + b\frac{\partial J}{\partial b} + c\frac{\partial J}{\partial c} + d\frac{\partial J}{\partial d} + f\frac{\partial J}{\partial f} + 2a_t\frac{\partial J}{\partial a_t} + a_x\frac{\partial J}{\partial a_x} + 2b_t\frac{\partial J}{\partial b_t} + b_x\frac{\partial J}{\partial b_x} + 2c_t\frac{\partial J}{\partial c_t} + c_x\frac{\partial J}{\partial c_x} + d_x\frac{\partial J}{\partial d_x} + f_x\frac{\partial J}{\partial f_x} = 0, \tag{38}$$

$$a\frac{\partial J}{\partial a_t} + b\frac{\partial J}{\partial b_t} + c\frac{\partial J}{\partial c_t} = 0. \tag{39}$$

等式(32)将不变量(26)简化为

$$J = J(a, b, c, d, f, a_x, a_t, b_x, b_t, c_x, c_t, d_x, f_x). \tag{40}$$

将(33)和(39)式代入(37)式得

$$2c\frac{\partial J}{\partial c_t} + \frac{\partial J}{\partial d_x} = 0,$$

解其特征方程得

$$\lambda_1 = d_x - \frac{c_t}{2c}, \tag{41}$$

因此，函数(40)变为

$$J = J(a, b, c, d, f, a_x, a_t, b_x, b_t, c_x, f_x, \lambda_1). \tag{42}$$

将(42)代入方程(35)并简化，可以得到

$$\frac{\partial J}{\partial f} + a\frac{\partial J}{\partial a_t} + 2b\frac{\partial J}{\partial b_t} = 0, \tag{43}$$

解其特征方程得

$$\lambda_2 = f - \frac{a_t}{a}, \lambda_3 = f - \frac{b_t}{2b}, \tag{44}$$

因此，函数(42)变为

$$J = J(a, b, c, d, a_x, b_x, c_x, f_x, \lambda_1, \lambda_2, \lambda_3). \tag{45}$$

将(45)代入方程(39)并简化，可以得到

$$\frac{\partial J}{\partial \lambda_1} + \frac{\partial J}{\partial \lambda_3} + 2\frac{\partial J}{\partial \lambda_2} = 0, \tag{46}$$

解其特征方程得

$$\lambda_4 = \lambda_2 - 2\lambda_3, \lambda_5 = \lambda_2 - 2\lambda_1, \tag{47}$$

因此，函数(45)变为

$$J = J(a, b, c, d, a_x, b_x, c_x, f_x, \lambda_4, \lambda_5). \tag{48}$$

将(48)代入方程(33)并简化，得

$$\frac{\partial J}{\partial d} + \left(\frac{a_x}{a} - \frac{b_x}{b}\right)\frac{\partial J}{\partial \lambda_4} + \left(\frac{a_x}{a} - \frac{c_x}{c}\right)\frac{\partial J}{\partial \lambda_5} = 0, \tag{49}$$

解其特征方程得

$$\lambda_6 = \lambda_4 - \left(\frac{a_x}{a} - \frac{b_x}{b}\right)d,$$

$$\lambda_7 = \lambda_5 - \left(\frac{a_x}{a} - \frac{c_x}{c}\right)d, \tag{50}$$

因此，函数(48)变为

$$J = J(a, b, c, a_x, b_x, c_x, f_x, \lambda_6, \lambda_7). \tag{51}$$

将(51)代入方程(34)并简化，得

$$a\frac{\partial J}{\partial a} + 2b\frac{\partial J}{\partial b} + a_x\frac{\partial J}{\partial a_x} + 2b_x\frac{\partial J}{\partial b_x} = 0, \tag{52}$$

解其特征方程得

$$\lambda_8 = \frac{b}{a^2}, \lambda_9 = \frac{a_x}{a}, \lambda_{10} = \frac{b_x}{a^2}, \tag{53}$$

因此，函数(51)变为

$$J = J(c, c_x, f_x, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}). \tag{54}$$

