

# 圈量子引力中面积与体积算符本征作用的估值 \*

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用统一处理几何算符作用的方法及所得的结果, 获得了一种完备的面积谱. 并对 Thiemann 的 Hamilton 约束中的欧氏项的作用给出了不同解释与结果. 在图式法处理抓作用中, 通过化简, 给出了抓任意三重组对任何  $n$  顶角作用的重耦矩阵表式. 用抓作用的移动法, 推算出了抓的任意三重组对任意价顶角作用的重耦矩阵的完整、精确的一般表式.

关键词: 体、面积算符作用的统一表述, 完备面积谱, 抓作用的化简, 重耦矩阵一般表式

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## 1. 引 言

面积与体积算符的本征作用与本征值, 在量子正则广义相对论的运动学与动力学中, 起中心作用. 它们不仅在 Planck 尺度上给出了空间面积与体积的量子化离散值<sup>[1]</sup>, 而且对面积与体积是量子化的平坦空间的描述与构成<sup>[2,3]</sup>, 对黑洞的熵及其表面的面积量子态与其周围的热力学交换关系的计算<sup>[4]</sup>, 自旋网的 Hamilton 动力学演化<sup>[5]</sup>, 以及面积量子对量子信息中的量子 qubit 的解释与可能的纠缠上<sup>[6]</sup>等都起着某种关键作用. 面积与体积的表式是由时空  $3+1$  分解后而得的 3 维空间  $\Sigma$  中的 Riemann 几何下各自的算式给出, 然而, 它们各自的量子化算符本身、使用及结果文献中却存在着不同.

对于面积算符而言, 存在着其中的圈算符中的抓是作用在自旋网的腿上<sup>[7]</sup>, 或是抓在腿中的圈线上<sup>[8,9]</sup>, 两种情况. 另外, 还存在一种不涉及抓作用, 而是利用面积算符与自旋网的腿相交时生出的  $SU(2)$  群生成元所构成的 Casimir 算子, 直接得到面积本征值<sup>[1]</sup>. 对体积算符而言, 也存在着其中的抓三重组是作用在腿上<sup>[7]</sup>, 或抓在腿中的圈线上<sup>[9]</sup>, 两种情况. 对于作用在腿上的情况, 还存在外腿封闭和外

腿不封闭而抓被等价移动, 两种作法<sup>[1,7]</sup>. 当然, 由于体、面积算符本征作用实施方式的不同, 所得相应本征值也不同. 本文通过体积算符对任意阶顶角作用的化简, 得到了全部相应重耦矩阵的表式. 用抓的移动法, 计算了任意抓三重组对任意高价顶角作用时精确的重耦矩阵.

此外, 本文用统一的将抓作用定义为抓在自旋网腿中的 Wilson 圈上的方法, 及其得到的面积与体积本征值谱<sup>[10,11]</sup>, 给出其进一步的结果. 即得到了一种完备的面积本征值谱, 同时证明通常的面积算符对腿的作用, 是其特殊情况. 利用得到的体积本征值, 计算出了直接可求得结果的 Thiemann 的 Hamilton 约束欧氏项对 3 顶角的作用系数的最终表式.

我们知道, 利用时空作  $3+1$  分解得到的空间  $\Sigma$  中的 2-形式提供量子化面积算符, 并在其与自旋网腿(或圈线)的交点插入群  $SU(2)$  生成元, 再利用生成元构成的 Casimir 算子的方法, 可求得面积本征值<sup>[1]</sup>. 但此方法中并不是利用抓算符参加计算, 而是用与在同一交的两次“抓作用”给出的 Casimir 算子的本征值的直接对应得到结果. 这一结果比较简单, 具有不同的意义, 但不是唯一的. 而且同一思路推广到体积算符本征作用时, 由于不出现  $SU(2)$  生成元的类似情况, 这种思路将不能延续下去.

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若认为体、面积算符中的抓是抓在腿中的 Wilson 圈上,则对于面积算符的作用,正如文献[9]指出,应考虑两种类型的抓作用,即  $(A) = \text{Diagram } A$  和  $(B) = \text{Diagram } B$

$= \text{Diagram } C$  对面积算符本征值的贡献,对体积算符而言,抓作用则应有如下三种类型  $(I) = \text{Diagram } I$ ,

$(II) = \text{Diagram } II$   $(III) = \text{Diagram } III$ . 这里指出,文献[7]在

计算面积算符作用时,采用的是抓作用在腿上的方法,而文献[9]采用的则是抓在腿中的圈线上,但在计算体积本征值的最后结果时,并未考虑Ⅲ型作用.正如文献[1]中指出,对面积与体积算符的颜色估值而言,不同的方法将会得到不同的结果.文献[7]与文献[9]的结果就不同,本文后面将进一步阐述这一问题.

这里将文献[7]中的重耦矩阵(去掉印误后,进一步)改写为

$$W_{(str)}^{(n)j_{t-2}j_{r-2}} = (-1) \lambda_{j_{t-2}}^{2i_{t-2}} P_s P_t P_r \begin{Bmatrix} j_{t-2} & P_t & j_{r-2} \\ i_{t-2} & P_t & i_{r-2} \\ 2 & 2 & 2 \end{Bmatrix} \times \frac{\delta_{i_1}^{j_1} \dots \delta_{i_{s-2}}^{j_{s-2}} \delta_{i_{r-1}}^{j_{r-1}} \dots \delta_{i_{n-3}}^{j_{n-3}} \text{Tet} \begin{Bmatrix} P_s & P_s & 2 \\ j_{t-2} & j_{t-2} & j_{s-2} \end{Bmatrix} \text{Tet} \begin{Bmatrix} i_{t-2} & j_{r-2} & j_{r-1} \\ P_r & P_r & 2 \end{Bmatrix} \Delta_{j_{t-2}} \Delta_{j_{r-2}}}{\mathcal{H}(2, j_{t-2}, i_{t-2}) \mathcal{H}(2, j_{r-2}, i_{r-2}) \mathcal{H}(j_{s-2}, P_s, j_{t-2}) \mathcal{H}(j_{t-2}, P_t, j_{r-2}) \mathcal{H}(j_{r-1}, P_r, j_{r-1})}, \quad (1)$$

上式为更普遍的结果,当  $s=2, t=3, r=4$  时,该式变为文献[7]只便于 4 价和个别 5 价顶角的计算)中的形式(符号参见文献[7]).用(1)式求出具体重耦矩阵后,将其乘以虚数  $i$ ,得到的矩阵将会有本征值,将其逐个求出后,并记以  $\Omega_{(str)}^{(n)}$ ,再利用文献[7]给出的求  $n$  顶角体积本征值公式

$$V^{(n)} = \sqrt{\sum_{\substack{s=0, \dots, n-3 \\ t=r+1, \dots, n-2 \\ r=t+1, \dots, n-1}} \frac{1}{16} \Omega_{(str)}^{(n)}}, \quad (2)$$

可最后求出体积本征值.

由(1)式中的  $9-j$  记号可知,此种方法只能用来计算抓三重组( $str$ )中的  $s, t, r$  是连续分布情况下的重耦矩阵贡献的体积值.对于高价顶角,其他三重组抓法贡献的重耦矩阵,并不能直接求得,这是(1)和

(2)式的局限.然而按体积算符的定义,所有毗邻高价顶角腿的任何三重组均应等价地对体积有贡献.

## 2. 体积算符抓三重组作用图的简化与重耦矩阵

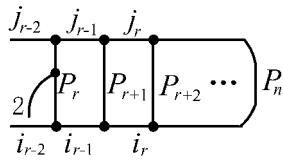
当抓三重组抓住的腿间有自由腿时,对于高于 5 价的顶角,由于直接得不到抓作用图与  $9-j$  记号的对应,故文献[7]中的上述方法不再适用.本文将推出一种抓对任意价顶角作用图的化简方法,使抓作用对应于  $9-j$  记号,并将整个图化为对 5 价或更低价顶角的作用,同时推算出它们的重耦矩阵表式.为此将体积算符抓作用的重耦矩阵方程给出如下:

$$P_s P_t P_r \begin{array}{c} j_1 \quad j_{s-2} \\ \hline \text{Diagram } A \\ \hline i_1 \quad i_{s-2} \\ \hline 2 \quad 2 \end{array} P_n = W_{(str)}^{(n)j_1 \dots j_{n-3}} (P_1, \dots, P_n) \begin{array}{c} j_1 \quad j_{s-2} \quad j_{t-2} \quad j_{r-2} \quad j_{n-3} \\ \hline \text{Diagram } B \\ \hline i_1 \quad j_{s-2} \quad j_{t-2} \quad j_{r-2} \quad j_{n-3} \\ \hline P_1 \quad P_s \quad P_t \quad P_r \quad P_{n-1} \\ \hline P_2 \quad \dots \quad \dots \quad \dots \quad P_n \end{array} \quad (3)$$

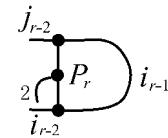
下面将就抓三重组( $s, t, r$ )的不同分布, 分别求出其重耦矩阵.

## 2.1. $r \leq n-1$ 并且 $s \geq 2$

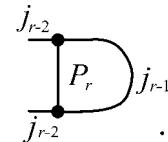
不失普遍性, 若首先考虑  $r \leq n-1$ , 则(3)式左、右两端的两个图中的右端分别为



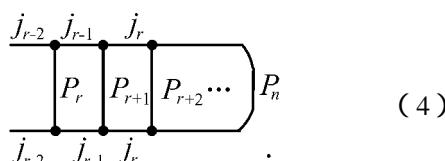
利用重耦理论的约化公式, 可将(4)变成



和

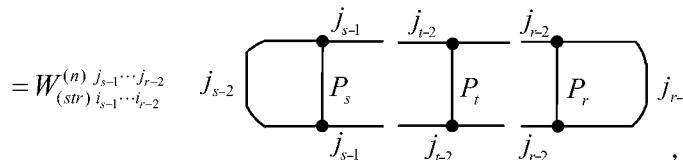
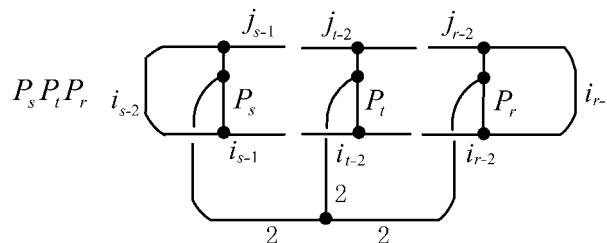


(5)



如果  $s \geq 2$ , 则(3)式等号两端两个图的左端亦做类似简化.

