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具有大范围运动和非线性变形的空间柔性梁的精确动力学建模*

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研究具有大范围运动和非线性变形的空间柔性梁的有限元动力学建模. 首先在精确描述空间柔性梁的非线性变形的基础上,采用有限元方法对梁结构进行离散,导出其动能、势能及外力对应的广义力,然后利用 Lagrange 方程建立了空间柔性梁的精确动力学方程. 该方程在原有一次耦合模型的基础上,增加了新的表征纵向、横向、侧向弯曲变形,以及扭转变形的耦合项,同时包含了变形运动与大范围运动之间的相互耦合项. 本建模方法和所得结论可为以后空间柔性梁的动力学特性分析作以参考.

关键词: 大范围运动, 非线性变形, 空间柔性梁, 精确动力学建模

PACC: 0320, 0547, 4610

1. 引 言

对于具有大范围运动和非线性变形的空间柔性梁,建立其刚-柔耦合系统动力学方程是进行其动力学研究的重要步骤[1—4]. 基于不同的变形模式,产生了不计大范围运动与变形运动相互耦合作用的运动弹性动力分析法(KED 法)、对耦合作用处理简单的零次近似模型、采用附加刚度或考虑变形耦合作用的一次耦合模型. 但无论何种模型,为了既合理描述柔性构件的变形,又使计算简单便于应用,在建模过程中都忽略了对系统影响较小的部分,因而所考虑的耦合因素有所欠缺[5,6]. 特别是现有的一些建模方法,直接套用结构动力学中的静边界条件,其计算精度是值得怀疑的[7,8],只有当选取的模态足够多时精度才会明显提高. 为了提高求解精度[9—11],本文将对具有大范围运动和非线性变形的空间柔性梁的有限元动力学建模方法进行研究,得到一套工程上可用的刚-柔耦合系统的精确动力学模型. 该模型在原有一次耦合模型的基础上,增加了新的表征纵向、横向、侧向弯曲变形,以及扭转变形的耦合项,同时包含了变形运动与大范围运动之间的相互耦合项.

2. 空间柔性梁的有限元离散

利用有限元方法将柔性梁离散,先将梁划分为 n 个单元, l_e 为单元长度,单元示意如图 1 所示, e^e 为单元坐标系. 设 $\boldsymbol{N}_{e,1}, \boldsymbol{N}_{e,2}, \boldsymbol{N}_{e,3}, \boldsymbol{N}_{e,4}$ 为第 e 个单元的形函数阵,该单元内任一点 a 对应的位移量依次为 s_0, v_0, w_0, φ,即轴向伸缩变形、横向、侧向弯曲变形以及扭转变形,则

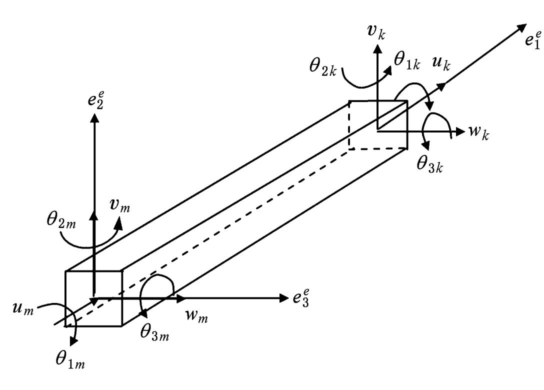

图 1 空间柔性梁单元示意图

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$$
\begin{aligned}
s_0(\bar{x},t) &= \boldsymbol{N}_{e,1}(\bar{x})\boldsymbol{q}_{e,f}(t),\\
v_0(\bar{x},t) &= \boldsymbol{N}_{e,2}(\bar{x})\boldsymbol{q}_{e,f}(t),\\
w_0(\bar{x},t) &= \boldsymbol{N}_{e,3}(\bar{x})\boldsymbol{q}_{e,f}(t),\\
\varphi(\bar{x},t) &= \boldsymbol{N}_{e,4}(\bar{x})\boldsymbol{q}_{e,f}(t),
\end{aligned}\tag{1}
$$

其中 $\bar{x}$ 为单元坐标系 e^e 下的纵坐标，$\boldsymbol{q}_{e,f}(t)$ 为单元节点的变形位移列阵. 设单元的两个节点为 m,k，则有

$$\boldsymbol{N}_{e,1}(\bar{x}) = [N_{ms}\quad 0\quad 0\quad 0\quad 0\quad 0\quad N_{ks}\quad 0\quad 0\quad 0\quad 0\quad 0],\tag{2}$$

$$\boldsymbol{N}_{e,2}(\bar{x}) = [0\quad N_{mv}\quad 0\quad 0\quad 0\quad N_{m\theta3}\quad 0\quad N_{kv}\quad 0\quad 0\quad 0\quad N_{k\theta3}],\tag{3}$$

$$\boldsymbol{N}_{e,3}(\bar{x}) = [0\quad 0\quad N_{mw}\quad 0\quad N_{m\theta2}\quad 0\quad 0\quad 0\quad N_{kw}\quad 0\quad N_{k\theta2}\quad 0],\tag{4}$$

$$\boldsymbol{N}_{e,4}(\bar{x}) = [0\quad 0\quad 0\quad N_{m\theta1}\quad 0\quad 0\quad 0\quad 0\quad 0\quad N_{k\theta1}\quad 0\quad 0],\tag{5}$$

$$\boldsymbol{q}_{e,f}(t) = [u_m\quad v_m\quad w_m\quad \theta_{1m}\quad \theta_{2m}\quad \theta_{3m}\quad u_k\quad v_k\quad m_k\quad \theta_{1k}\quad \theta_{2k}\quad \theta_{3k}]^{\mathrm{T}},\tag{6}$$

设 $\boldsymbol{q}_{f0}(t)$ 为总体变形位移列阵

$$\boldsymbol{q}_{e,f}(t) = \boldsymbol{B}_e\boldsymbol{q}_{f0}(t),\tag{7}$$

其中，$\boldsymbol{B}_e$ 为由单元编号决定的布尔(Boole)指示矩阵[12—14]

$$
\begin{matrix} \text{No.} & 1 & 2 & \cdots & m & \cdots & k & \cdots & n+1 \end{matrix}
$$

$$\boldsymbol{B}_e = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{I}_{6\times6} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{I}_{6\times6} & \cdots & \boldsymbol{0}\end{bmatrix}_{12\times6(n+1)}.\tag{8}$$