将(54)代入方程(36)并简化，可以得到

$$\lambda_8\frac{\partial J}{\partial \lambda_8} + \lambda_9\frac{\partial J}{\partial \lambda_9} + 2\lambda_{10}\frac{\partial J}{\partial \lambda_{10}} - 3c\frac{\partial J}{\partial c} + f_x\frac{\partial J}{\partial f_x} - 2c_x\frac{\partial J}{\partial c_x} = 0, \tag{55}$$

解其特征方程得

$$\lambda_{11}=\frac{\lambda_9}{\lambda_8}f_x,\lambda_{12}=\frac{\lambda_{10}}{\lambda_8^2},\lambda_{13}=c\lambda_8^3,$$

$$\lambda_{14}=c_x\lambda_8^2,\lambda_{15}=\frac{f_x}{\lambda_8},\tag{56}$$

因此,函数(54)变为

$$J=J(\lambda_6,\lambda_7,\lambda_{11},\lambda_{12},\lambda_{13},\lambda_{14},\lambda_{15}).\tag{57}$$

将(57)代入方程(38)并简化,可以得到

$$\lambda_7\frac{\partial J}{\partial\lambda_7}-\lambda_{14}\frac{\partial J}{\partial\lambda 14}+\lambda_6\frac{\partial J}{\partial\lambda_6}+\lambda_{12}\frac{\partial J}{\partial\lambda_{12}}-2\lambda_{13}\frac{\partial J}{\partial\lambda_{13}}$$

$$+2\lambda_{15}\frac{\partial J}{\partial\lambda_{15}}+\lambda_{11}\frac{\partial J}{\partial\lambda_{11}}=0,\tag{58}$$

解其特征方程得

$$\lambda_{16}=\lambda_{13}\lambda_7^2,\lambda_{17}=\lambda_{13}\lambda_6^2,\lambda_{18}=\lambda_{13}\lambda_{12}^2,$$

$$\lambda_{19}=\lambda_{13}\lambda_{11}^2,\lambda_{20}=\frac{\lambda_{14}^2}{\lambda_{13}},\lambda_{21}=\lambda_{13}\lambda_{15},\tag{59}$$

因此,我们可以得到方程(1)的等价变换的一阶微分不变量的一般形式

$$J=J(\lambda_{16},\lambda_{17},\lambda_{18},\lambda_{19},\lambda_{20},\lambda_{21}),\tag{60}$$

其中 $\lambda_{16},\lambda_{17},\lambda_{18},\lambda_{19},\lambda_{20}$ 和 λ_{21} 是 6 个独立的不变量,且它们分别为

$$\lambda_{16}=\frac{cb^3}{a^6}\left(f-\frac{a_t}{a}-2d_x+\frac{c_t}{c}-\frac{a_xd}{a}+\frac{c_xd}{c}\right)^2,$$

$$\lambda_{17}=\frac{cb^3}{a^6}\left(-f-\frac{a_t}{a}+\frac{b_t}{b}-\frac{a_xd}{a}+\frac{b_xd}{b}\right)^2,$$

$$\lambda_{18}=\frac{cb_x^2}{a^2b},\lambda_{19}=\frac{bca_x^2}{a^4},\lambda_{20}=\frac{bc_x^2}{ca^2},\lambda_{21}=\frac{cb^2f_x}{a^4}.$$

4. 群分类及应用

根据所求出的微分不变量,我们现对方程(1)进行群分类,并求出相应的精确解.