若抓三重组( $s, t, r$ )中的抓  $s$  和  $r$  同时满足  $r \leq n-1$  和  $s \geq 2$  则利用上面的结果(3)式可被化简为



(6)

此处, 内腿颜色  $i_{s-2} = j_{s-2}$ ,  $i_{r-1} = j_{r-1}$ . 利用如下公式

$$\frac{j_{r-2}}{2} \cdot P_r \cdot i_{r-1} = \frac{\text{Tet} \begin{bmatrix} i_{r-2} & j_{r-2} & i_{r-1} \\ P_r & P_r & 2 \end{bmatrix}}{\theta(2, i_{r-2}, j_{r-2})} \frac{j_{r-2}}{2} = X_1 \frac{j_{r-2}}{2} \quad (7)$$

和

$$i_{s-2} \frac{j_{s-1}}{2} \cdot P_s \cdot i_{s-1} = (-1) \lambda_{j_{s-1}} \frac{\text{Tet} \begin{bmatrix} P_s & P_s & 2 \\ j_{s-1} & i_{s-1} & j_{s-2} \end{bmatrix}}{\theta(2, j_{s-1}, i_{s-1})} \frac{j_{s-1}}{2} = Y_1 \frac{j_{s-1}}{2} \quad (8)$$

并将它们代入(6)式, 得到

$$= W_{(str) i_{s-1} \cdots i_{r-2}}^{(n) j_{s-1} \cdots j_{r-2}} j_{s-2} \left( \begin{array}{c} j_{s-1} \\ \text{---} \\ P_s \\ \text{---} \\ j_{s-1} \end{array} \right) \left( \begin{array}{c} j_{t-2} \\ \text{---} \\ P_t \\ \text{---} \\ j_{t-2} \end{array} \right) \left( \begin{array}{c} j_{r-2} \\ \text{---} \\ P_r \\ \text{---} \\ j_{r-2} \end{array} \right) j_{r-1} , \quad (9)$$

式中

The diagram shows a sequence of three hexagonal cells, each labeled  $P_s$ ,  $P_t$ , and  $P_r$  respectively. The cells are connected by vertical edges labeled  $j_{s-1}$ ,  $j_{t-2}$ , and  $j_{r-2}$ . The top and bottom edges of the hexagons are shown as thick lines.

$$= \theta(j_{s-2}, P_s, j_{s-1}) \frac{1}{\Delta_{j_{s-1}}} \theta(j_{s-1}, P_{s+1}, j_s) \cdots \frac{1}{\Delta_{j_{t-2}}} \theta(j_{t-2}, P_t, j_{t-1}) \frac{1}{\Delta_{j_{t-1}}} \cdots \\ \theta(j_{r-3}, P_{r-1}, j_{r-2}) \frac{1}{\Delta_{j_{r-2}}} \theta(j_{r-2}, P_r, j_{r-1}) = \Theta_1. \quad (10)$$

下面对(9)式中的图

进行估值.用四面体网和  $\theta$  网进行估值时,可得如下二式:

$$\begin{array}{c} j_{s-1} \\ \vdots \\ 2 \end{array} \begin{array}{c} j_{t-3} \\ \vdots \\ i_{s-1} \end{array} \begin{array}{c} j_{t-2} \\ \vdots \\ i_{t-3} \end{array} \begin{array}{c} P_{t-1} \\ \vdots \\ i_{t-2} \end{array} = \left( \prod_{l=s-1}^{t-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\theta(2, j_{l+1}, i_{l+1})} \right) \begin{array}{c} j_{t-2} \\ \vdots \\ i_{t-2} \end{array} = A_1 \begin{array}{c} j_{t-2} \\ \vdots \\ i_{t-2} \end{array} \quad (12)$$

和

$$\begin{aligned}
 & \text{Diagram 1: } \begin{array}{c} j_{t-1} \quad j_t \quad j_{r-2} \\ \hline P_{t+1} \\ i_{t-1} \quad i_t \quad i_{r-2} \end{array} \quad = \lambda_{j_{r-2}}^{2i_{r-2}} \quad \text{Diagram 2: } \begin{array}{c} j_{t-1} \quad j_t \quad j_{r-2} \\ \hline P_{t+1} \\ i_{t-1} \quad i_t \quad i_{r-2} \end{array} \quad \text{Diagram 3: } \begin{array}{c} j_{t-1} \quad j_{t+1} \quad 2 \\ \hline i_{t+1} \\ i_{t-1} \quad i_t \quad i_{t+2} \end{array} \\
 & = (-1) \lambda_{j_{r-1}}^{2i_{r-1}} \left( \prod_{l=t-1}^{r-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\theta(2, i_l, j_l)} \right) \begin{array}{c} j_{t-1} \\ \hline 2 \\ i_{t-1} \end{array} = B_1 \quad \begin{array}{c} j_{t-1} \\ \hline 2 \\ i_{t-1} \end{array} . \quad (13)
 \end{aligned}$$

将(12)和(13)式代入(11)式(11)式变成

$$\begin{array}{c} \text{Diagram of a 2x2 grid with nodes } j_{t-2}, j_{t-1}, P_t, i_{t-2}, i_{t-1} \text{ and boundary nodes 2.} \\ = A_l B_l \begin{pmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{pmatrix} \quad (14) \end{array}$$

令(9)中的因子

$$P_s P_t P_r X_1 Y_1 = Z_1 , \quad (15)$$

则将(10)(14)式和(15)式代入(9)式,得到此种情况下的重耦矩阵表式为

$$W_{(str)}^{j_{t-2}j_{t-1}} = \frac{A_1 B_1 Z_1}{\Theta_1} \begin{Bmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{Bmatrix}. \quad (16)$$

式中  $A_1, B_1, Z_1, \theta_1$  均为由颜色决定的可直接求出的固定数值。

## 2.2. $s = 1$ 或 $r = n$

当  $s = 1$  时 抓作用图(3)变为

$$= W_{(str)j_1 \cdots i_{n-3}}^{(n)j_1 \cdots j_{n-3}} \left( \begin{array}{c} j_1 \\ \hline P_1 \\ \hline P_2 \\ \hline i_1 \end{array} \right) \left( \begin{array}{c} j_{t-2} \\ \hline P_t \\ \hline j_{t-2} \end{array} \right) \left( \begin{array}{c} j_{r-2} \\ \hline P_r \\ \hline j_{r-2} \end{array} \right) \Bigg) P_n$$

应用 2.1 中的方法, 上式可化为

$$P_s P_t P_r \quad \text{Diagram (18)} \quad = W_{(str)i_1 \dots i_{n-3}}^{(n)j_1 \dots j_{n-3}} \quad \text{Diagram (18)}, \quad (18)$$

式中 颜色  $i_{r-1} = j_{r-1}$ . 仿照 2.1 中的计算, 并利用其中符号的类似定义 (18) 可写成

$$A_2 B_2 Z_2 \begin{Bmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{Bmatrix} = W_{(str)i_{t-2} i_{t-1}}^{(n)j_{t-2} j_{t-1}} \Theta_2, \quad (19)$$

式中

$$A_2 = \prod_{l=0}^{t-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\mathcal{O}(2, j_{l+1}, i_{l+1})} \quad (i_0 = j_0 = P_1), \quad (20)$$

$$B_2 = B_1, \quad (21)$$

$$\begin{aligned} \Theta_2 = & \mathcal{O}(P_1, P_2, j_1) \frac{1}{\Delta_{j_1}} \mathcal{O}(j_1, P_3, j_2) \dots \frac{1}{\Delta_{j_{t-2}}} \mathcal{O}(j_{t-2}, P_t, j_{t-1}) \frac{1}{\Delta_{j_{t-1}}} \dots \\ & \times \mathcal{O}(j_{r-3}, P_{r-1}, j_{r-2}) \frac{1}{\Delta_{j_{r-2}}} \mathcal{O}(j_{r-2}, P_r, j_{r-1}), \end{aligned} \quad (22)$$

$$Z_2 = P_s P_t P_r X_2, \quad (23)$$

而(23)中的

$$X_2 = X_1. \quad (24)$$

从而, 由(19)得到此种情况下的重耦矩阵表式为

$$W_{(str)i_{t-2} i_{t-1}}^{(n)j_{t-2} j_{t-1}} = \frac{A_2 B_2 Z_2}{\Theta_2} = \begin{Bmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{Bmatrix}. \quad (25)$$

另一方面, 当  $r = n$  时, 抓作用图为

$$= W_{(str)i_1 \cdots i_{n-3}}^{(n)j_1 \cdots j_{n-3}} \quad \begin{array}{c} \text{Diagram of a chain of } n \text{ nodes } P_1, P_s, P_t, P_{n-1}, P_n. \\ \text{Nodes } j_{s-2}, j_{t-2}, j_{n-3} \text{ are connected to } P_s, P_t, P_{n-1} \text{ respectively.} \\ \text{Nodes } j_{s-2}, j_{t-2}, j_{n-3} \text{ are also connected to } j_{s-2}, j_{t-2}, j_{n-3} \text{ respectively.} \end{array} \quad (26)$$

与前一情况类似, 求得的重耦矩阵的形式为

$$W_{(str)i_{t-2} i_{t-1}}^{(n)j_{t-2} j_{t-1}} = \frac{A_2 B_2 Z_2}{\Theta_2} = \begin{Bmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{Bmatrix}, \quad (27)$$

不过这里的  $A_2, B_2, Z_2$  和  $\Theta_2$  分别由下面的表式给出:

$$A_2 = A_1, \quad (28)$$

$$B_2 = (-1) \lambda_{j_{t-1}}^{i_{t-1}} \prod_{l=t-1}^{n-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\mathcal{K}(2, j_l, i_l)} \quad (i_{n-2} = j_{n-2} = P_n), \quad (29)$$

$$\begin{aligned} \Theta_2 = & \mathcal{K}(j_{s-2}, P_s, j_{s-1}) \frac{1}{\Delta_{j_{s-1}}} \mathcal{K}(j_{s-1}, P_{s+1}, j_s) \cdots \frac{1}{\Delta_{j_{t-2}}} \mathcal{K}(j_{t-2}, P_t, j_{t-1}) \frac{1}{\Delta_{j_{t-1}}} \cdots \\ & \times \mathcal{K}(j_{n-4}, P_{n-2}, j_{n-3}) \frac{1}{\Delta_{j_{n-3}}} \mathcal{K}(j_{n-3}, P_{n-1}, P_n), \end{aligned} \quad (30)$$

$$Z_2 = P_s P_t P_r Y_2, \quad (31)$$

(31)式中的

$$Y_2 = Y_1. \quad (32)$$

### 2.3. $s=0$ 并且 $r=n$

此种情况下 (3)式变为

$$\begin{array}{c} \text{Diagram of a chain of } n \text{ nodes } P_s, P_t, P_r, P_2, P_1, P_{n-1}, P_n. \\ \text{Nodes } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ are connected to } P_s, P_t, P_{n-1}, P_n \text{ respectively.} \\ \text{Nodes } i_1, i_{t-2}, i_{t-1}, i_{n-3} \text{ are connected to } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ respectively.} \\ \text{Nodes } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ are also connected to } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ respectively.} \end{array} \\ = W_{(str)i_1 \cdots i_{n-3}}^{(n)j_1 \cdots j_{n-3}} \quad \begin{array}{c} \text{Diagram of a chain of } n \text{ nodes } P_1, P_s, P_t, P_{n-1}, P_n. \\ \text{Nodes } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ are connected to } P_s, P_t, P_{n-1}, P_n \text{ respectively.} \\ \text{Nodes } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ are also connected to } j_1, j_{t-2}, j_{t-1}, j_{n-3} \text{ respectively.} \end{array} \quad (33)$$

将(12)和(13)式代入(33)式, 可得

$$W_{(str)i_{t-2} i_{t-1}}^{(n)j_{t-2} j_{t-1}} = \frac{A_3 B_3}{\Theta_3} = \begin{Bmatrix} j_{t-2} & P_t & j_{t-1} \\ i_{t-2} & P_t & i_{t-1} \\ 2 & 2 & 2 \end{Bmatrix}, \quad (34)$$

式中

$$A_3 = \prod_{l=1}^{t-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\mathcal{K}(2, j_{l+1}, i_{l+1})} = A_2, \quad (35)$$

$$B_3 = (-1) \lambda_{j_{t-1}}^{2i_{t-1}} \prod_{l=t-1}^{n-3} \frac{\text{Tet} \begin{bmatrix} j_l & j_{l+1} & 2 \\ i_{l+1} & i_l & P_{l+2} \end{bmatrix}}{\mathcal{K}(2, i_l, j_l)} = B_2, \quad (36)$$

$$\begin{aligned}\Theta_3 &= \mathcal{C}(P_1, P_2, j_1) \frac{1}{\Delta_{j_1}} \mathcal{C}(j_1, P_3, j_2) \dots \\ &\times \frac{1}{\Delta_{j_{t-2}}} \mathcal{C}(j_{t-2}, P_t, j_{t-1}) \frac{1}{\Delta_{j_{t-1}}} \dots \\ &\times \mathcal{C}(j_{n-4}, P_{n-2}, j_{n-3}) \frac{1}{\Delta_{j_{n-3}}} \mathcal{C}(j_{n-3}, j_{n-1}, P_n) \quad (37)\end{aligned}$$

易知  $A_3$  和  $B_3$  分别与(20)和(29)相等.