为表述方便，引入转化矩阵 $\boldsymbol{R}$[12]，$\boldsymbol{q}_{f0}(t)$ 可表示为

$$\boldsymbol{q}_{f0}(t) = \boldsymbol{R}\boldsymbol{q}_f(t),\tag{9}$$

$\boldsymbol{q}_f(t)$ 为独立的总体变形位移列阵，转化矩阵 $\boldsymbol{R}$ 根据边界条件确定. 例如对于两端固定结构，$\boldsymbol{R},\boldsymbol{q}_f(t)$ 分别为

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{0}_{6\times6(n-1)}\\ \boldsymbol{I}_{6(n-1)\times6(n-1)}\\ \boldsymbol{0}_{6\times6(n-1)}\end{bmatrix},\tag{10}$$

$$\boldsymbol{q}_f(t) = [u_2\quad v_2\quad w_2\quad \theta_{12}\quad \theta_{22}\quad \theta_{32}\quad \cdots\quad u_n\quad v_n\quad w_n\quad \theta_{1n}\quad \theta_{2n}\quad \theta_{3n}\quad]^{\mathrm{T}}.\tag{11}$$

将(9)，(7)式代入(1)式得

$$s_0(\bar{x},t) = \boldsymbol{N}_1(\bar{x})\boldsymbol{q}_f(t),v_0(\bar{x},t) = \boldsymbol{N}_2(\bar{x})\boldsymbol{q}_f(t),w_0(\bar{x},t) = \boldsymbol{N}_3(\bar{x})\boldsymbol{q}_f(t),\varphi(\bar{x},t) = \boldsymbol{N}_4(\bar{x})\boldsymbol{q}_f(t).\tag{12}$$

这样，柔性梁上任一点 a 在连体坐标系 e^1 下的变形位移列阵为

$$\boldsymbol{u}_{af} = \begin{bmatrix} u\\ v\\ w\end{bmatrix} = \begin{bmatrix} \boldsymbol{N}_1\boldsymbol{q}_f - y\dfrac{\partial\boldsymbol{N}_2}{\partial\bar{x}}\boldsymbol{q}_f - z\dfrac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}\boldsymbol{q}_f^{\mathrm{T}}H\boldsymbol{q}_f + \underline{y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_1^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_2}{\partial\bar{x}}\boldsymbol{q}_f - y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_3^{\mathrm{T}}}{\partial\bar{x}}\boldsymbol{N}_4\boldsymbol{q}_f}\\ \underline{+ z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_1^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\boldsymbol{q}_f + z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\boldsymbol{N}_4\boldsymbol{q}_f}\\ \boldsymbol{N}_2\boldsymbol{q}_f \underline{- \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_2}{\partial\bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4\boldsymbol{q}_f - z\boldsymbol{N}_4\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\boldsymbol{q}_f}\\ \boldsymbol{N}_3\boldsymbol{q}_f \underline{+ y\boldsymbol{N}_4\boldsymbol{q}_f - \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial\boldsymbol{N}_3^{\mathrm{T}}}{\partial\bar{x}}\dfrac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4\boldsymbol{q}_f}\end{bmatrix}.\tag{13}$$

与一次耦合模型相比增加了划线项，与零次耦合模型相比增加了含 H 的耦合形函数阵. 这里划线项意义相同. 其中耦合形函数为[12,13]

$$
\begin{aligned}
\boldsymbol{H} = {}& \boldsymbol{R}^{\mathrm{T}}B_e^{\mathrm{T}}\int_0^{\bar{x}}\left(\frac{\partial\boldsymbol{N}_{e,2}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,2}}{\partial\bar{x}} + \frac{\partial\boldsymbol{N}_{e,3}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,3}}{\partial\bar{x}}\right)\mathrm{d}\bar{x}\boldsymbol{B}_e\boldsymbol{R}\\
& + \sum_{i=1}^{e-1}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{B}_i^{\mathrm{T}}\int_0^{l_i}\left(\frac{\partial\boldsymbol{N}_{i,2}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{i,2}}{\partial\bar{x}} + \frac{\partial\boldsymbol{N}_{i,3}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{i,3}}{\partial\bar{x}}\right)\mathrm{d}\bar{x}\boldsymbol{B}_i\boldsymbol{R}.
\end{aligned}
$$

3. 空间柔性梁的离散动力学方程

3.1. 系统动能

梁非中线上任一点 a 的位置矢量在连体坐标系 e^1 下的坐标列阵为

$$
\boldsymbol{u}_a = \boldsymbol{u}_{a0} + \boldsymbol{u}_{af} = [\,u_{a1} \quad u_{a2} \quad u_{a3}\,]^{\mathrm{T}}
$$

$$
= \begin{bmatrix}
x + s_0 - \dfrac{1}{2}\displaystyle\int_0^x \left(\left(\frac{\partial v_0}{\partial x} \right)^2 + \left(\frac{\partial w_0}{\partial x} \right)^2 \right) \mathrm{d}x + \left(- \dfrac{\partial v_0}{\partial x} + \underline{\dfrac{\partial s_0}{\partial x}\dfrac{\partial v_0}{\partial x} - \dfrac{\partial w_0}{\partial x}\varphi} \right) y + \left(- \dfrac{\partial w_0}{\partial x} + \underline{\dfrac{\partial s_0}{\partial x}\dfrac{\partial w_0}{\partial x} + \dfrac{\partial v_0}{\partial x}\varphi} \right) z \\
y + v_0 + \underline{\left(- \dfrac{1}{2}\left(\dfrac{\partial v_0}{\partial x} \right)^2 - \dfrac{1}{2}\varphi^2 \right) y + \left(- \varphi - \dfrac{1}{2}\dfrac{\partial v_0}{\partial x}\dfrac{\partial w_0}{\partial x} \right) z} \\
z + w_0 + \underline{\left(\varphi - \dfrac{1}{2}\dfrac{\partial v_0}{\partial x}\dfrac{\partial w_0}{\partial x} \right) y + \left(- \dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x} \right)^2 - \dfrac{1}{2}\varphi^2 \right) z}
\end{bmatrix}. \tag{14}
$$

