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直立防波堤上部分反射的三阶双色双向水波理论*

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为充分克服以往直立防波堤上立波波浪力的内在局限性和刻画直立防波堤堤前的多色多向波,提出一种可包含经典二维长峰波解和三维单色单向短峰波解的部分反射的三阶双色双向水波理论,并从中推断出适用于高阶多色多向波系统的一项普适法则——偶数阶频率为零,奇数阶频率个数构成一等比数列.

关键词: 双色双向水波, 三阶理论, 普适法则, 部分反射波
PACC: 0340K, 0200, 9210H

1. 引 言

波浪力为近海、海洋建筑物上所遭受的一种主要荷载,在当今全球急剧发展的海洋工程实践中至关紧要. 如何确定最为常见的直立防波堤上的波浪力呢? 如果堤前处于非破碎波状态,通常就以理想、简单的波浪正入射直立堤上而发生全反射,最终导致的二维长峰立波波浪力为标准. 这里,存在着方便、实用甚至可靠的特色,但其内在缺陷是明显的. 例如, 立波波浪力确实为"最大、最危险"的荷载吗? 这并非想象或简单的理论、实验所能确定.

现在,面对着直立堤,可以明确以下两个基本物理事实:堤前波浪一般并非处于假定的正入射和全反射状态,而是通常的倾斜入射、倾斜部分反射;入射波、反射波并非受制于理想的单色单向限制,理应是多色多向的. 如果依据以上事实而得以谋划一理论,则其包容性、先进性是明确和必要的.

早在三十年前, Hsu 等[1] 就提出了一种可发生于直立堤前的经典三阶三维单色单向全反射短峰波理论. 正是借助该理论, Fenton[2] 后来发现:倾斜入射直立堤时而产生的波浪力显著大于相应的立波波浪力. 至今,短峰波已在多个方面得到大大拓展[3—8]. 其中就包括以两列自由交叉波传播为背景而于最近构造的三阶双色双向波理论[6]. 显见,目前这种多色多向波理论发展远远落后于二维长峰波[8—10] 和三维短峰波的进程. 其原因就在于前者理论构造的极大繁杂性. 若要真正反映一般或典型工程物理场景多色多向波运动的本质属性,则从双色双向波入手,就不失为一个实际的良好开端.

基于以上考虑,本文提出一种可比较实际地考虑直立堤前非破碎波入射和部分反射波态的三阶双色双向波理论,以此考察、确定多色多向波的一般结构特征.

2. 构 造

假定流体做无黏、不可压缩的无旋流体运动,其速度势为 $\phi(x,y,z,t)$. 采用如图 1 所示的直角坐标系: x 轴沿着直立防波堤方向, y 轴垂直于堤, z 轴竖直向上,原点位于静水面上. $z = \zeta(x,y,t)$ 和 $z = -h$ 分别代表自由表面和常水深海底, $y = 0$ 则表示直立堤. 假定两列波分别以与 y 轴成 θ_m, θ_n 的角度入射、反射,其角频率分别为 ω_m, ω_n, 并且入射波或反射波的平均波数和平均频率分别为 $\bar{k}$ 和 $\bar{\omega}$, 以此可引入如下无量纲化量:

$$(\tilde{x},\tilde{y},\tilde{z},\tilde{\zeta},\tilde{h}) = \bar{k}(x,y,z,\zeta,h),\quad \tilde{\phi} = \phi\sqrt{\frac{\bar{k}^3}{g}},$$

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$$\tilde{t} = \omega t, \tilde{\omega} = \frac{\omega}{\sqrt{g\bar{k}}} = \frac{\omega}{\bar{\omega}},$$

$$\tilde{k}_i = \frac{k_i}{\bar{k}}, \quad \tilde{\omega}_i = \frac{\omega_i}{\bar{\omega}}, \tag{1}$$