如上三种化简方法,可以对抓三重组  $(s, t, r)$  在被抓住  $n$  顶角的外腿之间,有或没有自由腿的各种情况下,直接求出抓作用的所有重耦矩阵.这样,可以得到体积算符的一个完整的谱为

$$V^{(n)} = \sqrt{\sum_{str} \frac{1}{16} \Omega_{str}^{(n)}}. \quad (38)$$

式中,对  $(s, t, r)$  的求和(与文献[7]不同)包含了  $\hat{V}$  对  $\Gamma$  的  $n$  顶角作用时的所有三重组的贡献.

### 3. 抓的移动方法得到的重耦矩阵

利用抓在作用图中的移动,通过图式计算,也可以得到求任何重耦矩阵的方法.本文通过推算,给出这种方法,并求出任意抓三重组  $(s, t, r)$  对 3 4 5 顶角作用的重耦矩阵的全部具体表式,以及对任意价顶角作用的精确的一般表式.

我们首先需要用重耦定理,定义抓的两种移动  $M_1$  和  $M_2$  如下:

$$\begin{aligned}M_1: \quad & \text{Diagram showing a horizontal line with points } i_{s-2}, i_{s-1}, \text{ and a vertical leg } P_s \text{ at } i_{s-1}. \text{ A curved arc connects } i_{s-2} \text{ to } i_{s-1}. \\ & = \sum_{j_{s-2}} \left\{ \begin{matrix} i_{s-1} & i_{s-2} & j_{s-2} \\ 2 & P_s & P_s \end{matrix} \right\} \text{Diagram showing a horizontal line with points } i_{s-2}, j_{s-2}, i_{s-1}, \text{ and a vertical leg } P_s \text{ at } i_{s-1}. \text{ A curved arc connects } i_{s-2} \text{ to } j_{s-2}. \\ & = \sum_{j_{s-2}} \left\{ \begin{matrix} i_{s-1} & i_{s-2} & j_{s-2} \\ 2 & P_s & P_s \end{matrix} \right\} \lambda^{2i_{s-2}} \text{Diagram showing a horizontal line with points } i_{s-2}, j_{s-2}, j_{s-1}, \text{ and a vertical leg } P_s \text{ at } j_{s-1}. \text{ A curved arc connects } i_{s-2} \text{ to } j_{s-2}. \quad (39)\end{aligned}$$

$$M_2: \quad \text{Diagram showing a horizontal line with points } i_{s-3}, i_{s-2}, j_{s-2}, \text{ and a vertical leg } P_{s-1} \text{ at } i_{s-2}. \text{ A curved arc connects } i_{s-3} \text{ to } i_{s-2}. \\ = \sum_{j_{s-3}} \left\{ \begin{matrix} j_{s-2} & 2 & j_{s-3} \\ i_{s-3} & P_{s-1} & i_{s-2} \end{matrix} \right\} \text{Diagram showing a horizontal line with points } i_{s-3}, j_{s-3}, j_{s-2}, \text{ and a vertical leg } P_{s-1} \text{ at } j_{s-2}. \text{ A curved arc connects } i_{s-3} \text{ to } j_{s-3}. \quad (40)$$

利用如上两种移动,可以把作用在外腿上的抓,移动到作用图的左端.

#### 3.1. 3 价顶角的重耦矩阵

此时抓三重组只有  $(1, 2, 3)$  一种,在抓三重组  $(1, 2, 3)$  作用下,抓三重组的作用将具有如下形式:

$$\begin{aligned}P_1 P_2 P_3 \sqrt{\frac{1}{\theta(P_1, P_2, P_3)}} & \quad \text{Diagram showing a horizontal line with points } P_1, P_2, P_3. \text{ A curved arc connects } P_1 \text{ to } P_2, \text{ and another arc connects } P_2 \text{ to } P_3. \\ & = W_{(123)}^{(3)} \sqrt{\frac{1}{\theta(P_1, P_2, P_3)}} \quad \text{Diagram showing a horizontal line with points } P_1, P_2, P_3. \quad (41)\end{aligned}$$

把上式左侧  $P_2$  上的抓,  $P_3$  上的抓,经如上  $M_1$  和  $M_2$  诸移动后(41)式变为

$$P_1 P_2 P_3 \sum_{j_0} \left\{ \begin{matrix} P_3 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{j_0}^{2B} (-1) \left\{ \begin{matrix} P_3 & 2 & P_1 \\ \tilde{j}_0 & P_2 & P_3 \end{matrix} \right\} \quad \begin{array}{c} \text{Diagram:} \\ \text{A directed graph with 5 nodes.} \\ \text{Top node: } P_2 \text{ (outgoing edges to } P_3 \text{ and } j_0). \\ \text{Middle node: } j_0 \text{ (outgoing edge to } P_3). \\ \text{Bottom-left node: } P_1 \text{ (outgoing edges to } \tilde{j}_0 \text{ and } P_3). \\ \text{Bottom-right node: } \tilde{j}_0 \text{ (outgoing edge to } P_1 \text{).} \\ \text{Left edge: } P_1 \text{ to } \tilde{j}_0 \text{ with label 2.} \\ \text{Right edge: } \tilde{j}_0 \text{ to } P_1 \text{ with label 2.} \\ \text{Top edge: } P_2 \text{ to } j_0 \text{ with label 2.} \\ \text{Bottom edge: } j_0 \text{ to } P_3 \text{ with label 2.} \end{array} \quad (42)$$

将上式左侧图的左端的抓作用图用 Tet 记号代替, 经等号两侧图形约化, 并解出上述方程, 得 3 顶角的重耦矩阵表式

$$W_{(123)}^{(3)} = P_1 P_2 P_3 \sum_{\tilde{j}_0} \left\{ \begin{matrix} P_3 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{\tilde{j}_0}^{2P_1} (-1) \left\{ \begin{matrix} P_3 & 2 & P_1 \\ \tilde{j}_0 & P_2 & P_3 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}. \quad (43)$$

### 3.2. 4 价顶角的重耦矩阵

对于规一化的4顶角 $v^4$ ，我们先以抓三重组(1, 2, 4)为例，计算抓对它的作用。这一作用可用下式给出：

$$\begin{aligned}
 & P_1 P_2 P_3 \sqrt{\frac{\Delta_i}{\theta(P_1, P_2, i) \theta(i, P_3, P_4)}} \quad \text{Diagram 1:} \\
 & \quad \text{Diagram 1 shows a horizontal line with four vertices labeled } P_1, P_2, P_3, P_4. \text{ The first vertex } P_1 \text{ has a loop labeled } 2 \text{ below it. The second vertex } P_2 \text{ has a loop labeled } 2 \text{ below it. The third vertex } P_3 \text{ has a label } i \text{ to its right. The fourth vertex } P_4 \text{ has a loop labeled } 2 \text{ below it.} \\
 & = \sum_j W_{(124)i}^{(4)j} \sqrt{\frac{\Delta_j}{\theta(P_1, P_2, j) \theta(j, P_3, P_4)}} \quad \text{Diagram 2:} \\
 & \quad \text{Diagram 2 shows a horizontal line with four vertices labeled } P_1, P_2, P_3, P_4. The first vertex } P_1 \text{ has a label } P_1 \text{ above it. The second vertex } P_2 \text{ has a label } P_2 \text{ above it. The third vertex } P_3 \text{ has a label } P_3 \text{ above it. The fourth vertex } P_4 \text{ has a label } P_4 \text{ above it. The vertex } P_3 \text{ is labeled } j \text{ below it.}
 \end{aligned} \tag{44}$$

在(44)式中,将抓在  $P_2$  上的抓,  $P_4$  上的抓移动后,使(44)式变成

$$\begin{aligned}
& P_1 P_2 P_4 \sqrt{\frac{\Delta_i}{\theta(P_1, P_2, i) \theta(i, P_3, P_4)}} \sum_{\tilde{j}_0} \left\{ \begin{matrix} i & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{\tilde{j}_0}^{2P_1} (-1) \\
& \sum_j \left\{ \begin{matrix} P_4 & 2 & j \\ i & P_3 & P_4 \end{matrix} \right\} \left\{ \begin{matrix} j & 2 & P_1 \\ \tilde{j}_0 & P_2 & i \end{matrix} \right\} \quad \text{Diagram} \\
& = \sum_j W_{(124)i}^{(4)j} \sqrt{\frac{\Delta_j}{\theta(P_1, P_2, j) \theta(j, P_3, P_4)}} \quad \text{Diagram} \quad . \quad (45)
\end{aligned}$$

式中,左侧中的求和  $\sum_j$  是按重耦定理对 4 顶角内腿所有着色进行,而右侧也有按矩阵相乘形式的对

指标  $j$  的求和. 故方程(45)的解为

$$\begin{aligned}
W_{(124)i}^{(4)i} &= NP_1 P_2 P_4 \sum_{j_0} \left\{ \begin{matrix} i & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \\
&\times \lambda_{\tilde{j}_0}^{2P_1} (-1) \left\{ \begin{matrix} P_4 & 2 & j \\ i & P_3 & P_4 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} j & 2 & P_1 \\ \tilde{j}_0 & P_2 & i \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (46) \\
&\times \left\{ \begin{matrix} P_4 & 2 & j \\ \tilde{j}_0 & P_3 & P_4 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} , \quad (49)
\end{aligned}$$

式中

$$N = \sqrt{\frac{\Delta_i \mathcal{H}(P_1, P_2, j) \mathcal{H}(j, P_3, P_4)}{\Delta_j \mathcal{H}(P_1, P_2, i) \mathcal{H}(i, P_3, P_4)}}. \quad (47)$$