利用有限元离散化后,得

$$
\boldsymbol{u}_a = [\,u_{a1} \quad u_{a2} \quad u_{a3}\,]^{\mathrm{T}}
$$

$$
= \begin{bmatrix}
x + \boldsymbol{N}_1\boldsymbol{q}_f - y\dfrac{\partial \boldsymbol{N}_2}{\partial \bar{x}}\boldsymbol{q}_f - z\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}\boldsymbol{q}_f^{\mathrm{T}}H\boldsymbol{q}_f \underline{+ y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_1^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_2}{\partial \bar{x}}\boldsymbol{q}_f - y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_3^{\mathrm{T}}}{\partial \bar{x}}\boldsymbol{N}_4\boldsymbol{q}_f} \\
\underline{+ z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_1^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}}\boldsymbol{q}_f + z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_2^{\mathrm{T}}}{\partial \bar{x}}\boldsymbol{N}_4\boldsymbol{q}_f} \\
y + \boldsymbol{N}_2\boldsymbol{q}_f \underline{- \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_2^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_2}{\partial \bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4\boldsymbol{q}_f - z\boldsymbol{N}_4\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_2^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}}\boldsymbol{q}_f} \\
z + \boldsymbol{N}_3\boldsymbol{q}_f \underline{+ y\boldsymbol{N}_4\boldsymbol{q}_f - \dfrac{1}{2}y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_2^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_3^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}}\boldsymbol{q}_f - \dfrac{1}{2}z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4\boldsymbol{q}_f}
\end{bmatrix}. \tag{15}
$$

利用有限元离散化后,得到任一点 a 的位置矢量在惯性系 e^0 下为

$$
\dot{\boldsymbol{r}}_a = \dot{\boldsymbol{r}}_0 - \boldsymbol{A}\tilde{\boldsymbol{u}}_a\boldsymbol{\omega} + \boldsymbol{A}\dot{\boldsymbol{u}}_{af}
$$

$$
= \dot{\boldsymbol{r}}_0 - \boldsymbol{A}\tilde{\boldsymbol{u}}_a\boldsymbol{\omega} + \boldsymbol{A}\begin{bmatrix}
\boldsymbol{N}_1 - y\dfrac{\partial \boldsymbol{N}_2}{\partial \bar{x}} - z\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}} - \boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{H} + y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{12} - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{t34} \\
+ z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{13} + z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{t24} \\
\boldsymbol{N}_2 - z\boldsymbol{N}_4 - y\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_2^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_2}{\partial \bar{x}} - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4 - z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{23} \\
\boldsymbol{N}_3 + y\boldsymbol{N}_4 - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{23} - z\boldsymbol{q}_f^{\mathrm{T}}\dfrac{\partial \boldsymbol{N}_3^{\mathrm{T}}}{\partial \bar{x}}\dfrac{\partial \boldsymbol{N}_3}{\partial \bar{x}} - z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4
\end{bmatrix}\dot{\boldsymbol{q}}_f, \tag{16}
$$

其中

$$
\boldsymbol{V}_{ml} = \left(\frac{\partial \boldsymbol{N}_m^{\mathrm{T}}}{\partial \bar{x}}\frac{\partial \boldsymbol{N}_l}{\partial \bar{x}} \right)^{\mathrm{T}} + \frac{\partial \boldsymbol{N}_m^{\mathrm{T}}}{\partial \bar{x}}\frac{\partial \boldsymbol{N}_l}{\partial \bar{x}}, \boldsymbol{V}_{tml} = \left(\frac{\partial \boldsymbol{N}_m^{\mathrm{T}}}{\partial \bar{x}}\boldsymbol{N}_l \right)^{\mathrm{T}} + \frac{\partial \boldsymbol{N}_m^{\mathrm{T}}}{\partial \bar{x}}\boldsymbol{N}_l, m, l = 1, \cdots, 4.
$$

柔性体动能为

$$
T = T_f = \frac{1}{2}\int_V \rho \dot{\boldsymbol{r}}_a^{\mathrm{T}}\dot{\boldsymbol{r}}_a \mathrm{d}V, \tag{17}
$$

其中,ρ 为密度,V 为体积,将(16)式代入(17)式,略去变形位移及其偏导数的三次以上项,为便于与一次、零次耦合模型相比较,采用与文献[12]相似的符号记法,得

$$T=\frac{1}{2}\begin{bmatrix}\dot{\boldsymbol{r}}_0^{\mathrm{T}} & \boldsymbol{\omega}^{\mathrm{T}} & \dot{\boldsymbol{q}}_f^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13}\\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23}\\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{r}}_0\\ \boldsymbol{\omega}\\ \dot{\boldsymbol{q}}_f\end{bmatrix}. \tag{18}$$

设 A,l 分别为梁的横截面积和长度,并考虑到坐标轴为对称轴,广义质量阵各分块的表达式为

$$\boldsymbol{M}_{11}=\rho Al\boldsymbol{I}_{3\times 3}, \tag{19}$$

$$\boldsymbol{M}_{12}=\boldsymbol{M}_{21}^{\mathrm{T}}=-A\tilde{\boldsymbol{s}}, \tag{20}$$

$$\boldsymbol{s}=[s_1 \quad s_2 \quad s_3]^{\mathrm{T}},$$

$$s_1=E_1+\boldsymbol{Y}_1\boldsymbol{q}_f-\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{C}\boldsymbol{q}_f/2,$$

$$s_2=\boldsymbol{Y}_2\boldsymbol{q}_f,\ s_3=\boldsymbol{Y}_3\boldsymbol{q}_f,$$

$$\boldsymbol{M}_{13}=\boldsymbol{M}_{31}^{\mathrm{T}}=A\begin{bmatrix}\boldsymbol{Y}_1-\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{C}\\ \boldsymbol{Y}_2\\ \boldsymbol{Y}_3\end{bmatrix}, \tag{21}$$

$$\boldsymbol{M}_{22}=\begin{bmatrix}d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{32} & d_{33}\end{bmatrix}, \tag{22}$$

式中

$$d_{11}=J_{22}+J_{33}+\boldsymbol{q}_f^{\mathrm{T}}(\boldsymbol{W}_{22}+\boldsymbol{W}_{33})\boldsymbol{q}_f\underline{-\boldsymbol{q}_f^{\mathrm{T}}(\overline{\boldsymbol{W}}_{22}+\overline{\boldsymbol{W}}_{33})\boldsymbol{q}_f},$$

$$d_{12}=d_{21}=\overline{\boldsymbol{Y}}_2\boldsymbol{q}_f-\boldsymbol{Z}_{12}\boldsymbol{q}_f-\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{W}_{21}\boldsymbol{q}_f\underline{-\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{02,12}\boldsymbol{q}_f+\boldsymbol{q}_f^{\mathrm{T}}(\boldsymbol{W}_{2,34}+\boldsymbol{W}_{3,43})\boldsymbol{q}_f},$$