情形 1　设 $a_x=b_x=c_x=f_x=0$, $d=\left(-\frac{a_t}{a}+\frac{b_t}{2b}+\frac{c_t}{2c}\right)x+m(t)$, $f=-\frac{a_t}{a}+\frac{b_t}{b}$,此时,方程(1)变为

$$u_t+a(t)uu_x+b(t)u^2u_x+c(t)u_{xxx}$$
$$+\left(\left(-\frac{a_t}{a}+\frac{b_t}{2b}+\frac{c_t}{2c}\right)x+m(t)\right)u_x$$
$$+\left(-\frac{a_t}{a}+\frac{b_t}{b}\right)u=0,\tag{61}$$

作变换

$$u=\frac{k_1a(t)}{b(t)}U(X(x,t),T(t)),$$

$$X=\frac{k_1a(t)}{\sqrt{b(t)c(t)}}x-k_2\int\frac{a(t)}{\sqrt{b(t)c(t)}}\mathrm{d}t,$$

$$T=k_2^3\int\frac{a^3(t)}{b(t)\sqrt{b(t)c(t)}}\mathrm{d}t,$$

其中 k_1,k_2 为任意常数.则方程(61)变为

$$U_T+\frac{k_1}{k_2^2}UU_X+\frac{k_1^2}{k_2^2}U^2U_X+U_{XXX}=0.\tag{62}$$

由文献[1]知方程(62)的解,故方程(61)的解为

$$u_1=\pm\frac{k_1a(t)}{b(t)}\left(\frac{A6k_2^2\sec(\sqrt{-A}(\xi+\xi_0))}{k_1\sqrt{1+6Ak_2^2}\pm k_1\sec(\sqrt{-A}(\xi+\xi_0))}\right),$$

$$\xi=\frac{k_1a(t)}{\sqrt{b(t)c(t)}}x-k_2\int\frac{a(t)}{\sqrt{b(t)c(t)}}\mathrm{d}t-Ak_2^3\int\frac{a^3(t)}{b(t)\sqrt{b(t)c(t)}}\mathrm{d}t\quad\left(6k_2^2<-\frac{1}{A}\right).$$

$$u_2=\pm\frac{k_1a(t)}{b(t)}\left(\frac{A6k_2^2\operatorname{csch}(\sqrt{A}(\xi+\xi_0))}{k_1\sqrt{-1-6Ak_2^2}\pm k_1\operatorname{csch}(\sqrt{A}(\xi+\xi_0))}\right),$$

$$\xi=\frac{k_1a(t)}{\sqrt{b(t)c(t)}}x-k_2\int\frac{a(t)}{\sqrt{b(t)c(t)}}\mathrm{d}t-Ak_2^3\int\frac{a^3(t)}{b(t)\sqrt{b(t)c(t)}}\mathrm{d}t\quad\left(6k_2^2>-\frac{1}{A}\right).$$

$$u_3=\pm\frac{k_1a(t)}{b(t)}\left(\frac{A6k_2^2\csc(\sqrt{-A}(\xi+\xi_0))}{k_1\sqrt{1+6Ak_2^2}\pm k_1\csc(\sqrt{-A}(\xi+\xi_0))}\right),$$

$$\xi=\frac{k_1a(t)}{\sqrt{b(t)c(t)}}x-k_2\int\frac{a(t)}{\sqrt{b(t)c(t)}}\mathrm{d}t-Ak_2^3\int\frac{a^3(t)}{b(t)\sqrt{b(t)c(t)}}\mathrm{d}t\quad\left(6k_2^2>-\frac{1}{A}\right).$$

情形 2　当方程(61)中的系数满足如下条件:$a(t)=l_1c(t)\mathrm{e}^{\int f(t)\mathrm{d}t}$, $b(t)=l_2c(t)\mathrm{e}^{2\int f(t)\mathrm{d}t}$,方程(61)化为

$$u_t+l_1c(t)\mathrm{e}^{\int f(t)\mathrm{d}t}uu_x+l_2c(t)\mathrm{e}^{2\int f(t)\mathrm{d}t}u^2u_x$$

$$+c(t)u_{xxx}+m(t)u_x+f(t)u=0, \quad (63)$$

作变换

$$T=\left(\frac{\lambda^2 l_2}{v}\right)^{3/2}\int c(t)\mathrm{d}t,$$

$$X=x\sqrt{\frac{l_2\lambda^2}{v}}+\sqrt{\frac{l_2\lambda^2}{v}}\times\int\left(\left(\frac{l_1^2}{4l_2}-\frac{l_2\mu^2\lambda^2}{4v^2}\right)c(t)-m(t)\right)\mathrm{d}t,$$

$$u=U\lambda\mathrm{e}^{-\int f(t)\mathrm{d}t}+\mathrm{e}^{-\int f(t)\mathrm{d}t}\left(\frac{\mu\lambda}{2v}-\frac{l_1}{2l_2}\right),$$

此时方程(63)化为方程(3).