其他的重耦矩阵表式,也可类似地求得. 现直接给出如下:

$$\begin{aligned}
W_{(123)i}^{(4)i} &= NP_1 P_2 P_3 \sum_{j_0} \left\{ \begin{matrix} i & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \\
&\times \lambda_{\tilde{j}_0}^{2P_1} \left\{ \begin{matrix} P_4 & i & j \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_j^{2i} \\
&\times \left\{ \begin{matrix} j & 2 & P_1 \\ \tilde{j}_0 & P_2 & i \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (48)
\end{aligned}$$

$$\begin{aligned}
W_{(134)i}^{(4)i} &= NP_1 P_3 P_4 \sum_{j} \left\{ \begin{matrix} P_4 & i & \tilde{j} \\ 2 & P_3 & P_3 \end{matrix} \right\} \\
&\times \lambda_j^{2i} \sum_{j_0} \left\{ \begin{matrix} \tilde{j} & 2 & \tilde{j}_0 \\ P_1 & P_2 & i \end{matrix} \right\} (-1)
\end{aligned}$$

$$W_{(234)i}^{(4)i} = NP_2 P_3 P_4 \sum_{j_0} \left\{ \begin{matrix} i & P_1 & \bar{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\}$$

$$\begin{aligned}
&\times \lambda_{\tilde{j}_0}^{2P_1} \sum_j \left\{ \begin{matrix} P_4 & i & \tilde{j} \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_j^{2i} \\
&\times \sum_{j_0} \left\{ \begin{matrix} \tilde{j} & 2 & \tilde{j}_0 \\ \bar{j}_0 & P_2 & i \end{matrix} \right\} (-1) \\
&\times \left\{ \begin{matrix} P_4 & 2 & j \\ \tilde{j} & P_3 & P_4 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} j & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j} \end{matrix} \right\} \\
&\times \frac{\text{Tet} \left[ \begin{matrix} \bar{j}_0 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} . \quad (50)
\end{aligned}$$

### 3.3.5 价顶角的重耦矩阵

对于5顶角,共有10种抓三重组的作用. 与4顶角的重耦矩阵求法类似,现在以三重组(3 4 5)为例,对归一化的5顶角  $v^5$  在该抓三重组作用下的重耦矩阵表式进行计算. 此种情况下,有

$$\begin{aligned}
&P_3 P_4 P_5 \sqrt{\frac{\Delta_{i_1} \Delta_{i_2}}{\theta(P_1, P_2, i_1) \theta(i_1, P_3, i_2) \theta(i_2, P_4, P_5)}} \quad \text{Diagram:} \\
&\quad \text{Diagram shows a 5-pointed star with vertices } P_1, P_2, P_3, P_4, P_5. \text{ Edges are labeled } i_1, i_2, \bar{i}_1, \bar{i}_2. \text{ Curved lines connect } P_1 \text{ to } P_3, P_2 \text{ to } P_4, \text{ and } P_3 \text{ to } P_5. \text{ The label } 2 \text{ is placed under the edge } P_4 \text{ to } P_5. \\
&= W_{(345)i_1 i_2}^{(5)j_1 j_2} \sqrt{\frac{\Delta_{j_1} \Delta_{j_2}}{\theta(P_1, P_2, j_1) \theta(j_1, P_3, j_2) \theta(j_2, P_4, P_5)}} \quad \text{Diagram:} \\
&\quad \text{Diagram shows a 5-pointed star with vertices } P_1, P_2, P_3, P_4, P_5. \text{ Edges are labeled } j_1, j_2, \bar{j}_1, \bar{j}_2. \text{ Curved lines connect } P_1 \text{ to } P_3, P_2 \text{ to } P_4, \text{ and } P_3 \text{ to } P_5. \text{ The label } 2 \text{ is placed under the edge } P_4 \text{ to } P_5. \\
&\quad . \quad (51)
\end{aligned}$$

$P_3$  上的抓,  $P_4$  上的抓和  $P_5$  上的抓,完成向左移动后(51)式变为

$$P_3 P_4 P_5 \sqrt{\frac{\Delta_{i_1} \Delta_{i_2}}{\theta(P_1, P_2, i_1) \theta(i_1, P_3, i_2) \theta(i_2, P_4, P_5)}} \sum_{j_1} \left\{ \begin{matrix} i_2 & i_1 & \bar{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_{j_1}^{2i_1}$$

$$\begin{aligned}
& \sum_{\bar{j}_0} \left\{ \begin{matrix} \bar{j}_1 & 2 & \bar{j}_0 \\ P_1 & P_2 & i_1 \end{matrix} \right\} \sum_{\bar{j}_2} \left\{ \begin{matrix} P_5 & i_2 & \tilde{j}_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda_{\bar{j}_2}^{2i_2} \sum_{\bar{j}_1} \left\{ \begin{matrix} \tilde{j}_2 & 2 & \tilde{j}_1 \\ \bar{j}_1 & P_3 & i_2 \end{matrix} \right\} \\
& \sum_{\bar{j}_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ \bar{j}_0 & P_2 & \bar{j}_1 \end{matrix} \right\} (-1) \sum_{j_2} \left\{ \begin{matrix} P_5 & 2 & j_2 \\ \tilde{j}_2 & P_4 & P_5 \end{matrix} \right\} \sum_{j_1} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & \tilde{j}_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & 2 & P_0 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \\
& \begin{array}{c} \text{Diagram showing a path from } P_1 \text{ to } P_5 \text{ through } P_2, P_3, P_4. \\ \text{Vertices are labeled } \bar{j}_0, \tilde{j}_0, j_0, j_1, j_2, \tilde{j}_1, \tilde{j}_2, \bar{j}_1, \bar{j}_0. \\ \text{Connections: } P_1 \rightarrow \bar{j}_0, \bar{j}_0 \rightarrow \tilde{j}_0, \tilde{j}_0 \rightarrow j_0, j_0 \rightarrow j_1, j_1 \rightarrow j_2, j_2 \rightarrow \tilde{j}_1, \tilde{j}_1 \rightarrow \tilde{j}_2, \tilde{j}_2 \rightarrow \bar{j}_1, \bar{j}_1 \rightarrow \bar{j}_0. \end{array} \\
& = W_{(345)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} \sqrt{\frac{\Delta_{\bar{j}_1} \Delta_{\bar{j}_2}}{\theta(P_1, P_2, j_1) \theta(j_1, P_3, j_2) \theta(j_2, P_4, P_5)}} \quad \begin{array}{c} \text{Diagram showing a path from } P_1 \text{ to } P_5 \text{ through } P_2, P_3, P_4. \\ \text{Vertices are labeled } j_1, j_2, \bar{j}_1, \bar{j}_2, \bar{j}_0, \tilde{j}_0, \tilde{j}_1, \tilde{j}_2, \bar{j}_0. \\ \text{Connections: } P_1 \rightarrow j_1, j_1 \rightarrow j_2, j_2 \rightarrow \bar{j}_1, \bar{j}_1 \rightarrow \bar{j}_2, \bar{j}_2 \rightarrow \bar{j}_0, \bar{j}_0 \rightarrow \tilde{j}_0, \tilde{j}_0 \rightarrow \tilde{j}_1, \tilde{j}_1 \rightarrow \tilde{j}_2, \tilde{j}_2 \rightarrow \bar{j}_0. \end{array} \quad (52)
\end{aligned}$$

解出如上方程, 得到重耦矩阵为

$$\begin{aligned}
W_{(345)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} &= NP_3 P_4 P_5 \sum_{\bar{j}_0 \bar{j}_0 \bar{j}_1 \bar{j}_1 \bar{j}_2} \\
&\times \left\{ \begin{matrix} i_2 & i_1 & \bar{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_{\bar{j}_1}^{2i_1} \left\{ \begin{matrix} \bar{j}_1 & 2 & \bar{j}_0 \\ P_1 & P_2 & i_1 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} P_5 & i_2 & \tilde{j}_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda_{\tilde{j}_2}^{2i_2} \left\{ \begin{matrix} \tilde{j}_2 & 2 & \tilde{j}_1 \\ \bar{j}_1 & P_3 & i_2 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ \bar{j}_0 & P_2 & \bar{j}_1 \end{matrix} \right\} (-1) \times \left\{ \begin{matrix} P_5 & 2 & j_2 \\ \tilde{j}_2 & P_4 & P_5 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & \tilde{j}_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \\
&\times \frac{\text{Tet} \left[ \begin{matrix} \bar{j}_0 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}, \quad (53)
\end{aligned}$$

式中

$$N = \sqrt{\frac{\Delta_{i_1} \Delta_{i_2} \mathcal{K}(P_1, P_2, j_1) \mathcal{K}(j_1, P_3, j_2) \mathcal{K}(j_2, P_4, P_5)}{\Delta_{j_1} \Delta_{j_2} \mathcal{K}(P_1, P_2, i_1) \mathcal{K}(i_1, P_3, i_2) \mathcal{K}(i_2, P_4, P_5)}}. \quad (54)$$

现将其余抓三重组作用的重耦矩阵, 给出如下:

$$W_{(123)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} = NP_1 P_2 P_3 \delta_{\bar{j}_2}^{i_2} \sum_{\bar{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{\bar{j}_0}^{2P_1}$$

$$\begin{aligned}
&\times \left\{ \begin{matrix} i_2 & i_1 & j_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_{j_1}^{2i_1} \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & i_1 \end{matrix} \right\} \\
&\times \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}, \quad (55)
\end{aligned}$$

$$\begin{aligned}
W_{(124)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} &= NP_1 P_2 P_4 \sum_{\bar{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{\tilde{j}_0}^{2P_1} \\
&\times \left\{ \begin{matrix} P_5 & i_2 & j_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda_{j_2}^{2i_2} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ i_1 & P_3 & i_2 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & i_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (56)
\end{aligned}$$

$$\begin{aligned}
W_{(125)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} &= NP_1 P_2 P_5 \sum_{\bar{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda_{\tilde{j}_0}^{2P_1} (-1) \\
&\times \left\{ \begin{matrix} P_5 & 2 & j_2 \\ i_2 & P_4 & P_5 \end{matrix} \right\} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ i_1 & P_3 & i_2 \end{matrix} \right\} \\
&\times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & i_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (57)
\end{aligned}$$

$$W_{(134)\bar{j}_1\bar{j}_2}^{(5)\bar{j}_1\bar{j}_2} = NP_1 P_3 P_4 \sum_{\bar{j}_1} \left\{ \begin{matrix} i_2 & i_1 & \tilde{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda_{\tilde{j}_1}^{2i_1}$$

$$\begin{aligned} & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ P_1 & P_2 & i_1 \end{matrix} \right\} \left\{ \begin{matrix} P_5 & i_2 & j_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda^{2i_2} \\ & \times \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & i_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \\ & \times \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}, \end{aligned} \quad (58)$$

$$\begin{aligned} W_{(135)\tilde{j}_1 i_2}^{(5)\tilde{j}_2} &= NP_1 P_3 P_5 \sum_{\tilde{j}_1} \left\{ \begin{matrix} i_2 & i_1 & \tilde{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda^{2i_1} \\ & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ P_1 & P_2 & i_1 \end{matrix} \right\} (-1) \left\{ \begin{matrix} P_5 & 2 & j_2 \\ i_2 & P_4 & P_5 \end{matrix} \right\} \\ & \times \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & i_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \\ & \times \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}, \end{aligned} \quad (59)$$

$$\begin{aligned} W_{(145)\tilde{j}_1 i_2}^{(5)\tilde{j}_2} &= NP_1 P_4 P_5 \sum_{\tilde{j}_2} \left\{ \begin{matrix} P_5 & i_2 & \tilde{j}_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda^{2i_2} \\ & \times \sum_{j_1} \left\{ \begin{matrix} \tilde{j}_2 & 2 & \tilde{j}_1 \\ i_1 & P_3 & i_2 \end{matrix} \right\} \\ & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ P_1 & P_2 & i_1 \end{matrix} \right\} \\ & \times (-1) \left\{ \begin{matrix} P_5 & 2 & j_2 \\ \tilde{j}_2 & P_4 & P_5 \end{matrix} \right\} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & \tilde{j}_2 \end{matrix} \right\} \\ & \times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} P_1 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}}, \end{aligned} \quad (60)$$