……

$$\boldsymbol{M}_{23}=\boldsymbol{M}_{32}^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{g}_1\\ \boldsymbol{g}_2\\ \boldsymbol{g}_3\end{bmatrix}, \tag{23}$$

$$\boldsymbol{g}_1=\boldsymbol{q}_f^{\mathrm{T}}(\boldsymbol{W}_{23}-\boldsymbol{W}_{32})\underline{+\overline{\overline{\boldsymbol{Y}}}_{34}+\overline{\overline{\boldsymbol{Y}}}_{24}+\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{3,23}-\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{2,23}},$$

$$\boldsymbol{g}_2=-\boldsymbol{q}_f^{\mathrm{T}}(\boldsymbol{W}_{13}-\boldsymbol{W}_{31})-\overline{\boldsymbol{Y}}_3-\boldsymbol{Z}_{13}\underline{+\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{3,13}+\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{t3,24}-\boldsymbol{q}_f^{\mathrm{T}}(\hat{\boldsymbol{W}}_{2,42}-\hat{\boldsymbol{W}}_{2,24})},$$

$$\boldsymbol{g}_3=-\boldsymbol{q}_f^{\mathrm{T}}(\boldsymbol{W}_{21}-\boldsymbol{W}_{12})+\boldsymbol{Z}_{12}+\overline{\boldsymbol{Y}}_2\underline{-\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{2,12}+\boldsymbol{q}_f^{\mathrm{T}}\overline{\boldsymbol{W}}_{t2,34}-\boldsymbol{q}_f^{\mathrm{T}}(\hat{\boldsymbol{W}}_{3,43}-\hat{\boldsymbol{W}}_{3,34})},$$

$$\boldsymbol{M}_{33}=\boldsymbol{W}_{11}+\boldsymbol{W}_{22}+\boldsymbol{W}_{33}+\overline{\boldsymbol{W}}_{22}+\overline{\boldsymbol{W}}_{33}\underline{+\boldsymbol{W}_{44}+\overline{\overline{\boldsymbol{W}}}_{34}+\overline{\overline{\boldsymbol{W}}}_{24}}, \tag{24}$$

其中

$$E_1=\int_V\rho x\mathrm{d}V=\rho Al^2/2,$$

$$J_{11}=\int_V\rho x^2\mathrm{d}V=\rho Al^3/3,\ J_{22}=\int_V\rho y^2\mathrm{d}V=\rho lI_{zz},\ J_{33}=\int_V\rho z^2\mathrm{d}V=\rho lI_{yy},$$

I_{yy},I_{zz} 为横截面的惯性矩 $I_{yy}=\int_A z^2\mathrm{d}A,\ I_{zz}=\int_A y^2\mathrm{d}A$,

$$\boldsymbol{W}_{m,k}=\int_V\rho\boldsymbol{N}_m^{\mathrm{T}}\boldsymbol{N}_k\mathrm{d}V=\boldsymbol{R}^{\mathrm{T}}\sum_{e=1}^{n}\boldsymbol{B}_e^{\mathrm{T}}\int_0^{l_e}\rho A\boldsymbol{N}_{e,m}^{\mathrm{T}}\boldsymbol{N}_{e,k}\mathrm{d}\bar{x}\boldsymbol{B}_e\boldsymbol{R},\ m,k=1,2,3,4,$$

$$\overline{\boldsymbol{W}}_{22}=\int_V\rho y^2\frac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_2}{\partial\bar{x}}\mathrm{d}V=\boldsymbol{R}^{\mathrm{T}}\sum_{e=1}^{n}\boldsymbol{B}_e^{\mathrm{T}}\int_0^{l_e}\rho I_{zz}\frac{\partial\boldsymbol{N}_{e,2}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,2}}{\partial\bar{x}}\mathrm{d}\bar{x}\boldsymbol{B}_e\boldsymbol{R},$$

$$\overline{W}_{33}=\int_V\rho z^2\frac{\partial\boldsymbol{N}_3^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_3}{\partial\bar{x}}\mathrm{d}V=\boldsymbol{R}^{\mathrm{T}}\sum_{e=1}^{n}\boldsymbol{B}_e^{\mathrm{T}}\int_0^{l_e}\rho I_{yy}\frac{\partial\boldsymbol{N}_{e,3}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,3}}{\partial\bar{x}}\mathrm{d}\bar{x}\boldsymbol{B}_e\boldsymbol{R},$$

$$\overline{W}_{02,23}=\int_V\rho y^2\frac{\partial N_2^{\mathrm T}}{\partial\bar x}\frac{\partial N_3}{\partial\bar x}\mathrm dV=R^{\mathrm T}\sum_{e=1}^n B_e^{\mathrm T}\int_0^{l_e}I_{zz}\frac{\partial N_{e,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,3}}{\partial\bar x}\mathrm d\bar xB_eR,$$

$$\overline{W}_{3,23}=\int_V\rho z^2V_{23}\mathrm dV=R^{\mathrm T}\sum_{e=1}^n B_e^{\mathrm T}\int_0^{l_e}I_{yy}\left(\left(\frac{\partial N_{e,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,3}}{\partial\bar x}\right)^{\mathrm T}+\frac{\partial N_{e,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,3}}{\partial\bar x}\right)\mathrm d\bar xB_eR,$$

……

$$\overline{\overline{Y}}_{24}=\int_V\rho y^2N_4\mathrm dV=\sum_{e=1}^n\int_0^{l_e}\rho I_{zz}N_{e,4}\mathrm d\bar xB_eR,$$

$$C=\int_V\rho H\mathrm dV=\sum_{e=1}^n\int_0^{l_e}\rho AH\mathrm d\bar x=R^{\mathrm T}\sum_{e=1}^n B_e^{\mathrm T}\int_0^{l_e}\int_0^{\bar x}\rho A\left(\frac{\partial N_{e,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,2}}{\partial\bar x}+\frac{\partial N_{e,3}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,3}}{\partial\bar x}\right)\mathrm d\bar x\mathrm d\bar xB_eR$$
$$+R^{\mathrm T}\sum_{e=1}^n\sum_{i=1}^{e-1}B_{\mathrm i}^{\mathrm T}\int_0^{l_e}\int_0^{l_i}\rho A\left(\frac{\partial N_{i,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{i,2}}{\partial\bar x}+\frac{\partial N_{i,3}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{i,3}}{\partial\bar x}\right)\mathrm d\bar x\mathrm d\bar xB_iR,$$

$$D=\int_V\rho xH\mathrm dV=\sum_{e=1}^n\int_0^{l_e}\rho A(x_e+\bar x)H\mathrm d\bar x$$
$$=R^{\mathrm T}\sum_{e=1}^n B_e^{\mathrm T}\int_0^{l_e}(x_e+\bar x)\int_0^{\bar x}\rho A\left(\frac{\partial N_{e,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,2}}{\partial\bar x}+\frac{\partial N_{e,3}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{e,3}}{\partial\bar x}\right)\mathrm d\bar x\mathrm d\bar xB_eR$$
$$+R^{\mathrm T}\sum_{e=1}^n\sum_{i=1}^{e-1}B_i^{\mathrm T}\int_0^{l_e}(x_e+\bar x)\int_0^{l_i}\rho A\left(\frac{\partial N_{i,2}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{i,2}}{\partial\bar x}+\frac{\partial N_{i,3}^{\mathrm T}}{\partial\bar x}\cdot\frac{\partial N_{i,3}}{\partial\bar x}\right)\mathrm d\bar x\mathrm d\bar xB_iR.$$