其中，ω 和 g 分别代表角频率和重力加速度，$k_i = \frac{2\pi}{L_i}$(i = m,n)，并且 k_i 的分量可表示为

$$k_{ix} = \frac{2\pi}{L_{ix}} = k_i \sin\theta_i = k_i i_1,$$

$$k_{iy} = \frac{2\pi}{L_{iy}} = k_i \cos\theta_i = k_i i_2. \tag{2}$$

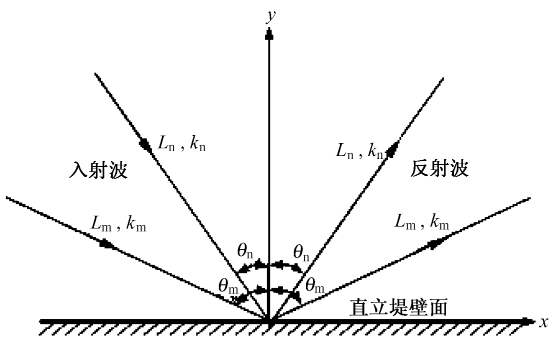

图 1　双色双向波入射、反射直立堤

现在，可将短峰波的控制方程及其边界条件无量纲化为（省略无量纲符号）

$$\nabla^2\phi = 0 \quad (-h \leqslant z \leqslant \zeta), \tag{3}$$

$$\zeta + \omega\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) = 0 \quad (z = \zeta), \tag{4}$$

$$\phi_z - \omega\zeta_t - (\phi_x\zeta_x + \phi_y\zeta_y) = 0 \quad (z = \zeta), \tag{5}$$

$$\phi_z = 0 \quad (z = -h), \tag{6}$$

其中，$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$，$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

将 ϕ,ζ 和 ω 进行如下摄动展开：

$$\phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots,$$

$$\zeta = \varepsilon\zeta_1 + \varepsilon^2\zeta_2 + \cdots,$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots. \tag{7}$$

将自由表面速度势在 $z = 0$ 处进行 Taylor 展开，可得下列各阶（r = 1,2,3）控制方程组，从而可依次求解.

$$\nabla^2\phi_r = 0 \quad (-h \leqslant z \leqslant 0), \tag{8}$$

$$\zeta_r + \omega_0\phi_{rt} = E_r \quad (z = 0), \tag{9}$$

$$\phi_{rz} + \omega_0^2\phi_{rtt} = F_r \quad (z = 0), \tag{10}$$

$$\phi_{rz} = 0 \quad (z = -h), \tag{11}$$

其中，

$$E_1 = F_1 = 0, \tag{12}$$

$$E_2 = -\omega_1\phi_{1t} - \omega_0\phi_{1tz}\zeta_1 - \frac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2 + \phi_{1z}^2), \tag{13}$$

$$F_2 = -2\omega_0\omega_1\phi_{1tt} - 2\omega_0 \nabla\phi_1 \cdot \nabla\phi_{1t} - \zeta_1(\phi_{1zz} + \omega_0^2\phi_{1ztt}), \tag{14}$$

$$\begin{aligned} E_3 = &-\omega_2\phi_{1t} - \omega_1(\phi_{2t} + \phi_{1zt}\zeta_1) \\ &-\omega_0\left(\phi_{1zt}\zeta_2 + \phi_{2zt}\zeta_1 + \frac{1}{2}\phi_{1zzt}\zeta_1^2\right) \\ &-\nabla\phi_1 \cdot \nabla\phi_2 - \zeta_1 \nabla\phi_1 \cdot \nabla\phi_{1z}, \end{aligned} \tag{15}$$

$$\begin{aligned} F_3 = &-2\omega_2\omega_0\phi_{1tt} \\ &-2\omega_1[\omega_0(\phi_{2tt} + \phi_{1ztt}\zeta_1) + \nabla\phi_1 \cdot \nabla\phi_{1t}] \\ &-2\omega_0[\nabla\phi_1 \cdot \nabla\phi_{2t} + \nabla\phi_2 \cdot \nabla\phi_{1t} \\ &+\zeta_1(\nabla\phi_{1z} \cdot \nabla\phi_{1t} + \nabla\phi_1 \cdot \nabla\phi_{1zt})] \\ &-\omega_1^2\phi_{1tt} - \omega_0^2\left(\zeta_1\phi_{2ztt} + \zeta_2\phi_{1ztt} + \frac{1}{2}\zeta_1^2\phi_{1zztt}\right) \\ &-\left(\zeta_1\phi_{2zz} + \zeta_2\phi_{1zz} + \frac{1}{2}\zeta_1^2\phi_{1zzz}\right) \\ &-\frac{1}{2}\nabla\phi_1 \cdot \nabla[(\nabla\phi_1)^2]. \end{aligned} \tag{16}$$