作变换

$$T=\left(\frac{\lambda_1 l_2}{q}\right)^{3/2}\int c(t)\mathrm{d}t,$$

$$X=x\sqrt{\frac{l_2\lambda_1}{q}}+\sqrt{\frac{l_2\lambda_1}{q}}\int\left(\frac{l_1^2}{4l_2}c(t)-m(t)\right)\mathrm{d}t,$$

$$u=U\lambda_1\mathrm{e}^{-\int f(t)\mathrm{d}t}-\frac{l_1}{2l_2}\mathrm{e}^{-\int f(t)\mathrm{d}t},$$

此时方程(63)化为方程(2),其中 l_1, $l_2\neq 0$, $q\neq 0$, λ_1 是任意常数.

由文献[3]知方程(3)的解,故方程(63)的解为

$$u_1=\frac{12}{\mu}\lambda\mathrm{e}^{-\int f(t)\mathrm{d}t}\left(1+\tanh\left(x\sqrt{\frac{l_2\lambda^2}{v}}+\sqrt{\frac{l_2\lambda^2}{v}}\int\left(\left(\frac{l_1^2}{4l_2}-\frac{4l_2\lambda^2}{v}\right)c(t)-m(t)\right)\mathrm{d}t\right)\right)-\frac{l_1}{2l_2}\mathrm{e}^{-\int f(t)\mathrm{d}t},$$

$$u_2=\frac{12}{\mu}\lambda\mathrm{e}^{-\int f(t)\mathrm{d}t}\left(1+\coth\left(x\sqrt{\frac{l_2\lambda^2}{v}}+\sqrt{\frac{l_2\lambda^2}{v}}\int\left(\left(\frac{l_1^2}{4l_2}-\frac{4l_2\lambda^2}{v}\right)c(t)-m(t)\right)\mathrm{d}t\right)\right)-\frac{l_1}{2l_2}\mathrm{e}^{-\int f(t)\mathrm{d}t},$$

$$u_3=\frac{24}{\mu}\lambda\mathrm{e}^{-\int f(t)\mathrm{d}t}\left(2+\tanh\left(x\sqrt{\frac{l_2\lambda^2}{v}}+\sqrt{\frac{l_2\lambda^2}{v}}\int\left(\left(\frac{l_1^2}{4l_2}-\frac{16l_2\lambda^2}{v}\right)c(t)-m(t)\right)\mathrm{d}t\right)\right.$$
$$\left.+\coth\left(x\sqrt{\frac{l_2\lambda^2}{v}}+\sqrt{\frac{l_2\lambda^2}{v}}\int\left(\left(\frac{l_1^2}{4l_2}-\frac{16l_2\lambda^2}{v}\right)c(t)-m(t)\right)\mathrm{d}t\right)\right)-\frac{l_1}{2l_2}\mathrm{e}^{-\int f(t)\mathrm{d}t}.$$

情形 3 当方程(61)中的系数满足如下条件:$b(t)=-a_1c(t)$,方程(61)化为

$$u_t+a(t)uu_x-a_1c(t)u^2u_x+c(t)u_{xxx}+\left(\left(-\frac{a_t}{a}+\frac{b_t}{b}\right)x+m(t)\right)u_x+\left(-\frac{a_t}{a}+\frac{b_t}{b}\right)u=0, \quad (64)$$