$$\begin{aligned} W_{(234)\tilde{j}_1 i_2}^{(5)\tilde{j}_2} &= NP_2 P_3 P_4 \sum_{\tilde{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda^{2P_1} \\ & \times \sum_{j_1} \left\{ \begin{matrix} i_2 & i_1 & \tilde{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda^{2i_1} \\ & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ \tilde{j}_0 & P_2 & i_2 \end{matrix} \right\} \end{aligned}$$

$$\begin{aligned} & \times \left\{ \begin{matrix} P_5 & i_2 & j_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda^{2i_2} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & i_2 \end{matrix} \right\} \\ & \times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} \tilde{j}_0 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (61) \end{aligned}$$

$$W_{(235)\tilde{j}_1 i_2}^{(5)\tilde{j}_2} = NP_2 P_3 P_5 \sum_{\tilde{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda^{2P_1}$$

$$\begin{aligned} & \times \sum_{j_1} \left\{ \begin{matrix} i_2 & i_1 & \tilde{j}_1 \\ 2 & P_3 & P_3 \end{matrix} \right\} \lambda^{2i_1} \\ & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ \tilde{j}_0 & P_2 & i_2 \end{matrix} \right\} (-1) \\ & \times \left\{ \begin{matrix} P_5 & 2 & j_2 \\ i_2 & P_4 & P_5 \end{matrix} \right\} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & i_2 \end{matrix} \right\} \\ & \times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} \tilde{j}_0 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (62) \end{aligned}$$

$$W_{(245)\tilde{j}_1 i_2}^{(5)\tilde{j}_2} = NP_2 P_4 P_5 \sum_{\tilde{j}_0} \left\{ \begin{matrix} i_1 & P_1 & \tilde{j}_0 \\ 2 & P_2 & P_2 \end{matrix} \right\} \lambda^{2P_1}$$

$$\begin{aligned} & \times \sum_{j_2} \left\{ \begin{matrix} P_5 & i_2 & \tilde{j}_2 \\ 2 & P_4 & P_4 \end{matrix} \right\} \lambda^{2i_2} \\ & \times \sum_{j_1} \left\{ \begin{matrix} \tilde{j}_2 & 2 & \tilde{j}_1 \\ i_1 & P_3 & i_2 \end{matrix} \right\} \\ & \times \sum_{j_0} \left\{ \begin{matrix} \tilde{j}_1 & 2 & \tilde{j}_0 \\ \tilde{j}_0 & P_2 & i_1 \end{matrix} \right\} (-1) \\ & \times \left\{ \begin{matrix} P_5 & 2 & j_2 \\ \tilde{j}_2 & P_4 & P_5 \end{matrix} \right\} \left\{ \begin{matrix} j_2 & 2 & j_1 \\ \tilde{j}_1 & P_3 & \tilde{j}_2 \end{matrix} \right\} \\ & \times \left\{ \begin{matrix} j_1 & 2 & P_1 \\ \tilde{j}_0 & P_2 & \tilde{j}_1 \end{matrix} \right\} \frac{\text{Tet} \left[ \begin{matrix} \tilde{j}_0 & \tilde{j}_0 & P_1 \\ 2 & 2 & 2 \end{matrix} \right]}{\Delta_{P_1}} \quad (63) \end{aligned}$$

### 3.4. 重耦矩阵的一般表式

为了求出  $n$  顶角重耦矩阵的一般表式, 需要知道移动  $M_1$  和  $M_2$  的一般表式, 下面就来讨论这一问题.

首先给出移动  $M_1$  的一般表式. 对于任意三重

组  $(s, t, r)$  中的抓  $P_s$  显然有  $1 \leq s < n-1$ 。由于  $s=1$  时,  $P_s$  上的抓并不需移动, 故有  $P_s$  上的抓被取下的统一表式

$$\Theta \begin{bmatrix} i_{s+1} & i_{s-2} & \bar{j}_{s-2} \\ 2 & P_s & P_s \end{bmatrix} = \begin{cases} 1 & s=1 \\ \sum_{j_{s-2}} \begin{bmatrix} i_{s-1} & i_{s-2} & \bar{j}_{s-2} \\ 2 & P_s & P_s \end{bmatrix} \lambda_{j_{s-2}}^{2i_{s-2}} & 1 < s < n-1. \end{cases} \quad (64)$$

对于  $P_t$  上的抓, 有  $1 < t < n$ , 故有该抓被  $M_1$  取下的统一表式

$$\Theta \begin{bmatrix} i_{t-1} & i_{t-2} & \bar{j}_{t-2} \\ 2 & P_t & P_t \end{bmatrix} = \sum_{j_{t-2}} \begin{bmatrix} i_{t-1} & i_{t-2} & \bar{j}_{t-2} \\ 2 & P_t & P_t \end{bmatrix} \lambda_{j_{t-2}}^{2i_{t-2}} \quad 1 < t < n. \quad (65)$$

$$\Theta \begin{bmatrix} i_{r-1} & i_{r-2} & \bar{j}_{r-2} \\ 2 & P_r & P_r \end{bmatrix} = \begin{cases} \sum_{j_{r-2}} \begin{bmatrix} i_{r-1} & i_{r-2} & \bar{j}_{r-2} \\ 2 & P_r & P_r \end{bmatrix} \lambda_{j_{r-2}}^{2i_{r-2}} & 2 < r < n \\ \lambda_{P_n}^{2P_n} = -1 & r = n. \end{cases} \quad (66)$$

其次, 给出移动  $M_2$  的一般表式。对于  $P_s$  上的抓, 当其被  $M_1$  取下后, 而被  $M_2$  移动时, 有统一表式

$$\prod_{a=2}^{s-1} \begin{bmatrix} \bar{j}_{a-1} & 2 & \bar{j}_{a-2} \\ i_{a-1} & P_a & i_{a-1} \end{bmatrix} = \prod_{a=2}^{s-1} \sum_{j_{a-2}} \begin{bmatrix} \bar{j}_{a-1} & 2 & \bar{j}_{a-2} \\ i_{a-2} & P_a & i_{a-1} \end{bmatrix} \quad s > 2. \quad (67)$$

$$W_{(str)i_1 \dots i_{n-3}}^{\bar{j}_0 \dots \bar{j}_{n-3}} = \left( \frac{\Delta_{i_1} \dots \Delta_{i_{n-3}} \mathcal{H}(P_1, P_2, j_1) \mathcal{H}(j_1, P_3, j_2) \mathcal{H}(j_2, P_4, j_3) \dots \mathcal{H}(j_{n-3}, P_{n-1}, P_n)}{\Delta_{j_1} \dots \Delta_{j_{n-3}} \mathcal{H}(P_1, P_2, i_1) \mathcal{H}(i_1, P_3, i_2) \mathcal{H}(i_2, P_4, i_3) \dots \mathcal{H}(i_{n-3}, P_{n-1}, i_n)} \right)^{1/2} P_s P_t P_r$$

$$\times \Theta \begin{bmatrix} i_{s-1} & i_{s-2} & \bar{j}_{s-2} \\ 2 & P_s & P_s \end{bmatrix} \prod_{a=2}^{s-1} \begin{bmatrix} \bar{j}_{a-1} & 2 & \bar{j}_{a-2} \\ i_{a-2} & P_a & i_{a-1} \end{bmatrix}$$

$$\times \Theta \begin{bmatrix} i_{t-1} & i_{t-2} & \bar{j}_{t-2} \\ 2 & P_t & P_t \end{bmatrix} \prod_{b=2}^{t-1} \begin{bmatrix} \bar{j}_{b-1} & 2 & \bar{j}_{b-2} \\ (ij^-)_{b-2} & P_b & (ij^-)_{b-1} \end{bmatrix}$$

$$\times \Theta \begin{bmatrix} i_{r-1} & i_{r-2} & \bar{j}_{r-2} \\ 2 & P_r & P_r \end{bmatrix} \prod_{c=2}^{r-1} \begin{bmatrix} j_{c-1} & 2 & j_{c-2} \\ (ij^-)_{c-2} & P_c & (ij^-)_{c-1} \end{bmatrix} \left( \prod_{c=r-1}^{n-2} \delta_{i_c}^{j_c} \right) \frac{\text{Tet} \begin{bmatrix} \bar{j}_0 & \bar{j}_0 & P_1 \\ 2 & 2 & 2 \end{bmatrix}}{\Delta_{P_1}}. \quad (70)$$

对于  $P_t$  上的抓, 当其被  $M_1$  取下后, 而被  $M_2$  向左移动时, 有统一表式

$$\prod_{b=2}^{t-1} \begin{bmatrix} \bar{j}_{b-1} & 2 & \bar{j}_{b-2} \\ (ij^-)_{b-2} & P_b & (ij^-)_{b-1} \end{bmatrix} = \prod_{b=2}^{t-1} \sum_{j_{b-2}} \begin{bmatrix} \bar{j}_{b-1} & 2 & \bar{j}_{b-2} \\ (ij^-)_{b-2} & P_b & (ij^-)_{b-1} \end{bmatrix} \quad t > 2, \quad (68)$$

式中

$$(ij^-)_{b-2} = \begin{cases} i_{b-2} & b > s+1, \\ \bar{j}_{b-2} & b \leq s+1, \end{cases}$$

$$(ij^-)_{b-1} = \begin{cases} i_{b-1} & b > s-1, \\ \bar{j}_{b-1} & b \leq s-1. \end{cases}$$

对于  $P_r$  上的抓, 则用于一般重耦矩阵的  $M_2$  的统一表式为

$$\prod_{c=2}^{r-1} \begin{bmatrix} j_{c-1} & 2 & j_{c-2} \\ (ij^-)_{c-2} & P_c & (ij^-)_{c-1} \end{bmatrix} = \prod_{c=2}^{r-1} \begin{bmatrix} j_{c-1} & 2 & j_{c-2} \\ (ij^-)_{c-2} & P_c & (ij^-)_{c-1} \end{bmatrix} \quad r > 2, \quad (69)$$

式中

$$(ij^-)_{c-2} = \begin{cases} i_{c-2} & c > t+1, \\ \bar{j}_{c-2} & c \leq t+1, \end{cases}$$

$$(ij^-)_{c-1} = \begin{cases} i_{c-1} & c > t-1, \\ \bar{j}_{c-1} & c \leq t-1. \end{cases}$$

并且各内腿颜色  $j$  为裸指标。

应用(64)–(69)式, 可求得体积算符  $\hat{V}$  对规范化  $n$  顶角作用的重耦矩阵一般表式为

任何  $n$  顶角,在任何抓三重组  $(s, t, r)$  作用下的重耦矩阵,可用上式直接算出,它是个一般表式(文献[7]中给出的有关结果疏漏过多,无法使用).这种方法得到的重耦矩阵表式,不含有  $9-j$  记号,亦可用(38)式得到完整的体积谱.

#### 4. 抓作用统一表述得到的完备面积谱

将抓作用定义为抓在自旋网腿上的图式计算,为了利用重耦理论公式,须将体积算符中带抓的三角圈  变成图 . 图  在参加抓作用整体计算时,携带了一个 3 顶角(2 2 2)和 3 个具有微长度抓臂的抓.由于它们的存在,可使抓作用利用  $9-j$  记号或移动法完成重耦矩阵的求得.但这些特性都是体积算符本身几何定义中所没有的.

