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一般反射短峰波的普适法则——倍频率通向短峰波*

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考虑环境均匀流效应，给出从垂直防波堤发生一般部分反射短峰波的三阶解析解. 据此推断出一项普适法则：倍频率通向短峰波.

关键词： 三阶短峰波，倍频率，部分反射，均匀流

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1. 引 言

最简单的非线性函数——二次函数抛物线，可揭示在动力系统中隐含的许多重要现象[1,2]，其中“倍周期通向混沌”尤为引人注目. 在纷繁复杂的广阔海洋三维表面波运动[3,4]中，最基本的短峰波(short-crested waves)可产生于多种海况和实际工程中. 例如，波浪倾斜入射垂直防波堤发生反射的状况. 短峰波应该可以表征实际三维波浪的某些典型特征和机理. 在最近 30 年，短峰波从理论[5—10]、数值[11,12]和实验[13,14]等方面已愈来愈受到普遍关注，以适应全球急剧发展的近海、海洋工程实践.

如果将目前广泛采用的理想全反射短峰波状态改成更贴近实际的一般部分反射短峰波情形，则势必更能彰显三维波浪运行的普遍规律. 这类似于二次函数参数取值范围的不同而导致曲线点的性态不同. 因此，本文立足于经典的全反射三阶纯表面短峰波理论[3]，拟将其扩展为可纳入普遍“波-流相互作用机制”[15,16]之基本的均匀流效应的一般部分反射高阶短峰波，以得其解. 并有望从中发现、提取出短峰波演变的某种普适基本特征，从而有助于广阔的近海、海洋工程实践.

2. 短峰波系统

水波可通常被刻画成无黏、不可压缩流体的无旋运动，短峰波可发端于以不同方向传播的两个二维波列的非线性相互作用. 现在建立直角坐标系 $O(x,y,z)$，其中平面 $z=0$ 位于平均水位(MWL)上，z 轴竖直向上. 波长和波数分别为 L,k 的入射波和反射波(反射系数为 r)与 y 轴夹角均为 θ，同时伴随着沿 x 轴(相当于垂直防波堤)的沿岸均匀流动 U. 设自由表面高度为 $\zeta(x,y,t)$，总速度势为 $\Phi(x,y,z,t)$，并且

$$\Phi(x,y,z,t)=Ux+\phi(x,y,z,t), \tag{1}$$

其中，ϕ 描述纯波运动. 该短峰波非线性系统的基本控制方程组如下：

$$\nabla^2\phi=0 \quad (-h\leqslant z\leqslant\zeta), \tag{2}$$

$$g\zeta+\phi_t+\frac{1}{2}|\nabla\phi|^2+U\phi_x=0 \quad (z=\zeta), \tag{3}$$

$$g\phi_z+\mathfrak{R}\phi=-2\nabla\phi\cdot\mathfrak{I}(\nabla\phi)$$
$$-\frac{1}{2}\nabla\phi\cdot\nabla[(\nabla\phi)^2] \quad (z=\zeta), \tag{4}$$

$$\phi_z=0 \quad (z=-h), \tag{5}$$

其中，g 和 h 分别为重力加速度和常水深，$\nabla=(\frac{\partial}{\partial x},$

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$\frac{\partial}{\partial y},\frac{\partial}{\partial z})$，$\mathfrak{R} = \frac{\partial^2}{\partial t^2} + 2U\frac{\partial^2}{\partial t\partial x} + U^2\frac{\partial^2}{\partial x^2}$，$\mathfrak{J} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$.