3.2. 系统势能

由变形位移-应变的非线性关系，利用 Cauchy-Green 应变张量形式，柔性梁上任一点 a 的正应变为

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}+\frac12\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right),\tag{25}$$

从而可得

$$\begin{aligned}\varepsilon_{xx}=&\frac{\partial s_0}{\partial x}-\frac12\left(\frac{\partial v_0}{\partial x}\right)^2-\frac12\left(\frac{\partial w_0}{\partial x}\right)^2-\frac{\partial^2v_0}{\partial x^2}y+\frac{\partial^2s_0}{\partial x^2}\cdot\frac{\partial v_0}{\partial x}y+\frac{\partial s_0}{\partial x}\cdot\frac{\partial^2v_0}{\partial x^2}y-\frac{\partial^2w_0}{\partial x^2}\varphi y-\frac{\partial w_0}{\partial x}\cdot\frac{\partial\varphi}{\partial x}y\\&-\frac{\partial^2w_0}{\partial x^2}z+\frac{\partial^2s_0}{\partial x^2}\cdot\frac{\partial w_0}{\partial x}z+\frac{\partial s_0}{\partial x}\cdot\frac{\partial^2w_0}{\partial x^2}z+\frac{\partial^2v_0}{\partial x^2}\varphi z+\frac{\partial v_0}{\partial x}\cdot\frac{\partial\varphi}{\partial x}z\\&+\frac12\left(\frac{\partial s_0}{\partial x}-\frac12\left(\frac{\partial v_0}{\partial x}\right)^2-\frac12\left(\frac{\partial w_0}{\partial x}\right)^2-\frac{\partial^2v_0}{\partial x^2}y+\frac{\partial^2s_0}{\partial x^2}\cdot\frac{\partial v_0}{\partial x}y+\frac{\partial s_0}{\partial x}\cdot\frac{\partial^2v_0}{\partial x^2}y-\frac{\partial^2w_0}{\partial x^2}\varphi y-\frac{\partial w_0}{\partial x}\cdot\frac{\partial\varphi}{\partial x}y\right.\\&\left.-\frac{\partial^2w_0}{\partial x^2}z+\frac{\partial^2s_0}{\partial x^2}\cdot\frac{\partial w_0}{\partial x}z+\frac{\partial s_0}{\partial x}\cdot\frac{\partial^2w_0}{\partial x^2}z+\frac{\partial^2v_0}{\partial x^2}\varphi z+\frac{\partial v_0}{\partial x}\cdot\frac{\partial\varphi}{\partial x}z\right)^2\\&+\frac12\left(\frac{\partial v_0}{\partial x}+\left(-\frac{\partial v_0}{\partial x}\cdot\frac{\partial^2v_0}{\partial x^2}-\varphi\frac{\partial\varphi}{\partial x}\right)y+\left(-\frac{\partial\varphi}{\partial x}-\frac12\frac{\partial^2v_0}{\partial x^2}\cdot\frac{\partial w_0}{\partial x}-\frac12\frac{\partial v_0}{\partial x}\cdot\frac{\partial^2w_0}{\partial x^2}\right)z\right)^2\\&+\frac12\left(\frac{\partial w_0}{\partial x}+\left(-\frac{\partial w_0}{\partial x}\cdot\frac{\partial^2w_0}{\partial x^2}-\varphi\frac{\partial\varphi}{\partial x}\right)z+\left(\frac{\partial\varphi}{\partial x}-\frac12\frac{\partial^2v_0}{\partial x^2}\cdot\frac{\partial w_0}{\partial x}-\frac12\frac{\partial v_0}{\partial x}\cdot\frac{\partial^2w_0}{\partial x^2}\right)z\right)^2.\end{aligned}\tag{26}$$

略去变形位移及其偏导数的二次以上耦合项，得

$$\varepsilon_{xx}=\frac{\partial s_0}{\partial x}-y\frac{\partial^2v_0}{\partial x^2}-z\frac{\partial^2w_0}{\partial x^2}.\tag{27}$$

柔性梁上任一点 a 的剪应变为

$$\varepsilon_{xy}=-\frac12z\left(\frac{\partial\varphi}{\partial x}+\frac12\frac{\partial^2v_0}{\partial x^2}\frac{\partial w_0}{\partial x}-\frac12\frac{\partial^2w_0}{\partial x^2}\frac{\partial v_0}{\partial x}\right),\tag{28}$$

$$\varepsilon_{xz}=-\frac12y\left(-\frac{\partial\varphi}{\partial x}-\frac12\frac{\partial^2v_0}{\partial x^2}\frac{\partial w_0}{\partial x}+\frac12\frac{\partial^2w_0}{\partial x^2}\frac{\partial v_0}{\partial x}\right).\tag{29}$$

略去变形位移及其偏导数的二次以上耦合项，得

$$\varepsilon_{xy} = -\frac{1}{2}z\frac{\partial\varphi}{\partial x}, \tag{30}$$

$$\varepsilon_{xz} = \frac{1}{2}y\frac{\partial\varphi}{\partial x}. \tag{31}$$

柔性梁为各向同性线性材料，应力-应变关系为

$$\sigma_{xx} = E\varepsilon_{xx}, \tag{32}$$

$$\tau_{xy} = 2G\varepsilon_{xy}, \tag{33}$$

$$\tau_{xz} = 2G\varepsilon_{xz}, \tag{34}$$

其中，E 为材料的杨氏弹性模量，$G = E/2(1+\mu)$，μ 为泊松比.

