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广义 Birkhoff 系统的 Birkhoff 对称性与守恒量*

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研究广义 Birkhoff 系统的 Birkhoff 对称性问题,并给出此情形下相应的守恒量.将力学系统的等效 Lagrange 函数的一个定理推广到广义 Birkhoff 系统,证明了在一定条件下与两组动力学函数 B, R_μ, Λ_μ 和 $\bar{B}, \bar{R}_\mu, \bar{\Lambda}_\mu$ 分别给出的广义 Birkhoff 方程相关联的矩阵 Λ 的各次幂的迹是系统的守恒量.举例说明结果的应用.

关键词:广义 Birkhoff 系统,Birkhoff 对称性,守恒量,矩阵迹
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1. 引 言

1966 年,Currie 和 Saletan 曾研究了单自由度力学系统的等效 Lagrange 函数问题,并指出在此情形下存在守恒量[1].1981 年,Hojman 和 Harleston 将此结果推广到了一般的多自由度系统[2].赵跃宇等将这种等效 Lagrange 函数问题,即对应于某一个 Lagrange 函数的运动微分方程的每一个解都满足从另一个 Lagrange 函数得到的运动微分方程,称为 Lagrange 对称性[3],并将结果进一步推广到完整非保守系统.梅凤翔等[4]将这一思想移植到 Birkhoff 系统,并称之为 Birkhoff 对称性.1993 年,Mei[5]提出了广义 Birkhoff 方程并研究了 Birkhoff 系统和广义 Birkhoff 系统的 Noether 对称性.因为通常的 Birkhoff 系统不容易构造,而广义 Birkhoff 方程的实现则较易,并且有更多的"自由度",因此,对广义 Birkhoff 系统动力学的研究有重要意义,并取得了一些成果[6—8].本文进一步研究广义 Birkhoff 系统的 Birkhoff 对称性及其相应的守恒量问题,文末给出了一个算例.

2. 广义 Birkhoff 系统的 Birkhoff 对称性的定义和判据

广义 Birkhoff 方程有如下形式[9]:

$$\Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = -\Lambda_\mu \quad (\mu = 1,\cdots,2n), \tag{1}$$

其中 $B = B(t, \boldsymbol{a})$ 称为 Birkhoff 函数,$R_\mu = R_\mu(t, \boldsymbol{a})$ 称为 Birkhoff 函数组,$\Lambda_\mu = \Lambda_\mu(t, \boldsymbol{a})$ 称为附加项.设系统的 Birkhoff 变量 $a^\mu (\mu = 1,\cdots,2n)$ 彼此独立,而

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{2}$$

称为 Birkhoff 张量.假设方程(1)非奇异,即设

$$\det(\Omega_{\mu\nu}) \neq 0 \tag{3}$$

则由方程(1)可解出所有 $\dot{a}^\mu$,有

$$\dot{a}^\mu = \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\nu\right) \quad (\mu = 1,\cdots,2n), \tag{4}$$

其中

$$\Omega^{\mu\nu}\Omega_{\nu\tau} = \delta_{\mu\tau}. \tag{5}$$

展开方程(4),记作

$$\dot{a}^\mu = h_\mu(t, \boldsymbol{a}) \quad (\mu = 1,\cdots,2n). \tag{6}$$

定义算子

$$S_\mu = \Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} + \Lambda_\mu. \tag{7}$$

如果 Birkhoff 函数变换为 $\bar{B} = \bar{B}(t, \boldsymbol{a})$,Birkhoff 函数组变换为 $\bar{R}_\mu = \bar{R}_\mu(t, \boldsymbol{a})$,附加项变换为 $\bar{\Lambda}_\mu = \bar{\Lambda}_\mu(t, \boldsymbol{a})$,则有

$$\bar{S}_\mu = \bar{\Omega}_{\mu\nu}\dot{a}^\nu - \frac{\partial \bar{B}}{\partial a^\mu} - \frac{\partial \bar{R}_\mu}{\partial t} + \bar{\Lambda}_\mu, \tag{8}$$