采用下列无量纲变量（ω 为角频率）：

$$(\tilde{x},\tilde{y},\tilde{z},\tilde{\zeta},\tilde{h}) = k(x,y,z,\zeta,h),\quad \tilde{t} = \omega t,\ \tilde{\phi} = \frac{k^2}{\sqrt{gk}}\phi,\quad \tilde{\omega} = \frac{\omega}{\sqrt{gk}},\quad \tilde{U} = \frac{U}{\tilde{\omega}}\sqrt{\frac{k}{g}}, \tag{6}$$

可将(2)—(5)式无量纲化为（省略无量纲符号）：

$$\nabla^2\phi = 0 \qquad (-h \leqslant z \leqslant \zeta), \tag{7}$$

$$\zeta + \omega\mathfrak{J}\phi + \frac{1}{2}|\nabla\phi|^2 = 0 \qquad (z = \zeta), \tag{8}$$

$$\phi_z + \omega^2\mathfrak{R}\phi = -2\omega\nabla\phi\cdot\mathfrak{J}(\nabla\phi) - \frac{1}{2}\nabla\phi\cdot\nabla[(\nabla\phi)^2] \qquad (z = \zeta), \tag{9}$$

$$\phi_z = 0 \qquad (z = -h), \tag{10}$$

为使短峰波解能够回归经典的二维长峰波波况，该解须满足下列行波和驻波条件

$$\int_0^{\pi}\int_0^{\pi}\zeta(x,y,t)\,\mathrm{d}y\mathrm{d}x = 0,\quad \int_0^{2\pi}\zeta(x,t)\,\mathrm{d}x = 0, \tag{11}$$

$$\nabla\phi(x,y,z,t+2\pi) = \nabla\phi(x,y,z,t),\quad \nabla\phi(x,z,t+2\pi) = \nabla\phi(x+2\pi,z,t), \tag{12}$$

$$\int_0^{\pi}\zeta(y,t)[r\sin(y-t) - \sin(y+t)]\,\mathrm{d}t\mathrm{d}y = 0, \tag{13}$$

$$\int_{-h}^{0}\int_0^{\pi}\int_0^{2\pi}\phi(y,z,t)[r\sin(y-t) - \sin(y+t)]\,\mathrm{d}t\mathrm{d}y\mathrm{d}z = \pi^2(1+r^2)(\tanh h)^{\frac{1}{2}}. \tag{14}$$

其中，(11)和(12)分别保证水体质量守恒和水波时空周期性，(13)和(14)式分别为水波的相位方程和振幅方程.

最后，为保证短峰波解的唯一性，该解尚需满足下面唯一性条件：

$$[jmU(\tanh h)^{\frac{1}{2}} \pm (1-mU)(\gamma_{ji}\tanh\gamma_{ji}h)^{\frac{1}{2}}]^2/\tanh h \neq l^2. \tag{15}$$

其中，i,j,l 为整数：$i \geqslant 0, j \geqslant 1, l \geqslant 1, \gamma_{ji} = \sqrt{(jm)^2 + (in)^2}, m = \sin\theta, n = \cos\theta$.

注意，上述条件(13)—(15)式已显著地扩展了经典的同类条件[5,17].

将关于(7)—(10)式的未知变量 ϕ,ζ,ω 依照波陡小参数 ε 展开为

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots,\quad \zeta = \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \varepsilon^3\zeta_3 + \cdots,\quad \omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots, \tag{16}$$

继而将自由表面方程(8)和(9)式的速度势 ϕ 在 $z = 0$ 处进行 Taylor 展开，由此可得到下列各阶（阶数 $i = 1,2,3,\cdots$）方程组：

$$\nabla^2\phi_i = 0 \qquad (-h \leqslant z \leqslant 0), \tag{17}$$

$$\zeta_i + \omega_0\mathfrak{J}\phi_i = E_i \qquad (z = 0), \tag{18}$$

$$\phi_{iz} + \omega_0^2\mathfrak{R}\phi_i = F_i \qquad (z = 0), \tag{19}$$

$$\phi_{iz} = 0 \qquad (z = -h), \tag{20}$$

其中，$E_1 = F_1 = 0, E_i, F_i (i \geqslant 2)$ 均由若干项构成，并且

$$F_2 = -\zeta_1(\phi_{1zz} + \omega_0^2\mathfrak{R}\phi_{1z}) - 2\omega_0\omega_1\mathfrak{R}\phi_1 - 2\omega_0\nabla\phi_1\cdot\mathfrak{J}(\nabla\phi_1), \tag{21}$$