为得到较为精确的动力学方程，在本章的应变能中考虑因轴向扭转而引起的剪切效应. 则梁的应变势能为

$$U = \frac{1}{2}\int_V(\sigma_{xx}\varepsilon_{xx} + 2\tau_{xy}\varepsilon_{xy} + 2\tau_{xz}\varepsilon_{xz})\mathrm{d}V. \tag{35}$$

将(32)—(34)式代入(35)式，得

$$U = \frac{1}{2}\int_V(E\varepsilon_{xx}^2 + 4G(\varepsilon_{xy}^2 + \varepsilon_{xz}^2))\mathrm{d}V. \tag{36}$$

将(27)，(30)，(31)式代入(36)式，得

$$U = \frac{1}{2}\int_0^l\left(EA\left(\frac{\partial s_0}{\partial x}\right)^2 + GI_{\mathrm{p}}\left(\frac{\partial\varphi}{\partial x}\right)^2 + EI_{zz}\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + EI_{yy}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2\right)\mathrm{d}x, \tag{37}$$

其中，I_p 为极惯性矩，$I_p = I_{yy} + I_{zz}$，将(12)式代入(37)式，得

$$U = \frac{1}{2}\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{K}_f\boldsymbol{q}_f, \tag{38}$$

$$\boldsymbol{K}_f = \sum_{e=1}^{n}\boldsymbol{R}^{\mathrm{T}}B_e^{\mathrm{T}}\int_0^{l_e}\left(EA\frac{\partial\boldsymbol{N}_{e,1}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,1}}{\partial\bar{x}} + EI_{zz}\frac{\partial^2\boldsymbol{N}_{e,2}^{\mathrm{T}}}{\partial\bar{x}^2}\cdot\frac{\partial^2\boldsymbol{N}_{e,2}}{\partial\bar{x}^2} + EI_{yy}\frac{\partial^2\boldsymbol{N}_{e,3}^{\mathrm{T}}}{\partial\bar{x}^2}\cdot\frac{\partial^2\boldsymbol{N}_{e,3}}{\partial\bar{x}^2} + GI_{pp}\frac{\partial\boldsymbol{N}_{e,4}^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_{e,4}}{\partial\bar{x}}\right)\mathrm{d}\bar{x}\boldsymbol{B}_e\boldsymbol{R}.$$

3.3. 外力对应的广义力

设在惯性系 e^0 下柔性梁的分布外力（含体力）坐标列阵为

$$f_w = [f_1 \quad f_2 \quad f_3]^{\mathrm{T}}. \tag{39}$$

设 $\psi = [\alpha \quad \beta \quad \gamma]^{\mathrm{T}}$ 为角速度 ω 在连体系 e^1 下对应的伪坐标. 系统的广义坐标为

$$\boldsymbol{q} = [\boldsymbol{r}_0^{\mathrm{T}} \quad \boldsymbol{\psi}^{\mathrm{T}} \quad \boldsymbol{q}_f^{\mathrm{T}}]^{\mathrm{T}}. \tag{40}$$

令 $\delta\boldsymbol{r}_0 \neq \boldsymbol{0}, \delta\boldsymbol{\psi} = \boldsymbol{0}, \delta\boldsymbol{q}_f = \boldsymbol{0}$，则外力做功为

$$\int_V \delta W_{f_w}^{\delta r_0}\mathrm{d}V = \int_V \delta\boldsymbol{r}_0^{\mathrm{T}}\cdot\boldsymbol{f}_w\mathrm{d}V, \tag{41}$$

得 $\boldsymbol{r}_0$ 对应的广义力为

$$\boldsymbol{Q}_{r_0} = \frac{\int_V \delta W_{f_w}^{\delta r_0}\mathrm{d}V}{\delta\boldsymbol{r}_0} = \int_V \boldsymbol{f}_w\mathrm{d}V. \tag{42}$$

令 $\delta\boldsymbol{r}_0 = \boldsymbol{0}, \delta\boldsymbol{\psi} \neq \boldsymbol{0}, \delta\boldsymbol{q}_f = \boldsymbol{0}$，梁上任一点的位移矢量在惯性坐标系下为

$$\delta\boldsymbol{u}_\psi = \boldsymbol{A}(\delta\boldsymbol{\psi}\times\boldsymbol{u}_a), \tag{43}$$

则外力做功为

$$\begin{aligned}\int_V \delta W_{f_w}^{\delta\psi}\mathrm{d}V &= \int_V \delta\boldsymbol{u}_\psi^{\mathrm{T}}\cdot f_w\mathrm{d}V \\ &= \int_V(\delta\boldsymbol{\psi}\times\boldsymbol{u}_a)^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}_w\mathrm{d}V \\ &= -\int_V(\tilde{\boldsymbol{u}}_a\delta\boldsymbol{\psi})^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}f_w\mathrm{d}V \\ &= -\int_V\delta\boldsymbol{\psi}^{\mathrm{T}}\tilde{\boldsymbol{u}}_a^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}_w\mathrm{d}V \\ &= -\int_V\delta\boldsymbol{\psi}^{\mathrm{T}}\tilde{u}_a^{\mathrm{T}}\boldsymbol{f}'_w\mathrm{d}V,\end{aligned} \tag{44}$$

其中，$\boldsymbol{f}'_w = \boldsymbol{A}^{\mathrm{T}}\boldsymbol{f}_w = [f'_1 \quad f'_2 \quad f'_3]^{\mathrm{T}}$ 为外力在连体系下的坐标列阵.

从而得到 ψ 对应的广义力为

$$\begin{aligned}\boldsymbol{Q}_\psi &= \frac{\int_V \delta W_{f_w}^{\delta\psi}\mathrm{d}V}{\delta\psi^{\mathrm{T}}} = \int_V \tilde{\boldsymbol{u}}_a^{\mathrm{T}}\boldsymbol{f}'_w\mathrm{d}V \\ &= [f_\alpha \quad f_\beta \quad f_\gamma]^{\mathrm{T}},\end{aligned} \tag{45}$$

其中，$f_\alpha = \int_V(-u_{a3}f'_2 + u_{a2}f'_3)\mathrm{d}V$，$f_\beta = \int_V(u_{a3}f'_1 - u_{a1}f'_3)\mathrm{d}V$，$f_\beta = \int_V(-u_{a2}f'_1 + u_{a1}f'_2)\mathrm{d}V$.