其中 a_1 是任意的常数.方程(64)是变系数 KdV-mKdV方程.根据文献[5]的讨论,该方程的 Jacobi 椭圆函数解为:$u_1=\frac{a(t)}{-a_1c(t)}\left\{-\frac{1}{2}\pm\sqrt{\frac{6\lambda^2r^2}{a_1}}\mathrm{sn}\left[\lambda\frac{a(t)}{-a_1c(t)}x+g(t),r\right]\right\}$,当 $r\to 1$ 时,方程(64)有孤子解:$u_2=\frac{a(t)}{-a_1c(t)}\left\{-\frac{1}{2}\pm\sqrt{\frac{6\lambda^2r^2}{a_1}}\tanh\left[\lambda\frac{a(t)}{-a_1c(t)}x+g(t)\right]\right\}$.其中 λ 是任意常数,$r(0<r<1)$ 是 sn 的模.

情形 4 当方程(61)中的系数满足如下条件:$b(t)=0$,方程(61)化为

$$u_t+a(t)uu_x+c(t)u_{xxx}+\left(\left(-\frac{a_t}{a}+\frac{c_t}{2c}\right)x+m(t)\right)u_x-\frac{a_t}{a}u=0, \quad (65)$$

方程(64)是变系数 mKdV 方程.根据文献[12]的讨论,我们可以相应得到方程(65)的解.

情形 5 设 $a_x=b_x=c_x=d_x=f_x=0$,$f=\frac{a_t}{a}-\frac{c_t}{c}$,此时,方程(1)变为

$$u_t+a(t)uu_x+b(t)u^2u_x+c(t)u_{xxx}+d(t)u_x+\left(\frac{a_t}{a}-\frac{c_t}{c}\right)u=0, \quad (66)$$

作变换

$$u=\frac{c(t)}{a(t)}U(X(x,t),T(t)),$$

$$X=\lambda x-\lambda\int d(t)\mathrm{d}t,\ T=\lambda\int\frac{b(t)c^2(t)}{a^2(t)}\mathrm{d}t,$$

方程(66)化为

$$U_T+\frac{a^2(t)}{b(t)c(t)}UU_X+U^2U_X+\frac{a^2(t)\lambda^2}{b(t)c(t)}U_{XXX}=0. \quad (67)$$

由文献[6]我们可得方程(67)的解,进而可以确定方程(66)的解.

鉴于篇幅的原因,我们没有将常系数 KdV 方程、KdV-mKdV 方程的解一一列出,故在本文中只有选择性地得到了若干变系数方程的显式解.

5. 结 论

本文得到了一类变系数 Gardner 方程的等价变换的无穷小生成元. 应用不变量检验,得知方程(1)没有零阶微分不变量但含有一阶微分不变量. 作为微分不变量的应用,对方程(1)作了群分类并通过等价变换将一般的变系数 Gardner 方程映射为常系数 mKdV 方程,KdV-mKdV 方程. 同时,也得到了方程(1)的精确解.

衷心感谢聊城大学刘希强教授的关心和帮助.

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Differential invariants and group classification of variable coefficient generalized Gardner equation*

Guo Mei-Yu† Gao Jie

(*School of Mathematics Sciences, Liaocheng University, Liaocheng* 252059, *China*)

(Received 4 June 2008; revised manuscript received 12 February 2009)

Abstract

By using Lie's invariance infinitesimal criterion, we obtain the continuous equivalence transformations of a class of nonlinear Gardner equations with variable coefficients. Starting from the equivalence algebra, we construct the differential invariants of order one and make group classification. Finally some general class of variable coefficient nonlinear Gardner equations can be mapped to constant-coefficient mKdV equation and KdV-mKdV equation. In particular, some exact solutions of the Gardner equation with variable coefficients are obtained.

Keywords: Lie's invariance infinitesimal criterion, differential invariants, group classification, variable coefficient generalized Gardner equation

PACC: 0220

* Project supported by the Natural Science Foundation of Shandong Province, China (Grant Nos. 2004ZX16, Q2005A01).

† E-mail: skyguomeiyu@163.com