如将抓作用定义为抓在 Wilson 圈上,将不需如上变形所携带的特性参加作用.计算是直接来自微体积和微面积算符的定义,以及圈算符本身.根据是,在几何算符微尺度作用图像中,抓直接接触的首先是圈线,而非整条腿.并且,这可以实现将体、面积算符的作用做统一表述.显然,前种方法将可能给抓作用计算带进附加信息;而后种将是原本的和自然的方法. Penrose 曾指出的就是应当用后种方法完成自旋网在抓作用下的从头到尾的计算<sup>[15]</sup>. Borissor 在计算 Hamilton 约束时也指出,目前尚缺少这种更完整的代数方法的直接表式<sup>[16]</sup>. 按此想法,文献[10]和文献[11],用这种统一的方法处理体、面积算符中的抓作用,用一致的算法各自求得了曲面面积和体积的本征值谱.在求得面积本征值时,考虑抓对

$$\begin{aligned} \hat{A}_e^2 \parallel^p &= 2^{-1} \int_{S_e \otimes S_e} d^2\sigma d^2\tau n_a(\sigma) n_b(\tau) \hat{T}^{ab} [\alpha_{\sigma\tau}](\sigma, \tau) \parallel^p \\ &= 2^{-3} l_0^4 \int_{S_e \otimes S_e} d^2\sigma d^2\tau n_a(\sigma) \Delta^a [\beta_e, \sigma] n_b(\tau) \Delta^b [\beta_e, \tau] [P]^{p-1}, \\ &\quad \boxed{\quad} + (P^2 - P) \parallel^{p-2} \boxed{\quad}, \end{aligned} \quad (74)$$

式中  $l_0^2 = 16\pi\hbar G_{\text{Newton}}/c^3 = 16\pi l_{\text{Planck}}^2$ . 引入文献[10]证明的结果:面积算符的两个抓对腿中(A)(B)两种类型圈线的作用均为本征作用,本征值为 2. 则将面积分归一后(74)式可写成

$$\hat{A}_e^2 \parallel^p = 2^{-3} l_0^4 [2P \parallel^p + 2(P^2 - P) \parallel^p],$$

腿中圈线的作用是(A)(B)两种类型.在求体积本征值时,考虑抓对  $n$  顶角毗邻的所有腿中圈线的作用,并且作用方式有(I)(II)(III)三种.这一面积和体积本征值的求得,进一步得到了更加简明和深入的结果(下面将阐述这一问题).所得的面积本征值谱为<sup>[10]</sup>

$$A(S) = l_0^2 \sum j, \quad (71)$$

这里指出,  $n$  顶角的体积是个顶角不变量,它的完整体积本征值谱为<sup>[11]</sup>

$$V^{(n)} = 2^{-3/2} l_0^3 \sum_v \sqrt{\sum_i G_v^i}, \quad (72)$$

式中,  $G_v^i$  为自旋网  $\Gamma$  中的任意价顶角  $v$  经 3 顶角展开  $i = (i_1, \dots, i_{n-3})$  而得的顶角图中所有圈线上的全部抓法数.对于 3 顶角,有

$$G_{v_3} = P_1 P_2 P_3, \quad (73)$$

式中  $P_1, P_2, P_3$  为三条腿的颜色数.(72)式与被抓住的三条腿是否连续无关,适用于任意  $n$  价顶角,也无需经过重耦矩阵的手续,而是直接得到体积本征值.

本节首先用与文献[7]和[10]不同的,但仍是抓在圈线上的方法,再次求得(71),以确证该结果是个合理的候选者.然后,用同样的手段求出面积本征值的完备谱,再证明(71)式的结果是其特殊情况.从而说明,这种方法将导致进而得到一种完备的面积谱.

#### 4.1. 求面积本征值谱 $A(j) = l_0^2 j$

我们考虑(A)与(B)两种类型抓作用,将有微面积算符  $\hat{A}_e$  对腿态  $\parallel^p$  的作用为(符号参阅文献[7]):

$$\begin{aligned} &= 2^{-2} l_0^4 P^2 \parallel^p, \\ \text{从而有} \quad \hat{A}_e \parallel^p &= 2^{-1} l_0^2 P \parallel^p = l_0^2 j \parallel^p, \\ \text{即有} \quad A(j) &= l_0^2 j, \end{aligned} \quad (75)$$

亦即

$$A(S) = l_0^2 \sum j.$$

(75)式为得到的面积本征值.下面将看到,该式将是更具物理意义的结果的一部分.

## 4.2. 完备的面积谱

熟知,如上的面积算符的定义,并不包含对顶点在微曲面  $S_\epsilon$  上,而"躺在"  $S_\epsilon$  上的腿的作用.但当我们发现,用本节的作法,可以得到面积算符对包含有

这种腿在内的自旋网进行作用,从而得到自旋网对处于任意位形的曲面面积值的完整激发.

为此取 3 顶角 , 令微面积  $S_\epsilon$  交于该 3 顶

角的顶点,并且腿  $r$  躺在  $S_\epsilon$  之上.这样,面积算符对腿  $p, q, r$  均以顶点作为交,从而定义面积算符  $\hat{A}_{S_\epsilon}$  对 3 顶角 

$$\hat{A}_{S_\epsilon}^2 \left| \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle = \left| \zeta \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle + \left| \zeta \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle + \left| \zeta \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle. \quad (76)$$

利用 3 顶角在体积算符作用下的圈线走向展开图 I, II, III, 并按文献 11 实施这里的抓作用的分配,同时,对上式右侧的三个抓均定义为对腿中圈线有 (A) (B) 两种类型作用,通过(74)式,由(76)式可得到

$$\begin{aligned} \hat{A}_{S_\epsilon}^2 \left| \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle &= 2^{-2} 2^{-2} l_0^4 [(i_{pq} + i_{pr})^2 + (i_{pq} + i_{qr})^2 \\ &\quad + 2(i_{pq}^2 + i_{pq}i_{gr} + i_{pq}i_{pr} + i_{pr}i_{gr})] \left| \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle, \end{aligned}$$

式中等号右侧  $2^{-2}$  因子的出现,是由于当  $r = 0$  时 3 顶角应退化为一条腿,即有  $i_{pq} = p = q$ . 为避免(76)式中微小面积片与腿重复相交而重复取面积,须乘以  $2^{-1}$ . 又因此式是面积平方算符的作用,故而导致乘以因子  $2^{-2}$ . 从而得到的完备面积谱由下式给出:

$$\hat{A}_{S_\epsilon} \left| \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle = 2^{-1} 2^{-1} l_0^2 [(i_{pq} + i_{pr})^2 + (i_{pq} + i_{qr})^2 + 2(i_{pq}^2 + i_{pq}i_{gr} + i_{pq}i_{pr} + i_{pr}i_{gr})]^{1/2} \left| \begin{array}{c} p \\ q \\ \hline r \end{array} \right\rangle, \quad (77)$$

易知,当  $r = 0$  时,此时有  $i_{rp} = i_{rq} = 0$ ,  $p = q = i_{pq}$ , 则(77)式变成

$$\begin{aligned} A(j) &= 2^{-1} 2^{-1} l_0^2 \sqrt{(i_{pq}^2 + i_{pq}^2 + 2i_{pq})} \\ &= 2^{-1} l_0^2 p = l_0^2 j \end{aligned}$$

上式即(75)式.从而本文得到的结果是(77)式是一种新的完备面积谱,是通常面积谱(75)式的一种合理扩充.

## 5. Hamilton 约束方程中体积算符对 3 顶角的作用

周知,自旋网仅仅是由与背景无关的量,即自旋形成的代数关系来描述.用自旋网自身的编织,

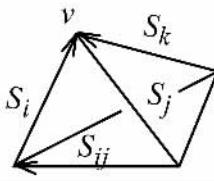
就是 3 维空间.自旋网作为量子态,必须满足 Hamilton 约束(H-约束).自旋网态满足 H-约束方程时,将使其叉生新的 3 顶角和腿,而这是独立于自旋网本身构成的 3 维空间之外的,这一经历便是时间.所以,时空本身是一种动力学组合关系.目前 H-约束存在不止一种形式,但都得到了使顶角演化,曲面在时空中叉分的结果.这是圈量子引力自旋网时空动力学的核心问题之一.

在体积算符直接参加作用的 Thiemann 的 H-约束形式中,存在的欧氏项  $H^E$  对自旋网  $\Gamma$  的作用,已得到了某些结果<sup>[16]</sup>.由于  $H^E$  中存在有体积算符  $\hat{V}$ ,某些文献在计算  $\hat{V}$  对 3 顶角作用时,将其看成是后者的消灭算符,从而使约束  $H^E$  对 3 顶角的某些作

用被视为零.本节将用文献[11]得到的3顶角的体积值不为零的做法,重新执行 $H^E$ 的计算.并证明这一结果为零并非是体积算符消灭所致.为此,我们通过对 $\Sigma$ 进行三角剖分,而得到 $\Gamma$ .

### 5.1. 体积算符中的 holonomy 对3顶角的腿的作用导致 $H^E$ 的两个作用为零

为此将 $\Sigma$ 做以微四面体为胞腔的三角剖分 $T$ ,令得到的自旋网为 $\Gamma$ ,任意胞腔微四面体为 $\Delta$ .将 $\Delta$ 的一顶点 $v$ ,以及与 $v$ 毗邻的 $\Delta$ 的三条棱取为剖分得到的自旋网 $\Gamma$ 的相应顶点,以及与该顶点毗邻的三条腿. $\hat{H}^E$ 定义在 $\Delta$ 上的算符将通过 $\Delta$ 的如下解析图



对 $\Gamma$ 中以 $v$ 为顶点的3顶角( $p, q, r$ )进行作用.该作用不失普遍性,将由如下两种形式的表式定义(去掉 $H^E$ 的上标)<sup>[16]</sup>:

$$\begin{aligned} \hat{H}_\Delta^{(1)} | p, q, r &= \hat{H}_\Delta^{(1)} | p, q, r \\ &= a\epsilon^{ijk} \text{Tr} \left( \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \hat{h}_{s_k} [\hat{h}_{s_k}^{-1}, \hat{V}] \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (78)$$

和

$$\begin{aligned} \hat{H}_\Delta^{(2)} | p, q, r &= \hat{H}_\Delta^{(2)} | p, q, r \\ &= a\epsilon^{ijk} \text{Tr} \left( \hat{h}_{s_k} [\hat{h}_{s_k}^{-1}, \hat{V}] \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (79)$$

式中, $a = \frac{2}{3} \frac{i}{l_0^2}$ , $h$ 为 holonomy, $\alpha_{ij} = s_i s_{ij} s_j^{-1}$ , $\hat{V}$ 为微四面体体积算符.