令 $\delta\boldsymbol{r}_0 = \boldsymbol{0}, \delta\boldsymbol{\psi} = \boldsymbol{0}, \delta\boldsymbol{q}_f \neq 0$，则外力做功为

$$\begin{aligned}\int_V \delta W_{f_w}^{\delta q_f}\mathrm{d}V &= \int_V(\boldsymbol{A}\delta\boldsymbol{u}_{af})^{\mathrm{T}}\cdot\boldsymbol{f}_w\mathrm{d}V \\ &= \int_V\delta\boldsymbol{q}_f^{\mathrm{T}}\Bigg(\Bigg(\boldsymbol{N}_1 - y\frac{\partial\boldsymbol{N}_2}{\partial\bar{x}} - z\frac{\partial\boldsymbol{N}_3}{\partial\bar{x}} - \boldsymbol{q}_f^{\mathrm{T}}H + y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{12} - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{t34} + z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{13} + z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{t24}\Bigg)^{\mathrm{T}}f'_1 \\ &\quad + \Bigg(\boldsymbol{N}_2 - z\boldsymbol{N}_4 - y\boldsymbol{q}_f^{\mathrm{T}}\frac{\partial\boldsymbol{N}_2^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_2}{\partial\bar{x}} - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4 - z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{23}\Bigg)^{\mathrm{T}}f'_2 \\ &\quad + \Bigg(\boldsymbol{N}_3 + y\boldsymbol{N}_4 - z\boldsymbol{q}_f^{\mathrm{T}}\frac{\partial\boldsymbol{N}_3^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial\boldsymbol{N}_3}{\partial\bar{x}} - z\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{N}_4^{\mathrm{T}}\boldsymbol{N}_4 - y\boldsymbol{q}_f^{\mathrm{T}}\boldsymbol{V}_{23}\Bigg)^{\mathrm{T}}f'_2\Bigg)\mathrm{d}V.\end{aligned} \tag{46}$$

从而得到 $\boldsymbol{q}_f$ 对应的广义力为

$$Q_{q_f}=\frac{\int_V \delta W_{f_w}^{\delta q_f}\mathrm{d}V}{\delta q_f^{\mathrm{T}}}$$

$$=\int_V\Bigg(\bigg(N_1-y\frac{\partial N_2}{\partial\bar{x}}-z\frac{\partial N_3}{\partial\bar{x}}-q_f^{\mathrm{T}}H+yq_f^{\mathrm{T}}V_{12}-yq_f^{\mathrm{T}}V_{t34}+zq_f^{\mathrm{T}}V_{13}+zq_f^{\mathrm{T}}V_{t24}\bigg)^{\mathrm{T}}f'_1$$

$$+\bigg(N_2-zN_4-yq_f^{\mathrm{T}}\frac{\partial N_2^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial N_2}{\partial\bar{x}}-yq_f^{\mathrm{T}}N_4^{\mathrm{T}}N_4-zq_f^{\mathrm{T}}V_{23}\bigg)^{\mathrm{T}}f'_2$$

$$+\bigg(N_3+yN_4-zq_f^{\mathrm{T}}\frac{\partial N_3^{\mathrm{T}}}{\partial\bar{x}}\cdot\frac{\partial N_3}{\partial\bar{x}}-zq_f^{\mathrm{T}}N_4^{\mathrm{T}}N_4-yq_f^{\mathrm{T}}V_{23}\bigg)^{\mathrm{T}}f'_2\Bigg)\mathrm{d}V, \qquad (47)$$

则外力对应的广义力为

$$Q=[Q_{r_0}^{\mathrm{T}}\quad Q_{\psi}^{\mathrm{T}}\quad Q_{q_f}^{\mathrm{T}}]^{\mathrm{T}}. \qquad (48)$$

3.4. 梁动力学方程

将(18),(38),(48)式代入 Lagrange 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial T}{\partial\dot{q}}\Big)-\frac{\partial T}{\partial q}+\frac{\partial U}{\partial q}=Q, \qquad (49)$$

得运动规律未知的刚-柔耦合系统的动力学方程为

$$M\ddot{q}=Q_v \qquad (50)$$

$$M=\begin{bmatrix}M_{11}&M_{12}&M_{13}\\M_{21}&M_{22}&M_{23}\\M_{31}&M_{32}&M_{33}\end{bmatrix}, \qquad (51)$$

$$Q_v=[Q_{f1}^{\mathrm{T}}+Q_{r_0}^{\mathrm{T}}\quad Q_{f2}^{\mathrm{T}}+Q_{\psi}^{\mathrm{T}}\quad Q_{f3}^{\mathrm{T}}+Q_q^{\mathrm{T}}+Q_{q_f}^{\mathrm{T}}]^{\mathrm{T}}, \qquad (52)$$

$$Q_{f1}=-(G_{11}\dot{r}_0+G_{12}\omega+G_{13}\dot{q}_f), \qquad (53)$$

$$Q_{f2}=-(G_{21}\dot{r}_0+G_{22}\omega+G_{23}\dot{q}_f), \qquad (54)$$

$$Q_{f3}=-(G_{31}\dot{r}_0+G_{32}\omega+G_{33}\dot{q}_f)-K_dq_f, \qquad (55)$$

式中

$$G_{11}=0_{3\times3},G_{12}=G_{217}^{\mathrm{T}}=-A\tilde{\omega}\tilde{s}-A\dot{\tilde{s}},G_{32}=[\dot{g}_1^{\mathrm{T}}\quad\dot{g}_2^{\mathrm{T}}\quad\dot{g}_3^{\mathrm{T}}],$$

$$G_{13}=A\tilde{\omega}\begin{bmatrix}Y_1-q_f^{\mathrm{T}}C\\Y_2\\Y_3\end{bmatrix}+A\begin{bmatrix}-\dot{q}_f^{\mathrm{T}}C\\0\\0\end{bmatrix},G_{22}=\begin{bmatrix}\dot{h}_{11}&\dot{h}_{12}&\dot{h}_{13}\\\dot{h}_{21}&\dot{h}_{22}&\dot{h}_{23}\\\dot{h}_{31}&\dot{h}_{32}&\dot{h}_{33}\end{bmatrix}+[\dot{r}_0^{\mathrm{T}}A_{\psi}\tilde{s}]\ ,G_{23}=\begin{bmatrix}\dot{g}_1\\\dot{g}_2\\\dot{g}_3\end{bmatrix}-\left[\dot{r}_0^{\mathrm{T}}A_{\psi}\begin{bmatrix}Y_1-q_f^{\mathrm{T}}C\\Y_2\\Y_3\end{bmatrix}\right],$$

$$G_{31}=G_{13}^{\mathrm{T}}-[\omega_2Y_3^{\mathrm{T}}-\omega_3Y_2^{\mathrm{T}}-C\dot{q}_f\quad-\omega_1Y_3^{\mathrm{T}}+\omega_3(Y_1^{\mathrm{T}}-Cq_f)\quad\omega_1Y_2^{\mathrm{T}}-\omega_2(Y_1^{\mathrm{T}}-Cq_f)]A^{\mathrm{T}},$$

$$G_{33}=-(\omega_1(W_{23}-W_{32}+\overline{W}_{3,23}-\overline{W}_{2,23})-\omega_2(W_{13}-W_{31}-\overline{W}_{3,13}-\overline{W}_{t3,24}+\hat{W}_{2,42}-\hat{W}_{2,24})$$
$$-\omega_3(W_{21}-W_{12}-\overline{W}_{2,12}-\overline{W}_{t2,34}+\hat{W}_{3,43}-\hat{W}_{3,34})),$$