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其中

$$\bar{\Omega}_{\mu\nu} = \frac{\partial \bar{R}_\nu}{\partial a^\mu} - \frac{\partial \bar{R}_\mu}{\partial a^\nu}, \quad \det(\bar{\Omega}_{\mu\nu}) \neq 0. \tag{9}$$

定义　对于广义 Birkhoff 系统，如果由动力学函数 $B = B(t, \boldsymbol{a})$，$R_\mu = R_\mu(t, \boldsymbol{a})$，$\Lambda_\mu = \Lambda_\mu(t, \boldsymbol{a})$ 得到的运动微分方程

$$S_\mu = 0 \tag{10}$$

的每一个解都满足由动力学函数 $\bar{B} = \bar{B}(t, \boldsymbol{a})$，$\bar{R}_\mu = \bar{R}_\mu(t, \boldsymbol{a})$，$\bar{\Lambda}_\mu = \bar{\Lambda}_\mu(t, \boldsymbol{a})$ 确定的运动微分方程

$$\bar{S}_\mu = 0, \tag{11}$$

且反之亦然，则相应不变性称为广义 Birkhoff 系统的 Birkhoff 对称性.

由(8)式和方程(11)，可得

$$\dot{a}^\mu = \bar{\Omega}^{\mu\nu}\left(\frac{\partial \bar{B}}{\partial a^\nu} + \frac{\partial \bar{R}_\nu}{\partial t} - \bar{\Lambda}_\nu\right), \tag{12}$$

将方程(12)代入(7)式，则由方程(10)给出

$$\Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\frac{\partial \bar{B}}{\partial a^\rho} + \frac{\partial \bar{R}_\rho}{\partial t} - \bar{\Lambda}_\rho\right) = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \Lambda_\mu, \tag{13}$$

于是有

判据　对于广义 Birkhoff 系统，如果动力学函数 B, R_μ, Λ_μ 和 $\bar{B}, \bar{R}_\mu, \bar{\Lambda}_\mu$ 满足关系式(13)，则相应不变性为系统的 Birkhoff 对称性.

3. Birkhoff 对称性导致的守恒量

将(13)式代入(7)式，有

$$S_\mu = \Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\bar{\Omega}_{\rho l}\dot{a}^l - \frac{\partial \bar{B}}{\partial a^\rho} - \frac{\partial \bar{R}_\rho}{\partial t} + \bar{\Lambda}_\rho\right) = \Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\bar{S}_\rho, \tag{14}$$

即

$$\bar{S}_\mu = \Lambda_\mu^\nu S_\nu, \tag{15}$$

其中

$$\Lambda_\mu^\nu = \bar{\Omega}_{\mu\rho}\Omega^{\rho\nu}, \tag{16}$$

且 $\det\Lambda = \det(\Lambda_\mu^\nu) \neq 0$.

根据(16)式，关系式(13)可以写为

$$\frac{\partial \bar{B}}{\partial a^\mu} + \frac{\partial \bar{R}_\mu}{\partial t} - \bar{\Lambda}_\mu = \Lambda_\mu^\nu\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\nu\right). \tag{17}$$

将方程(17)两边同时对 a^ρ 求偏导数，有

$$\frac{\partial^2 \bar{B}}{\partial a^\mu \partial a^\rho} + \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} - \frac{\partial \bar{\Lambda}_\mu}{\partial a^\rho} = \frac{\partial \Lambda_\mu^\nu}{\partial a^\rho}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} - \Lambda_\nu\right) + \Lambda_\mu^\nu\left(\frac{\partial^2 B}{\partial a^\nu \partial a^\rho} + \frac{\partial^2 R_\nu}{\partial t \partial a^\rho} - \frac{\partial \Lambda_\nu}{\partial a^\rho}\right). \tag{18}$$

根据(16)式，有

$$\Lambda_\mu^\nu \Omega_{\nu l} = \bar{\Omega}_{\mu l}. \tag{19}$$

将(19)式两边同时对 a^ρ 求偏导数，有

$$\frac{\partial \Lambda_\mu^\nu}{\partial a^\rho}\Omega_{\nu l} = \frac{\partial \bar{\Omega}_{\mu l}}{\partial a^\rho} - \Lambda_\mu^\nu \frac{\partial \Omega_{\nu l}}{\partial a^\rho}. \tag{20}$$