$$\begin{aligned} F_3 = &-\left(\zeta_2\phi_{1zz} + \frac{1}{2}\zeta_1^2\phi_{1zzz} + \zeta_1\phi_{2zz}\right) \\ &- \omega_0^2\left(\zeta_2\mathfrak{R}\phi_{1z} + \frac{1}{2}\zeta_1^2\mathfrak{R}\phi_{1zz} + \zeta_1\mathfrak{R}\phi_{2z}\right) \\ &- 2\omega_0\omega_1(\mathfrak{R}\phi_2 + \zeta_1\mathfrak{R}\phi_{1z}) \\ &- (\omega_1^2 + 2\omega_0\omega_2)\mathfrak{R}\phi_1 \\ &- 2\omega_0\{\nabla\phi_1\cdot\mathfrak{J}[\nabla(\phi_2 + \zeta_1\phi_{1z})] \\ &+ \nabla(\phi_2 + \zeta_1\phi_{1z})\cdot\mathfrak{J}(\nabla\phi_1)\} \\ &- 2\omega_1\nabla\phi_1\cdot\mathfrak{J}(\nabla\phi_1) - \frac{1}{2}\nabla\phi_1\cdot\nabla[(\nabla\phi_1)^2]. \end{aligned} \tag{22}$$

3. 解析解和推论

求解前三阶方程组(17)—(20)式，依次可得

$$\zeta_1 = \cos(X - Y) + r\cos(X + Y), \tag{23}$$

$$\phi_1 = [\sin(X - Y) + r\sin(X + Y)]\cosh Z/\tau\cosh h, \tag{24}$$

$$\tanh h = \omega_0^2(1-mU)^2 \equiv \tau^2; \tag{25}$$

$$\zeta_2 = \alpha_{02}^2\cos 2Y + \alpha_{20}^2\cos 2X + \alpha_{22}^2[\cos 2(X - Y) + r^2\cos 2(X + Y)], \tag{26}$$

$$\phi_2 = \lambda_0^2 t + \gamma_0^2 + \beta_{20}^2\cosh 2mZ\sin 2X + \beta_{22}^2[\sin 2(X - Y)$$

$$+ r^2\sin 2(X+Y)]\cosh 2Z, \tag{27}$$

$$\omega_1 = 0, \tag{28}$$

$$\begin{aligned}\phi_3 =& \beta_{13}^3[\sin(X-3Y) + r\sin(X+3Y)]\cosh\gamma_{13}Z \\ &+ \beta_{31}^3[\sin(3X-Y) + r\sin(3X+Y)]\cosh\gamma_{31}Z \\ &+ \beta_{33}^3[\sin 3(X-Y) + r^3\sin 3(X+Y)]\cosh 3Z,\end{aligned} \tag{29}$$

$$\begin{aligned}\zeta_3 =& \alpha_{11}^3\cos(X-Y) + \overline{\alpha}_{11}^3\cos(X+Y) \\ &+ \alpha_{13}^3[\cos(X-3Y) + r\cos(X+3Y)] \\ &+ \alpha_{31}^3[\cos(3X-Y) + r\cos(3X+Y)] \\ &+ \alpha_{33}^3[\cos 3(X-Y) + r^3\cos 3(X+Y)],\end{aligned} \tag{30}$$

$$\begin{aligned}&\omega_{21}(1-mU) \\ &= \frac{1}{16}(9\tau^{-7} - 10\tau^{-3} + 9\tau) \\ &- \frac{r^2}{2}m\tau^2\omega_m^2K_1 + \frac{r^2}{8}(\tau^4 + 4m^2 - 1)K_2 \\ &- \frac{r^2}{4}[(m^2-n^2)^2\tau^{-3} + 4(n^2-m^2)\tau - \tau^5],\end{aligned} \tag{31a}$$

$$\begin{aligned}&\omega_{22}(1-mU) \\ &= \frac{r^2}{16}(9\tau^{-7} - 10\tau^{-3} + 9\tau) \\ &- \frac{1}{2}m\tau^2\omega_m^2K_1 + \frac{1}{8}(\tau^4 + 4m^2 - 1)K_2 \\ &- \frac{1}{4}[(m^2-n^2)^2\tau^{-3} + 4(n^2-m^2)\tau - \tau^5].\end{aligned} \tag{31b}$$