其中

$$\dot{d}_{11}=2q_f^{\mathrm{T}}(W_{22}+W_{33})\dot{q}_f-2q_f^{\mathrm{T}}(\overline{W}_{33}+\overline{W}_{22})\dot{q}_f,$$

$$\dot{d}_{12}=\dot{h}_{21}=\overline{Y}_2\dot{q}_f-Z_{12}\dot{q}_f-q_f^{\mathrm{T}}bW_{21}\dot{q}_f-q_f^{\mathrm{T}}b\overline{W}_{02,12}\dot{q}_f+q_f^{\mathrm{T}}(b\hat{W}_{2,34}+b\hat{W}_{3,34})\dot{q}_f,$$

……

$$\dot{g}_3=-(\dot{q}_f^{\mathrm{T}}(W_{21}-W_{12})+\dot{q}_f^{\mathrm{T}}(\overline{W}_{2,12}-\overline{W}_{t2,34})+\dot{q}_f^{\mathrm{T}}(\hat{W}_{3,43}-\hat{W}_{3,34})),$$

式中

$$bW_{21}=W_{21}+W_{21}^{\mathrm{T}},$$

$$K_d=K_f-(\omega_1^2(W_{22}+W_{33}-\overline{W}_{33}-\overline{W}_{22})+\omega_1\omega_2(-bW_{21}-b\overline{W}_{02,12}+b\hat{W}_{2,34}+b\hat{W}_{3,34})$$
$$+\omega_1\omega_3(-bW_{31}-b\overline{W}_{03,13}+b\hat{W}_{2,24}-b\hat{W}_{3,24})$$
$$+\frac{1}{2}\omega_2^2(2W_{11}+2W_{33}+2\overline{W}_{22}+2\overline{W}_{33}-b\,\overline{\overline{W}}_{34}+b\,\overline{\overline{W}}_{24}-2D-2\overline{W}_{33})$$
$$+\omega_2\omega_3\Big(-bW_{32}+\frac{1}{2}(b\overline{W}_{03,23}+b\overline{W}_{02,23})\Big)$$
$$+\frac{1}{2}\omega_3^2(-2\overline{W}_{22}+b\,\overline{\overline{W}}_{34}-b\,\overline{\overline{W}}_{24}-2D+2(W_{11}+W_{22}+\overline{W}_{22}+\overline{W}_{33}))),$$

$$Q_q = \omega_1\omega_2(\bar{Y}_2^{\mathrm{T}} - Z_{12}^{\mathrm{T}}) + \omega_1\omega_3(\bar{Y}_3^{\mathrm{T}} - Z_{13}^{\mathrm{T}}) + \omega_2^2 Z_{11}^{\mathrm{T}} + \omega_2\omega_3(\bar{\bar{\bar{Y}}}_{34}^{\mathrm{T}} - \bar{\bar{\bar{Y}}}_{24}^{\mathrm{T}}) + \omega_3^2 Z_{11}^{\mathrm{T}}.$$

4. 计算及其分析

本文通过对不同回转半径的大范围运动空间柔性梁进行计算发现，随着回转半径的增加，柔性梁基点的加速度逐渐增加，一次近似模型与本文模型在一个方向的变形出现较明显的差异. 经分析表明，对于作大范围运动的空间柔性梁，除了在梁的纵向变形中考虑横向、侧向的变形耦合外，还应在任一点各方向的变形中，计及几何非线性变形造成的弯曲、扭转引起的横向、侧向的变形耦合. 计算表明，在梁具有较大的空间运动时，几何非线性变形的因素会影响柔性梁的动力学特性.

计算结果表明，由于本文在系统动能中考虑了轴向伸缩、横向和侧向弯曲的相互耦合作用，以及扭转效应，利用非线性精确变形模式得到的有限元离散模型，各项更加完备. 此外，基于非线性精确变形得到的广义质量阵，新增了一些表示变形耦合以及扭转效应的项，即划线项. 上述诸项可视为对文献[12]中广义质量阵的补充和完善.

通过比较一次耦合模型和本文精确模型，发现两种模型均在纵向伸长中保留了横向变形引起的偏导数的二次耦合项，因而都具有附加项 $\boldsymbol{C}$，$\boldsymbol{D}$. 而这两项是零次近似模型所忽略的，对于有较大运动速度的刚－柔耦合结构，附加项使得结构的刚度增加，因而零次近似模型失效. 此外由于本文精确模型下，考虑了较多的变形模式，使得广义力中包含了较完整的柔性变形以及柔性变形和大范围运动的相互耦合项.

5. 结　　论

1. 本文根据柔性梁的精确非线性变形模式，采用有限元方法对空间柔性梁进行有限元离散，建立了具有大范围运动和非线性变形的空间柔性梁的精确动力学方程. 本文模型在原有一次耦合模型的基础上，增加了新的表征纵向、横向、侧向弯曲变形，以及扭转变形的耦合项，完备了方程的形式. 所得到的方程可用于研究非惯性系下的结构动力学问题，也可用于大范围运动为未知的刚柔耦合问题.

2. 常规的有限元模型，是一种零次的近似模型，即用线性关系来拟合变形，对于大柔性高速多体系统，该方法的准确性值得怀疑. 而通过本文的研究，可以用一次或二次近似模型来拟合变形，有助于研究像航天器这样的大柔性高速系统.

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Exact dynamic modeling of a spatial flexible beam with large overall motion and nonlinear deformation*

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Abstract

In this paper, the dynamic modeling theory of a spatial flexible beam, which undergoes large overall motion and nonlinear deformation, is investigated. As we know, in spacecraft and space station, there are a lot of flexible appendices so the dynamic modeling of a flexible beam is essential. Yet the existing models, in our opinion, lack several important coupling terms. This paper supplies these important coupling terms. Based on the new approach of deformation of fully geometrically nonlinear beam model developed, the finite element method is used for the system discretization and the coupling dynamic equations of flexible beam are obtained by Lagrange's equations. The complete expression of stiffness matrix and all coupling terms are included in the dynamic equations. The second order coupling terms between rigid large overall motion, arc length stretch, lateral flexible deformation kinematics and torsional deformation terms are included in the present exact coupling model to expand the theory of one-order coupling model. The dynamic modeling method in this paper is of theoretical significance and has reference value for the rigid-flexible coupling system dynamic investigation .

Keywords: large overall motion, nonlinear deformation, patial flexible beam, exact dynamic modeling

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