将(2)式和(9)式代入(20)式，整理可得

$$\begin{aligned}\frac{\partial \Lambda_\mu^\nu}{\partial a^\rho}\Omega_{\nu l} &= \frac{\partial^2 \bar{R}_l}{\partial a^\mu \partial a^\rho} - \frac{\partial^2 \bar{R}_\mu}{\partial a^l \partial a^\rho} - \Lambda_\mu^\nu\left(\frac{\partial^2 R_l}{\partial a^\nu \partial a^\rho} - \frac{\partial^2 R_\nu}{\partial a^l \partial a^\rho}\right) \\ &= \frac{\partial^2 \bar{R}_l}{\partial a^\mu \partial a^\rho} - \frac{\partial^2 \bar{R}_\rho}{\partial a^\mu \partial a^l} + \frac{\partial^2 \bar{R}_\rho}{\partial a^\mu \partial a^l} - \frac{\partial^2 \bar{R}_\mu}{\partial a^l \partial a^\rho} \\ &\quad - \Lambda_\mu^\nu\left(\frac{\partial^2 R_l}{\partial a^\nu \partial a^\rho} - \frac{\partial^2 R_\rho}{\partial a^\nu \partial a^l} + \frac{\partial^2 R_\rho}{\partial a^\nu \partial a^l} - \frac{\partial^2 R_\nu}{\partial a^l \partial a^\rho}\right) \\ &= \frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu} + \frac{\partial \bar{\Omega}_{\mu\rho}}{\partial a^l} - \Lambda_\mu^\nu\left(\frac{\partial \Omega_{\rho l}}{\partial a^\nu} + \frac{\partial \Omega_{\nu\rho}}{\partial a^l}\right) \\ &= \frac{\partial \Lambda_\mu^\nu}{\partial a^l}\Omega_{\nu\rho} + \frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu} - \Lambda_\mu^\nu \frac{\partial \Omega_{\rho l}}{\partial a^\nu}.\end{aligned} \tag{21}$$

由方程(6)和(7)式，方程(10)给出

$$\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t} - \Lambda_\mu = \Omega_{\mu\nu}h_\nu. \tag{22}$$

将(22)式代入方程(18)，有

$$\frac{\partial^2 \bar{B}}{\partial a^\mu \partial a^\rho} + \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} - \frac{\partial \bar{\Lambda}_\mu}{\partial a^\rho} = \frac{\partial \Lambda_\mu^\nu}{\partial a^\rho}\Omega_{\nu l}h_l + \Lambda_\mu^\nu\left(\frac{\partial^2 B}{\partial a^\nu \partial a^\rho} + \frac{\partial^2 R_\nu}{\partial t \partial a^\rho} - \frac{\partial \Lambda_\nu}{\partial a^\rho}\right). \tag{23}$$

将(21)式代入(23)式，有

$$\begin{aligned}&\frac{\partial \Lambda_\mu^\nu}{\partial a^l}\Omega_{\nu\rho}h_l + \frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu}h_l - \Lambda_\mu^\nu \frac{\partial \Omega_{\rho l}}{\partial a^\nu}h_l + \Lambda_\mu^\nu\left(\frac{\partial^2 B}{\partial a^\nu \partial a^\rho} + \frac{\partial^2 R_\nu}{\partial t \partial a^\rho} - \frac{\partial \Lambda_\nu}{\partial a^\rho}\right) \\ &- \frac{\partial^2 \bar{B}}{\partial a^\mu \partial a^\rho} - \frac{\partial^2 \bar{R}_\mu}{\partial t \partial a^\rho} + \frac{\partial \bar{\Lambda}_\mu}{\partial a^\rho} = 0.\end{aligned} \tag{24}$$

将(19)式对 t 求偏导数，有

$$\frac{\partial \Lambda_\mu^\nu}{\partial t}\Omega_{\nu\rho} + \Lambda_\mu^\nu \frac{\partial \Omega_{\nu\rho}}{\partial t} = \frac{\partial \bar{\Omega}_{\mu\rho}}{\partial t}, \tag{25}$$