3. 三阶解

3.1. 第一阶解

$$\zeta_1 = a_m\cos\alpha_m + b_m\cos\beta_m + a_n\cos\alpha_n + b_n\cos\beta_n, \tag{17}$$

$$\begin{aligned} \phi_1 = &\frac{\cosh k_m Z}{\omega_m \cosh k_m h}(a_m\sin\alpha_m + b_m\sin\beta_m) \\ &+ \frac{\cosh k_n Z}{\omega_n \cosh k_n h}(a_n\sin\alpha_n + b_n\sin\beta_n) \\ &(Z = z + h), \end{aligned} \tag{18}$$

$$\omega_0^2 = \tanh h, \tag{19}$$

其中，

$$\alpha_i = k_i(i_1 x - i_2 y) - \frac{\omega_i}{\omega_0}t,$$

$$\beta_i = k_i(i_1 x + i_2 y) - \frac{\omega_i}{\omega_0}t,$$

$$\omega_i^2 = k_i \tanh k_i h. \tag{20}$$

显见，第一阶解呈一对称形式，两列波各自独立地线性叠加，蕴含着单色单向短峰波第一阶解[1]. 为确定该短峰波高阶解的某些系数而针对特定二维立波运动所得到的若干附加条件[11]，应该也

是确定该双色双向波高阶解相应某些系数的附加条件.这里只需对这些附加条件中的振幅方程做出局部修正,即

$$\int_{-h}^{0}\int_{0}^{\pi}\int_{0}^{2\pi}\phi_s(y,z,t)\sin t\cos y\mathrm{d}t\mathrm{d}y\mathrm{d}z$$
$$=-2c_s\pi^2(\tanh h)^{1/2},\tag{21}$$

其中,ϕ_s 和 c_s 依次表征立波的速度势和振幅.最后,为保证该双色双向波高阶解的唯一性,需满足如下条件[11]:

$$\frac{p\tanh ph}{\tanh h}\neq j^2\quad(p\geqslant 2,\ j\geqslant 1).\tag{22}$$

3.2. 第二阶解

$$\begin{aligned}\zeta_2 =\ & G_{2m}(a_m^2\cos2\alpha_m+b_m^2\cos2\beta_m)\\
&+G_{2n}(a_n^2\cos2\alpha_n+b_n^2\cos2\beta_n)\\
&+G_{mm}^+a_mb_m\cos(\alpha_m+\beta_m)\\
&+G_{nn}^+a_nb_n\cos(\alpha_n+\beta_n)\\
&+G_{mm}^-a_mb_m\cos(\alpha_m-\beta_m)\\
&+G_{nn}^-a_nb_n\cos(\alpha_n-\beta_n)\\
&+G_{mn}^+[a_ma_n\cos(\alpha_m+\alpha_n)\\
&+b_mb_n\cos(\beta_m+\beta_n)]\\
&+G_{mn}^-[a_ma_n\cos(\alpha_m-\alpha_n)\\
&+b_mb_n\cos(\beta_m-\beta_n)]\\
&+G_{nm}^+[a_nb_m\cos(\alpha_n+\beta_m)\\
&+b_na_m\cos(\beta_n+\alpha_m)]\\
&+G_{nm}^-[a_nb_m\cos(\alpha_n-\beta_m)\\
&+b_na_m\cos(\beta_n-\alpha_m)],\end{aligned}\tag{23}$$