将上二式中的反称方括号展开,有

$$\hat{H}_\Delta^{(1)} = \hat{H}_\Delta^{(1)}_1 + \hat{H}_\Delta^{(1)}_2, \quad (80)$$

和

$$\hat{H}_\Delta^{(2)} = \hat{H}_\Delta^{(2)}_1 + \hat{H}_\Delta^{(2)}_2, \quad (81)$$

式中

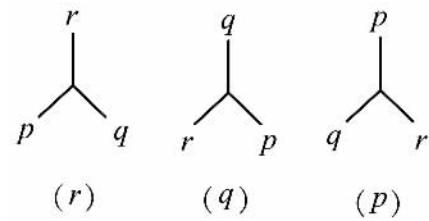
$$\begin{aligned} \hat{H}_\Delta^{(1)}_1 | p, q, r &= \hat{H}_\Delta^{(1)}_1 | pqr \\ &= a\epsilon^{ijk} \text{Tr} \left( \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \hat{h}_{s_k} \hat{V} \hat{h}_{s_k}^{-1} \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (82)$$

$$\begin{aligned} \hat{H}_\Delta^{(1)}_2 | p, q, r &= \hat{H}_\Delta^{(1)}_2 | pqr \\ &= a\epsilon^{ijk} \text{Tr} \left( \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \hat{V} \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (83)$$

$$\begin{aligned} \hat{H}_\Delta^{(2)}_1 | p, q, r &= \hat{H}_\Delta^{(2)}_1 | pqr \\ &= -a\epsilon^{ijk} \text{Tr} \left( \hat{h}_{s_k} \hat{V} \hat{h}_{s_k}^{-1} \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (84)$$

$$\begin{aligned} \hat{H}_\Delta^{(2)}_2 | p, q, r &= \hat{H}_\Delta^{(2)}_2 | pqr \\ &= a\epsilon^{ijk} \text{Tr} \left( \hat{V} \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2} \right) \\ &\quad \times | p, q, r \rangle, \end{aligned} \quad (85)$$

下面以与其他文献不同方式,经计算证明(85)式、(83)式的作用为零.由于每一3顶角 $| p, q, r \rangle$ 对 $\hat{H}_{\Delta_v}$ 的作用有如下三种贡献:



故在(85)式中的反对称 $\alpha$ 圈定义的 $\hat{\beta} = \frac{\hat{h}_{\alpha_{ij}} - \hat{h}_{\alpha_{ji}}}{2}$ 对

$(r)$ 、 $(p)$ 、 $(q)$ 三种位形的作用中,当 $\hat{\beta}$ 与腿 $p, q, r$ 分别合成后<sup>[17]</sup>,进行对 $(r)$ 、 $(p)$ 、 $(q)$ 求和时,这种合成将被消去.

现以腿 $p$ 为例,给予证明. $\hat{\beta}$ 对腿 $p$ ,将通过 holonomy $s_i$ 和 $s_i^{-1}$ 对 $(r)$ 和 $(q)$ 两种位形进行合成.易知,将每一合成的结果求和,可得到如下为零的结果:

$$\hat{\beta} \parallel^p \rangle = \frac{1}{2} \left| \begin{array}{c} p \\ \uparrow \\ S_i \end{array} \right\rangle - \left| \begin{array}{c} p \\ \downarrow \\ S_i^{-1} \end{array} \right\rangle_{(r)} + \frac{1}{2} \left| \begin{array}{c} S_i^{-1} \\ \downarrow \\ p \end{array} \right\rangle - \left| \begin{array}{c} S_i \\ \uparrow \\ p \end{array} \right\rangle_{(q)} = 0 \quad (86)$$

对于腿  $q, r$  也有同样结果, 故得(85)式为 0. 对于(83)式, 因  $\hat{V}$  对  $|p, q, r$  是本征作用, 本征值已由(73)式求出为常数  $pqr$ , 故可同理应用如上证明方法, 有(83)式为 0.

本文得到的结果是(83)与(85)式为 0 与  $\hat{V}$  是否消灭 3 顶角无关, 而是  $\hat{\beta}$  对 3 顶角  $|p, q, r$  的作用中, 对位形求和后, 自行相消的结果. 即该二式表示的 Hamilton 约束欧氏项对 3 顶角作用中, 有下式存在

$$\hat{\beta}|p, q, r = \hat{\beta}(|p, q, r^{(r)} + |p, q, r^{(p)} + |p, q, r^{(q)})$$

$$\begin{aligned} \hat{H}_{\Delta}^{(1)}|p, q, r &= \hat{H}_{\Delta}^{(1)}(|p, q, r^{(r)} + |p, q, r^{(p)} + |p, q, r^{(q)}) \\ &= a\varepsilon^{ijk}(\hat{\beta}\hat{h}_{s_k}^{-1}\hat{V}\hat{h}_{s_k}^{-1}(|p, q, r^{(r)} + |p, q, r^{(p)} + |p, q, r^{(q)})). \end{aligned} \quad (88)$$

### 5.2.1. 计算 $\hat{H}_{\Delta}^{(1)}|p, q, r^{(r)}$

(88)式中  $\hat{h}_{s_k}^{-1}$  与位形  $r$  的合成, 由下式给出

$$\begin{aligned} \hat{h}_{s_k}^{-1} \quad r \\ \text{---} \quad \text{---} \quad \text{---} \\ p \quad q \end{aligned} = \sum_{\varepsilon \pm 1} \alpha_{\varepsilon}(r) \quad \begin{aligned} r + \varepsilon \\ \text{---} \quad \text{---} \quad \text{---} \\ p \quad q \end{aligned} \quad (89)$$

式中  $\alpha_+(r) = 1, \alpha_-(r) = -\frac{r}{r+1}$ . 利用(72)式, 得  $\hat{V}$  对上式的作用为

$$\begin{aligned} \hat{V}\hat{h}_{s_k}^{-1} \quad r \\ \text{---} \quad \text{---} \quad \text{---} \\ p \quad q \end{aligned} = 2^{-\frac{3}{2}}l_0^3 \sum_{\varepsilon \pm 1} \alpha_{\varepsilon}(r) \sqrt{pq(r+\varepsilon)} \quad \begin{aligned} r + \varepsilon \\ \text{---} \quad \text{---} \quad \text{---} \\ p \quad q \end{aligned} \quad (90)$$

上式左侧的图与 holonomy  $\hat{h}_{s_k}^{-1}$  相乘后, 得到右侧的图. 故该图中带有小腿(颜色为 1)的顶角将经受  $\hat{V}$  的作用. 若认为此顶角为 3 顶角, 且认为  $\hat{V}$  对其作用为零, 则将得到该 Hamilton 约束的全部作用皆为零的结果. 为避免此情况发生, 通常文献将此顶角当作 4 顶角<sup>[16]</sup>.  $\hat{V}$  对其作用, 采用对 4 顶角作用充当. 但用认为抓三元组不抓在小腿上的方式, 计算重耦矩阵. 即抓三元组只抓在右图  $\hat{V}_v \left| \begin{smallmatrix} p & \text{---} & \text{---} \\ q & \text{---} & \text{---} \end{smallmatrix} \right. \right\rangle$  中

$$= 0, \quad (87)$$

据上述结果, 我们有

$$\hat{H}_{\Delta}^{(1)} = \hat{H}_{\Delta}^{(1)}_1, \hat{H}_{\Delta}^{(2)} = \hat{H}_{\Delta}^{(2)}_1$$

## 5.2. 用体积算符对 3 顶角作用的本征值直接给出 $H^E$ 的作用系数

由  $\Delta$  提供的  $H^E$  对 3 顶角  $|p, q, r$  的不为零的独立作用, 应是(82)和(84)式.

在(82)式中, 将封闭成圈的图的迹  $\text{Tr}$  去掉后, 可得

$$\begin{aligned} \hat{H}_{\Delta}^{(1)}|p, q, r &= \hat{H}_{\Delta}^{(1)}(|p, q, r^{(r)} + |p, q, r^{(p)} + |p, q, r^{(q)}) \\ &= a\varepsilon^{ijk}(\hat{\beta}\hat{h}_{s_k}^{-1}\hat{V}\hat{h}_{s_k}^{-1}(|p, q, r^{(r)} + |p, q, r^{(p)} + |p, q, r^{(q)})). \end{aligned} \quad (88)$$

的虚线圈的外侧. 然而这与得到这种作法时, 为了使抓作用对应  $9-j$  记号, 必须使小腿(加长)也与其他腿一样参加封闭的处理(见文献 41[7]), 不一致. 本文利用(72)式的结果, 避免了这一种情况(即, 它就是个 3 顶角, 这与  $H$ -约束作用后产生的是 3 顶角又分动力学演化相一致). 并可用来探讨体、面积算符统一描述的后果.

$\hat{\beta}$  对(90)式中右图的作用, 经计算为

$$\begin{aligned}
 \hat{\beta} &= \sum_{\varepsilon=\pm 1} \sum_{\tilde{\varepsilon}=\pm 1} \alpha_\varepsilon(p) \alpha_{\tilde{\varepsilon}}(q) \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } r+\varepsilon, \text{ the bottom edge is } r, \text{ the left edge is } p, \text{ the right edge is } q. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } r. \end{array} \\
 & \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } r+\varepsilon, \text{ the bottom edge is } r, \text{ the left edge is } p, \text{ the right edge is } q. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } r. \end{array} \\
 & \left[ \frac{p}{r} \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } p+\bar{\varepsilon}, \text{ the bottom edge is } 1, \text{ the left edge is } p, \text{ the right edge is } p+\bar{\varepsilon}. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } p+\bar{\varepsilon}. \end{array} \quad \frac{q}{r} \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } q+\tilde{\varepsilon}, \text{ the bottom edge is } 1, \text{ the left edge is } q, \text{ the right edge is } q+\tilde{\varepsilon}. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } q+\tilde{\varepsilon}. \end{array} \right] \quad \frac{p+\bar{\varepsilon}}{r} \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } p+\bar{\varepsilon}, \text{ the bottom edge is } q+\tilde{\varepsilon}, \text{ the left edge is } p+\bar{\varepsilon}, \text{ the right edge is } q+\tilde{\varepsilon}. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } p+\bar{\varepsilon}, q+\tilde{\varepsilon}. \end{array} \\
 & \quad \text{Diagram: } \begin{array}{c} \text{A triangle with vertices } p, q, r. \text{ The top vertex is } r, \text{ the left vertex is } p, \text{ the right vertex is } q. \end{array} \quad (91)
 \end{aligned}$$

最后得到

$$\begin{aligned}
 \hat{H}_{\Delta_v}^{(1)} |p, q, r\rangle^{(r)} &= -\alpha \text{Tr} (\hat{\beta} \hat{h}_{s_k} \hat{V} \hat{h}_{s_k}^{-1}) \quad \text{Diagram: } \begin{array}{c} \text{A triangle with vertices } p, q, r. \text{ The top vertex is } r, \text{ the left vertex is } p, \text{ the right vertex is } q. \end{array} \\
 &= \frac{2il_0}{3} \sum_{\bar{\varepsilon}, \tilde{\varepsilon}=\pm 1} H_{JK_{\bar{\varepsilon}\tilde{\varepsilon}}}^{(1)} (p, \bar{\varepsilon}, q, \tilde{\varepsilon}, r) \\
 & \quad \text{Diagram: } \begin{array}{c} \text{A triangle with vertices } p, q, r. \text{ The top vertex is } r, \text{ the left vertex is } p, \text{ the right vertex is } q. \end{array} \quad (92)
 \end{aligned}$$