即

$$\frac{\partial \Lambda_\mu^\nu}{\partial t}\Omega_{\nu\rho} + \Lambda_\mu^\nu\left(\frac{\partial^2 R_\rho}{\partial a^\nu \partial t} - \frac{\partial^2 R_\nu}{\partial a^\rho \partial t}\right)$$

$$-\frac{\partial^2 \bar{R}_\rho}{\partial a^\mu \partial t}+\frac{\partial^2 \bar{R}_\mu}{\partial a^\rho \partial t}=0. \tag{26}$$

将(22)式对 a^ν 求偏导数,有

$$\frac{\partial \Omega_{\rho l}}{\partial a^\nu}h_l+\Omega_{\rho l}\frac{\partial h_l}{\partial a^\nu}-\frac{\partial^2 B}{\partial a^\rho \partial a^\nu}-\frac{\partial^2 R_\rho}{\partial t \partial a^\nu}+\frac{\partial \Lambda_\rho}{\partial a^\nu}=0. \tag{27}$$

同理,有

$$\frac{\partial \bar{\Omega}_{\rho l}}{\partial a^\mu}h_l+\bar{\Omega}_{\rho l}\frac{\partial h_l}{\partial a^\mu}-\frac{\partial^2 \bar{B}}{\partial a^\rho \partial a^\mu}-\frac{\partial^2 \bar{R}_\rho}{\partial t \partial a^\mu}+\frac{\partial \bar{\Lambda}_\rho}{\partial a^\mu}=0. \tag{28}$$

将(24)式和(26)式相加,并利用(27)和(28)式,有

$$\frac{\bar{\mathrm{d}}\Lambda^\nu_\mu}{\mathrm{d}t}\Omega_{\nu\rho}-\bar{\Omega}_{\rho l}\frac{\partial h_l}{\partial a^\mu}-\frac{\partial \bar{\Lambda}_\rho}{\partial a^\mu}+\frac{\partial \bar{\Lambda}_\mu}{\partial a^\rho}+\Lambda^\nu_\mu\Omega_{\rho l}\frac{\partial h_l}{\partial a^\nu}+\Lambda^\nu_\mu\frac{\partial \Lambda_\rho}{\partial a^\nu}-\Lambda^\nu_\mu\frac{\partial \Lambda_\nu}{\partial a^\rho}=0, \tag{29}$$

其中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+h_\mu\frac{\partial}{\partial a^\mu}. \tag{30}$$

因此,当广义 Birkhoff 系统的附加项 $\Lambda_\mu,\bar{\Lambda}_\mu$ 满足

$$\frac{\partial \Lambda_\mu}{\partial a^\rho}=\frac{\partial \Lambda_\rho}{\partial a^\mu}, \tag{31}$$

$$\frac{\partial \bar{\Lambda}_\mu}{\partial a^\rho}=\frac{\partial \bar{\Lambda}_\rho}{\partial a^\mu} \tag{32}$$

时,由(29)式可得到一矩阵微分方程

$$\frac{\bar{\mathrm{d}}\Lambda}{\mathrm{d}t}=-H\Lambda+\Lambda H, \tag{33}$$

其中 $H=\left(\frac{\partial h_\nu}{\partial a^\mu}\right)$ 为 $2n$ 阶矩阵. 于是,利用矩阵迹的性质[10],我们有

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\mathrm{tr}\Lambda)=\mathrm{tr}\left(\frac{\bar{\mathrm{d}}\Lambda}{\mathrm{d}t}\right)=-\mathrm{tr}(H\Lambda)+\mathrm{tr}(\Lambda H)=0. \tag{34}$$

利用方程(33)和(34)式,容易得出

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}(\mathrm{tr}\Lambda^m)=0, \tag{35}$$

其中 m 为正整数. 由(35)式,得

$$\mathrm{tr}\Lambda^m=\mathrm{const}. \tag{36}$$

于是有

定理 1 对于广义 Birkhoff 系统,如果附加项 $\Lambda_\mu,\bar{\Lambda}_\mu$ 满足关系(31)和(32),则系统的 Birkhoff 对称性导致守恒量(36).