其中

$$\begin{aligned}X &= mx - t, \\ Y &= ny, \\ Z &= z + h,\end{aligned} \tag{32}$$

$$\alpha_{02}^2 = \frac{r}{2}[\tau^2 - (m^2-n^2)\tau^{-2}], \tag{33a}$$

$$\alpha_{20}^2 = \frac{r}{2}[3\tau^2 - (m^2-n^2)\tau^{-2} + \tau K_2], \tag{33b}$$

$$\alpha_{22}^2 = \frac{1}{4}(3\tau^{-6} - \tau^{-2}), \tag{33c}$$

$$\lambda_0^2 = \frac{1+r^2}{4\omega_0}(\tau^2 - \tau^{-2}), \tag{34a}$$

$$\beta_{20}^2 = \frac{rK_2}{4\cosh 2mh}, \tag{34b}$$

$$\beta_{22}^2 = \frac{3(\tau^{-7}-\tau)}{8\cosh 2h}, \tag{34c}$$

$$K_2 = \frac{(1+\omega_m^4)[(2m^2-2n^2+1)\tau^{-3} - 3\tau]}{1 + \omega_m^4 - m(\omega_m/\tau)^2}, \tag{34d}$$

$$K_1 = \frac{K_2}{1+\omega_m^4}, \tag{34e}$$

$$\omega_m^2 = \tanh mh; \tag{34f}$$

$$\begin{aligned}\beta_{13}^3 =& \frac{r}{8}[(\gamma_{13}\tanh\gamma_{13}h - \tau^2)\cosh\gamma_{13}h]^{-1} \\ &\times[-3\tau^{-7} + 8\tau^{-3} - 3\tau + 2\tau^5 \\ &+ m^2(-6\tau^{-7} + 4\tau^{-3} - 10\tau) \\ &+ n^2(6\tau^{-7} - 4\tau^{-3} - 2\tau) \\ &+ 4n^2(m^2-n^2)\tau^{-3}],\end{aligned} \tag{35a}$$

$$\begin{aligned}\beta_{31}^3 =& \frac{r}{8}[(\gamma_{31}\tanh\gamma_{31}h - 9\tau^2)\cosh\gamma_{31}h]^{-1} \\ &\times[-9\tau^{-7} + 64\tau^{-3} - 59\tau + 6\tau^5 + 40m\tau^2\omega_m^2K_1 \\ &+ 2(\tau^4 - 8m^2 - 1)K_2 + 4m^2(m^2-n^2)\tau^{-3} \\ &+ m^2(-18\tau^{-7} + 4\tau^{-3} - 8\tau) \\ &+ n^2(18\tau^{-7} - 4\tau^{-3} + 24\tau)],\end{aligned} \tag{35b}$$

$$\begin{aligned}\beta_{33}^3 =& \frac{1}{8}[(\sinh 3h - 3\tau^2)\cosh 3h]^{-1} \\ &\times(-9\tau^{-7} + 22\tau^{-3} - 13\tau),\end{aligned} \tag{35c}$$

$$\begin{aligned}\alpha_{11}^3 =& \frac{1}{8}(-3\tau^{-8} + 9\tau^{-4} - 3) + \frac{r^2}{2}m\tau\omega_m^2K_1 \\ &+ \frac{r^2}{4}(\tau^3 - m^2\tau^{-1})K_2 \\ &+ \frac{r^2}{4}[4\tau^4 - 3(m^2-n^2) + 2] + \frac{\omega_2}{\omega_0},\end{aligned} \tag{36a}$$