$$\begin{aligned}\phi_2 =\ & \beta_0+\alpha_0t+F_{2m}(a_m^2\sin2\alpha_m\\
&+b_m^2\sin2\beta_m)\cosh k_{2m}Z\\
&+F_{2n}(a_n^2\sin2\alpha_n+b_n^2\sin2\beta_n)\cosh k_{2n}Z\\
&+F_{mm}^+a_mb_m\sin(\alpha_m+\beta_m)\cosh k_{mm}Z\\
&+F_{nn}^+a_nb_n\sin(\alpha_n+\beta_n)\cosh k_{nn}Z\\
&+F_{mn}^+[a_ma_n\sin(\alpha_m+\alpha_n)\\
&+b_mb_n\sin(\beta_m+\beta_n)]\cosh k_{mn}^+Z\\
&+F_{mn}^-[a_ma_n\sin(\alpha_m-\alpha_n)\\
&+b_mb_n\sin(\beta_m-\beta_n)]\cosh k_{mn}^-Z\\
&+F_{nm}^+[a_nb_m\sin(\alpha_n+\beta_m)\\
&+b_na_m\sin(\beta_n+\alpha_m)]\cosh k_{nm}^+Z\\
&+F_{nm}^-[a_nb_m\sin(\alpha_n-\beta_m)\\
&+b_na_m\sin(\beta_n-\alpha_m)]\cosh k_{nm}^-Z,\end{aligned}\tag{24}$$

$$\omega_1=0,\tag{25}$$

其中,β_0 为一常量.各传递函数和有关的波数如下(对于具有同等表达形式的两个对称项,仅列出涉及下标 m 的项):

$$\alpha_0=\frac{G_{00}^m(a_m^2+b_m^2)+G_{00}^n(a_n^2+b_n^2)}{\omega_0},\tag{26}$$

$$G_{00}^m=\Lambda_{21}(\omega_m,\gamma_m^+),\tag{27}$$

$$G_{2m}=\Lambda_{22}(\omega_m,\gamma_m^+)+2\omega_mF_{2m}\cosh k_{2m}h,\tag{28}$$

$$G_{mm}^+=\Lambda_{22}(\omega_m,\gamma_m^-)+2\omega_mF_{2m}\cosh k_{2m}h,\tag{29}$$

$$G_{mm}^-=\Lambda_{21}(\omega_m,\gamma_m^-),\tag{30}$$

$$\begin{aligned}G_{mn}^\pm=\ &\Lambda_{23}(\omega_m,\pm\omega_n,\gamma_{mn}^+)\\&+(\omega_m\pm\omega_n)F_{mn}^\pm\cosh k_{mn}^\pm h,\end{aligned}\tag{31}$$

$$\begin{aligned}G_{nm}^\pm=\ &\Lambda_{23}(\pm\omega_n,\omega_m,\gamma_{nm}^-)\\&+(\omega_n\pm\omega_m)F_{nm}^\pm\cosh k_{nm}^\pm h,\end{aligned}\tag{32}$$

$$F_{2m}=\frac{A_{2m}}{2k_{2m}\sinh k_{2m}h-8\omega_m^2\cosh k_{2m}h},\tag{33}$$

$$F_{mm}^+=\frac{A_{mm}^+}{k_{mm}^+\sinh k_{mm}^+h-4\omega_m^2\cosh k_{mm}^+h},\tag{34}$$

$$F_{mn}^\pm=\frac{A_{mn}^\pm}{k_{mn}^\pm\sinh k_{mn}^\pm h-(\omega_m\pm\omega_n)^2\cosh k_{mn}^\pm h},\tag{35}$$

$$F_{nm}^\pm=\frac{A_{nm}^\pm}{k_{nm}^\pm\sinh k_{nm}^\pm h-(\omega_n\pm\omega_m)^2\cosh k_{nm}^\pm h},\tag{36}$$

$$k_{2m}=2k_m,\tag{37}$$

$$k_{mm}^+=2k_mm_1,\tag{38}$$

$$k_{mn}^\pm=\sqrt{k_m^2+k_n^2\pm2\gamma_{mn}^+},\tag{39}$$

$$k_{nm}^\pm=\sqrt{k_m^2+k_n^2\pm2\gamma_{nm}^-},\tag{40}$$

其中,所引入的函数为

$$A_{2m}=\Gamma_{22}(\omega_m,k_m,\gamma_m^+),\tag{41}$$

$$A_{mm}^+=\Gamma_{22}(\omega_m,k_m,\gamma_m^-),\tag{42}$$

$$\Gamma_{22}(\omega_m,k_m,\gamma_m^+)=\frac{3\omega_m^4-k_m^2-2\gamma_m^+}{\omega_m},\tag{43}$$