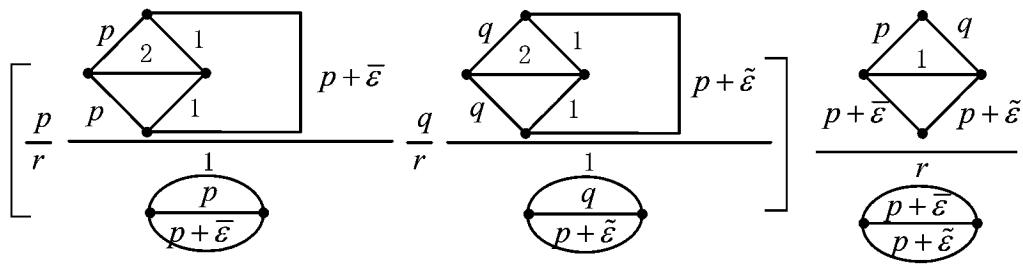
式中, 作用系数

$$H_{JK_{\bar{\varepsilon}\tilde{\varepsilon}}}^{(1)} = \sum_{\varepsilon=\pm 1} 2^{-\frac{3}{2}} \sqrt{pq(r+\varepsilon)} A_1(r, \varepsilon) B_1(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r), \quad (93)$$

这里,  $JK$  为颜色为  $p$  和  $q$  的两只腿组成的二元组, 而

$$A_1(r, \varepsilon) = \alpha_\varepsilon(r) \quad \text{Diagram: } \begin{array}{c} \text{A rectangle with a diamond inside. The top edge is } r+\varepsilon, \text{ the bottom edge is } r, \text{ the left edge is } r, \text{ the right edge is } r. \text{ The diamond has vertices } 1, 2, 1, 2. \\ \text{Below the rectangle is an oval with a horizontal line through it, labeled } r. \end{array} = \varepsilon \frac{r}{r+1}, \quad (94)$$

$$B_1(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r) = \alpha_\varepsilon(p) \alpha_{\tilde{\varepsilon}}(q)$$



$$= \begin{cases} \frac{p - q}{2r}, & \text{若 } \bar{\varepsilon} = +1, \tilde{\varepsilon} = +1, \\ -\frac{(q - p + r)(2 + p + q)}{4r(1 + q)}, & \text{若 } \bar{\varepsilon} = -1, \tilde{\varepsilon} = +1, \\ \frac{(p - q + r)(2 + p + q)}{4r(1 + p)}, & \text{若 } \bar{\varepsilon} = +1, \tilde{\varepsilon} = -1, \\ \frac{(p - q)(p + q - r)(2 + p + q + r)}{8r(1 + p)(1 + q)}, & \text{若 } \bar{\varepsilon} = -11, \tilde{\varepsilon} = -1. \end{cases} \quad (95)$$

令

$$\hat{D}_{wJK_{\tilde{\varepsilon}\tilde{\varepsilon}}} \left| \begin{array}{c} r \\ \diagup \quad \diagdown \\ q \quad p \\ \diagdown \quad \diagup \\ J \end{array} \right\rangle^K = \left| \begin{array}{c} r \\ \diagup \quad \diagdown \\ q + \tilde{\varepsilon} \quad q \\ \diagdown \quad \diagup \\ p + \tilde{\varepsilon} \quad p \\ \diagdown \quad \diagup \\ J \end{array} \right\rangle^K,$$

则(92)式可写成

$$\hat{H}_{\Delta}^{(\pm)} |p, q, r^{(\pm)}\rangle = \frac{2il_0}{3} \sum_{\substack{\epsilon \\ \tilde{\epsilon}}} \hat{H}_{JK_{\epsilon\tilde{\epsilon}}}^{(\pm)} \hat{D}_{vJK_{\epsilon\tilde{\epsilon}}} |p, q, r^{(\pm)}\rangle. \quad (96)$$

### 5.2.2. 计算 $\hat{H}_{\mathbb{A}}^{(2)_1} | p, q, r \rangle^{(r)}$

在(84)式中,有

$$\hat{h}_{s_k}^{-1} \hat{\beta} = \hat{h}_{s_k}^{-1} \alpha_{ij} - \hat{h}_{s_k}^{-1} \alpha_{ji}$$

$$= \sum_{\varepsilon, \tilde{\varepsilon}} \sum_{\tilde{\varepsilon} = \pm 1} \sum_{J=r, r+\varepsilon} \alpha_\varepsilon(r) \begin{Bmatrix} r & r+\varepsilon & J \\ 1 & 2 & 1 \end{Bmatrix} B_2(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r, J) \quad (97)$$


$\hat{h}_{s_i} \hat{V}$ 对(97)式的作用为

$$\begin{aligned}
 & \hat{h}_{s_k} \hat{V} \hat{h}_{s_k}^{-1} \hat{\beta} = 2^{-\frac{3}{2}} l_0^3 \sqrt{(p+\bar{\varepsilon})(q+\tilde{\varepsilon})(r+\varepsilon)} \sum_{\varepsilon, \bar{\varepsilon}, \tilde{\varepsilon}=\pm 1} \sum_{J=r, r+2} \alpha_\varepsilon(r) B'_2(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r, J) \\
 & \text{Diagram:} \\
 & \text{Top: A vertex with three legs labeled } r, p, q. \\
 & \text{Bottom: A rectangle with vertices labeled } r+\varepsilon, 1, r, 2. \text{ The left side is labeled } J. \\
 & \text{Bottom Left: A circle with a horizontal line through the center, labeled } r, J, 2. \\
 & \text{Bottom Right: A circle with a horizontal line through the center, labeled } 1, J, r+\varepsilon. \\
 & \text{Right: A circle with a horizontal line through the center, labeled } 1, r, r+\varepsilon. \\
 & \text{Bottom: A circle with a horizontal line through the center, labeled } r, r, 2. \\
 & \text{Bottom Right: A vertex with three legs labeled } r, p, q. \\
 & \text{.}
 \end{aligned} \tag{98}$$

现在定义作用系数

$$H_{JK_{\bar{\varepsilon}\tilde{\varepsilon}}}^{(2)} = \sum_{\varepsilon=\pm 1} 2^{-3/2} \sqrt{(p+\bar{\varepsilon})(q+\tilde{\varepsilon})(r+\varepsilon)} B'_2(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r, r) \alpha_\varepsilon(r)$$

$$\begin{aligned}
 & \text{Diagram:} \\
 & \text{Top: A rectangle with vertices labeled } r+\varepsilon, 1, r, 2. \\
 & \text{Bottom: A circle with a horizontal line through the center, labeled } r, r, 2. \\
 & \text{.}
 \end{aligned} \tag{99}$$

$$= \sum_{\varepsilon=\pm 1} 2^{-3/2} \sqrt{(p+\bar{\varepsilon})(q+\tilde{\varepsilon})(r+\varepsilon)} A_1(r, \varepsilon) B_1(p, \bar{\varepsilon}; q, \tilde{\varepsilon}; r) .$$

则由(98)式,有

$$\begin{aligned}
 & \hat{H}_\Delta^{(2)_i} |p, q, r\rangle^{(r)} = -a \text{Tr} (\hat{h}_{s_k} \hat{V} \hat{h}_{s_k}^{-1} \hat{\beta}) \\
 & \text{Diagram:} \\
 & \text{Top: A vertex with three legs labeled } r, p, q. \\
 & \text{Bottom: A circle with a horizontal line through the center, labeled } r, r, 2. \\
 & \text{.} \\
 & = \frac{2il_0}{3} \sum_{\bar{\varepsilon}, \tilde{\varepsilon}=\pm 1} H_{JK_{\bar{\varepsilon}\tilde{\varepsilon}}}^{(2)} \hat{D}_{wJK_{\bar{\varepsilon}\tilde{\varepsilon}}} |p, q, r\rangle^{(r)}
 \end{aligned} \tag{100}$$

总括如上结果,得到微四面体  $\Delta$  上定义的  $H^E$  对  $\Delta$  顶角和棱构成的自旋网的总作用为

$$\hat{H}_\Delta^{(i)} |p, q, r\rangle^{(r)} = \frac{2il_0}{3} \sum_{JK \in \mathcal{C}_v} \sum_{\bar{\varepsilon}, \tilde{\varepsilon}=\pm 1} [H_{JK_{\bar{\varepsilon}\tilde{\varepsilon}}}^{(i)} \hat{D}_{wJK_{\bar{\varepsilon}\tilde{\varepsilon}}} |p, q, r\rangle^{(r)}] \quad (i=1, 2) \tag{101}$$

式中,  $\mathcal{C}_v$  为毗邻顶角  $v$  的腿二元组的集合.

由(93)和(99)式知,作用系数  $H^1$  和  $H^2$  的区别,仅在于  $\Delta$  贡献的 3 顶角体积值的区别.由于和文献[16]不同,本文体积值不需通过重耦矩阵的间

接渠道,从而避免了相应计算由于无法得到数值结果而中断;而是本文的体积值直接出现在(96)和(100)式之中,故该结果更便于用来研究四面体  $\Delta$  中

的体积的颜色估值,3 顶角力学演化生成的曲面  
又分,以及自旋泡沫跃迁间的关系的直接计算.

## 6. 小结

在抓三重组是作用在连续分布的三条腿的情况下(1)式给出的是体积算符对任意  $n$  顶角作用的重耦矩阵的更一般表式. 当抓三重组对毗邻  $n$  顶角的腿的作用可连续亦可不连续分布时(1)式不能直接使用. 本文通过化简得到的重耦矩阵表式,可方便地用于任意抓三重组对任意价顶角作用的计算. 同时,抓的移动法也是种用于此种情况下的计算方式,算出的重耦矩阵(70)式是精确可用的. 这两种方法的作用与文献[7]不同,可通过本文给出的(38)式,求得  $n$  顶角贡献的完整体积谱.

对于 Hamilton 约束的作用,本文除用不同方法证明了  $H^E$  的 4 种作用中有两种为零外,还避免了

所得的作用系数表式  $H_v^{(i)}$  中仍存在(不是最后结果的)重耦矩阵,仍需视具体情况进一步求体积本征值,才能达到最后结果的状况,而使(95)(99)式成为 Thiemann 的 Hamilton 约束欧氏项  $H^E$  作用的本身不需再计算的最后结果,并便于直接应用于其他场核.

面积和体积本征值,在一些文献中常被提及与使用,而本文利用的这两种本征值谱,其作用机制是统一的,且具有直接可得到结果的优点. 特别是对于体积本征值,不需经过重耦矩阵→其本征值→它们贡献的体积值的途径求得. 也与抓三重组抓住的三条腿是否连续无关. 由于只利用 Poisson 括号与双元恒等式,而不需利用图式计算,同时已证明它是 3 顶角展开不变的,故这种方法带来的是更加贴近空间量子化的原本信息. 而将已有面积谱(75)式作为特例的完备面积谱(77)式的求得,是如上这种统一描述几何算符作用的方法带来的更有启发意义的新结果.

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# Evaluation of eigenactions of area and volume operators in loop quantum gravity<sup>\*</sup>

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## Abstract

Using a method and its results which deals with the geometrical operators consistently, a complete area spectrum is obtained. At the same time, a different interpretation and result is given for the action of the Euclidean term of Thiemann's Hamiltonian constraint. In the graphical scheme of the grasping action, using the simplification of any triple of grasping actions of the volume operator, the associated recoupling matrices are demonstrated. Employing the movements of grasps, a complete and exact general expression of recoupling matrix for any triple of grasp actings on any valent vertex is detailed.

**Keywords**: consistent description of actions of area and volume operators, complete area-spectrum, simplification of grasping actions, general expressions of recoupling matrices

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