$$\begin{aligned}\overline{\alpha}_{11}^3 =& \frac{r^3}{8}(-3\tau^{-8} + 9\tau^{-4} - 3) \\ &+ \frac{r}{2}m\tau\omega_m^2K_1 + \frac{r}{4}(\tau^3 - m^2\tau^{-1})K_2 \\ &+ \frac{r}{4}[4\tau^4 - 3(m^2-n^2) + 2] + \frac{r\omega_2}{\omega_0},\end{aligned} \tag{36b}$$

$$\begin{aligned}\alpha_{13}^3 =& \frac{r}{8}[-3(m^2-n^2)\tau^{-8} + 9\tau^{-4} - 5 \\ &- 6m^2 + 2\tau^4] + \tau\beta_{13}^3\cosh\gamma_{13}h,\end{aligned} \tag{36c}$$

$$\begin{aligned}\alpha_{31}^3 =& \frac{r}{8}[-3(m^2-n^2)\tau^{-8} + 21\tau^{-4} - 15 \\ &+ 6n^2 + 6\tau^4] + \frac{3r}{2}m\tau\omega_m^2K_1 \\ &+ \frac{r}{4}(\tau^3 - m^2\tau^{-1})K_2 + 3\tau\beta_{31}^3\cosh\gamma_{31}h,\end{aligned} \tag{36d}$$

$$\alpha_{33}^3 = \frac{1}{8}(-3\tau^{-8} + 21\tau^{-4} - 15)$$

$$+\frac{-27\tau^{-6}+66\tau^{-2}-39\tau^{2}}{8(\tanh 3h-3\tau^{2})}. \tag{36e}$$

由上述前三阶频率的个数特征和控制系统方程组(17)—(20)式,可发现和推断:1) 偶数阶频率 $\omega_1,\omega_3,\omega_5,\cdots$,均依次来源于下列项:$-2\omega_0\omega_1\Re\phi_1$, $-2\omega_0\omega_3\Re\phi_1$, $-2\omega_0\omega_5\Re\phi_1,\cdots$. 因而,从消除各自唯一长期项的条件出发,便得 $\omega_1=\omega_3=\omega_5=\cdots=0$. 2) 奇数阶频率 $\omega_2,\omega_4,\omega_6,\cdots$,均依次来源于下列项:$-(2\omega_0\omega_2+\omega_1^2)\Re\phi_1$, $-(2\omega_0\omega_4+\omega_2^2)\Re\phi_1$, $-(2\omega_0\omega_6+2\omega_2\omega_4)\Re\phi_1,\cdots$,以及相应各阶其他以 ϕ_1 中的两项为因子的项. 注意:$\Re\phi_1=\Re\phi_{1zz}=\Re\phi_{1zzzz}=\cdots$. 由于"$\omega_2,\omega_4,\omega_6,\cdots$"各自相关于求解方程中的两个不同长期项,则由消除长期项的条件,再纳入一阶频率 ω_0,则奇数阶频率数目自成一个倍频率无穷序列:$1,2,2^2,2^3,\cdots$,从而构成一般反射高阶短峰波系统的一般法则:倍频率通向短峰波.

4. 结　　论

近 30 年兴起的三维短峰波研究,几乎还受制于理想的全反射状态,并且在高阶理论上往往"只见树木,不见森林",缺乏概括、简洁的"基本和定性"观点. 本文的落脚点就在于实际的"部分反射"短峰波上,以经典的短峰波理论[5]为出发点,涉及普遍的波-流相互作用机制,给出前三阶解析解,进而分析和推断高阶短峰波系统,意外发现了"倍频率通向短峰波"的普适法则. 这将积极促进对近海和海洋工程中波浪力的确定和优化.

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A universal law on general reflected short-crested waves —— frequency-doubling route to short-crested waves*

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Abstract

Considering ambient uniform currents, a third-order analytic solution for short-crested waves arising from a general partial reflection from a vertical breakwater is presented, thus reaching a universal law: frequency-doubling route to short-crested waves.

Keywords: third-order short-crested waves, frequency-doubling, partial reflection, uniform currents

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