$$\gamma_m^\pm=k_m^2(m_1^2\pm m_2^2),\tag{44}$$

$$A_{mn}^\pm=\Gamma_{23}(\omega_m,\pm\omega_n,k_m,\pm k_n,\pm\gamma_{mn}^+),\tag{45}$$

$$A_{nm}^\pm=\pm\Gamma_{23}(\pm\omega_n,\omega_m,\pm k_n,k_m,\pm\gamma_{nm}^-),\tag{46}$$

$$\begin{aligned}&\Gamma_{23}(\omega_m,\omega_n,k_m,k_n,\gamma_{mn}^+)\\
&=\omega_m\omega_n(\omega_m+\omega_n)+\frac{1}{2}\left(\frac{\omega_m^4-k_m^2}{\omega_m}+\frac{\omega_n^4-k_n^2}{\omega_n}\right)\\
&\quad-\gamma_{mn}^+\left(\frac{1}{\omega_m}+\frac{1}{\omega_n}\right),\end{aligned}\tag{47}$$

$$\gamma_{mn}^{\pm} = \gamma_{nm}^{\pm} = k_m k_n (m_1 n_1 \pm m_2 n_2), \tag{48}$$

$$\Lambda_{21}(\omega_m, \gamma_m^{+}) = \frac{\omega_m^4 - \gamma_m^{+}}{4\omega_m^2}, \tag{49}$$

$$\Lambda_{22}(\omega_m, \gamma_m^{+}) = \frac{3\omega_m^4 - \gamma_m^{+}}{4\omega_m^2}, \tag{50}$$

$$\begin{aligned} &\Lambda_{23}(\omega_m, \omega_n, \gamma_{mn}^{+}) \\ &= \frac{1}{2}\left(\omega_m^2 + \omega_n^2 + \omega_m\omega_n - \frac{\gamma_{mn}^{+}}{\omega_m\omega_n}\right). \end{aligned} \tag{51}$$

可以看出，第二阶解——ζ_2 与 ϕ_2 相互对称地存在对应项。通过四种基本位相 $\alpha_m, \beta_m, \alpha_n, \beta_n$，两列波以下述四种形式组合在一起：同波同相和：$2\alpha_m$，$2\beta_m, 2\alpha_n, 2\beta_n$；同波异相和差：$\alpha_m \pm \beta_m, \alpha_n \pm \beta_n$；异波同函数相和差：$\alpha_m \pm \alpha_n, \beta_m \pm \beta_n$；异波异函数相和差：$\alpha_n \pm \beta_m$；$\alpha_m \pm \beta_n$.

实际上，这四种基本的第二阶位相组合可预先判知，并可称为对称性、位相决定第二阶波.