如果附加项 $\Lambda_\mu=\bar{\Lambda}_\mu=0(\mu=1,\cdots,2n)$,则系统成为通常的 Birkhoff 系统. 在此情形下条件(31),(32)自然成立,于是定理 1 成为

定理 2 对于通常的 Birkhoff 系统,系统的 Birkhoff 对称性导致守恒量(36).

定理 2 是文献[4]给出的,但文献[4]的证明有误.

4. 算 例

例 已知某二阶广义 Birkhoff 系统的动力学函数为

$$B=(a^1)^2+a^1a^2+(a^2)^2, R_1=a^2,$$
$$R_2=-a^1, \Lambda_1=a^2, \Lambda_2=a^1, \tag{37}$$

相应的广义 Birkhoff 方程给出

$$-2(\dot{a}^2+a^1)=0, 2(\dot{a}^1-a^2)=0. \tag{38}$$

实际上,方程(38)的解也可以由另外的动力学函数导出. 如取

$$\bar{B}=\frac{1}{3}(a^1)^3\cos t-\frac{1}{3}(a^2)^3\sin t+a^1a^2\cos t,$$
$$\bar{R}_1=a^1a^2\cos t, \bar{R}_2=a^1a^2\sin t,$$
$$\bar{\Lambda}_1=a^2\cos t, \bar{\Lambda}_2=a^1\cos t, \tag{39}$$

与动力学函数(39)相应的广义 Birkhoff 方程为

$$(a^2\sin t-a^1\cos t)(\dot{a}^2+a^1)=0,$$
$$(a^2\sin t-a^1\cos t)(-\dot{a}^1+a^2)=0. \tag{40}$$

比较方程(38)和(40)可以发现,它们两者具有相同解.

(2)式给出

$$(\Omega_{\mu\nu})=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}, \tag{41}$$

而

$$(\bar{\Omega}^{\nu\rho})=\frac{1}{a^1\cos t-a^2\sin t}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{42}$$

由(41),(40)式,易知条件(13)成立. 根据本文给出的判据,动力学函数(37)和(39)相应于系统的 Birkhoff 对称性.

(16)式给出

$$(\Lambda^\nu_\mu)=\begin{pmatrix}0 & a^2\sin t-a^1\cos t\\ a^1\cos t-a^2\sin t & 0\end{pmatrix}\begin{pmatrix}0 & \frac{1}{2}\\ -\frac{1}{2} & 0\end{pmatrix}$$
$$=\begin{pmatrix}-\frac{(a^2\sin t-a^1\cos t)}{2} & 0\\ 0 & \frac{(a^1\cos t-a^2\sin t)}{2}\end{pmatrix}. \tag{43}$$

容易验证,函数 $\Lambda_1,\Lambda_2,\bar{\Lambda}_1,\bar{\Lambda}_2$ 满足条件(31),

(32)，根据定理 1，我们有

$$I = a^1 \cos t - a^2 \sin t = \text{const.} \tag{44}$$

这是由上述 Birkhoff 对称性导致的一个守恒量，此守恒量完全取决于 Birkhoff 函数组 R_1，R_2 和 $\bar{R}_1$，$\bar{R}_2$.

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Birkhoff symmetries and conserved quantities of generalized Birkhoffian systems *

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Abstract

The problem of Birkhoff symmetry for generalized Birkhoffian systems is studied, and the corresponding conserved quantities are given. A theorem known for nonsingular equivalent Lagrangians is generalized to the generalized Birkhoffian systems. We prove that under certain conditions the matrix Λ, which is related with the generalized Birkhoffian equations obtained from two groups of dynamical functions B, R_μ, Λ_μ and $\bar{B}$, $\bar{R}_\mu$, $\bar{\Lambda}_\mu$, has the property that the traces of all its integer powers are the conserved quantities of the system. An example is given to illustrate the application of the results.

Keywords: generalized Birkhoffian system, Birkhoff symmetry, conserved quantity, trace of matrix

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