3.3. 第三阶解

$$\begin{aligned}
\zeta_3 = {} & G_{m0} a_m \cos\alpha_m + G_{n0} a_n \cos\alpha_n + G_{0m} b_m \cos\beta_m \\
& + G_{0n} b_n \cos\beta_n + G_{3m}(a_m^3 \cos 3\alpha_m + b_m^3 \cos 3\beta_m) \\
& + G_{3n}(a_n^3 \cos 3\alpha_n + b_n^3 \cos 3\beta_n) \\
& + G_{m2m}^{-}[a_m b_m^2 \cos(\alpha_m - 2\beta_m) \\
& + b_m a_m^2 \cos(\beta_m - 2\alpha_m)] \\
& + G_{n2n}^{-}[a_n b_n^2 \cos(\alpha_n - 2\beta_n) \\
& + b_n a_n^2 \cos(\beta_n - 2\alpha_n)] \\
& + G_{2mm}^{+}[a_m^2 b_m \cos(2\alpha_m + \beta_m) \\
& + b_m^2 a_m \cos(2\beta_m + \alpha_m)] \\
& + G_{2nn}^{+}[a_n^2 b_n \cos(2\alpha_n + \beta_n) \\
& + b_n^2 a_n \cos(2\beta_n + \alpha_n)] \\
& + G_{m2n}^{+}[a_m a_n^2 \cos(\alpha_m + 2\alpha_n) \\
& + b_m b_n^2 \cos(\beta_m + 2\beta_n)] \\
& + G_{m2n}^{-}[a_m a_n^2 \cos(\alpha_m - 2\alpha_n) \\
& + b_m b_n^2 \cos(\beta_m - 2\beta_n)] \\
& + G_{2mn}^{+}[a_m^2 a_n \cos(2\alpha_m + \alpha_n) \\
& + b_m^2 b_n \cos(2\beta_m + \beta_n)] \\
& + G_{2mn}^{-}[a_m^2 a_n \cos(2\alpha_m - \alpha_n) \\
& + b_m^2 b_n \cos(2\beta_m - \beta_n)] \\
& + G_{n2m}^{+}[a_n b_m^2 \cos(\alpha_n + 2\beta_m) \\
& + b_n a_m^2 \cos(\beta_n + 2\alpha_m)] \\
& + G_{n2m}^{-}[a_n b_m^2 \cos(\alpha_n - 2\beta_m) \\
& + b_n a_m^2 \cos(\beta_n - 2\alpha_m)] \\
& + G_{2nm}^{+}[a_n^2 b_m \cos(2\alpha_n + \beta_m) \\
& + b_n^2 a_m \cos(2\beta_n + \alpha_m)] \\
& + G_{2nm}^{-}[a_n^2 b_m \cos(2\alpha_n - \beta_m) \\
& + b_n^2 a_m \cos(2\beta_n - \alpha_m)] \\
& + G_{mnm}^{+}[a_m a_n b_m \cos(\alpha_m + \alpha_n - \beta_m) \\
& + b_m b_n a_m \cos(\beta_m + \beta_n - \alpha_m)] \\
& + G_{mnm}^{-}[a_m a_n b_m \cos(\alpha_m - \alpha_n - \beta_m) \\
& + b_m b_n a_m \cos(\beta_m - \beta_n - \alpha_m)] \\
& + G_{mnn}^{+}[a_m a_n b_n \cos(\alpha_m + \alpha_n - \beta_n) \\
& + b_m b_n a_n \cos(\beta_m + \beta_n - \alpha_n)] \\
& + G_{mnn}^{-}[a_m a_n b_n \cos(\alpha_m - \alpha_n + \beta_n) \\
& + b_m b_n a_n \cos(\beta_m - \beta_n + \alpha_n)] \\
& + G_{mmn}^{+}[a_m b_m b_n \cos(\alpha_m + \beta_m + \beta_n) \\
& + a_m b_m a_n \cos(\alpha_m + \beta_m + \alpha_n)] \\
& + G_{mmn}^{-}[a_m b_m b_n \cos(\alpha_m + \beta_m - \beta_n) \\
& + a_m b_m a_n \cos(\alpha_m + \beta_m - \alpha_n)] \\
& + G_{nmn}^{+}[a_n b_m b_n \cos(\alpha_n + \beta_m \\
& + \beta_n) + a_n a_m b_n \cos(\alpha_n + \alpha_m + \beta_n)] \\
& + G_{nmn}^{-}[a_n b_m b_n \cos(\alpha_n - \beta_m + \beta_n) \\
& + a_n a_m b_n \cos(\alpha_n - \alpha_m + \beta_n)],
\end{aligned} \tag{52}$$

$$\begin{aligned}
\phi_3 = {} & F_{3m}(a_m^3 \sin 3\alpha_m + b_m^3 \sin 3\beta_m) \cosh 3k_m Z \\
& + F_{3n}(a_n^3 \sin 3\alpha_n + b_n^3 \sin 3\beta_n) \cosh 3k_n Z \\
& + F_{m2m}^{-}[a_m b_m^2 \sin(\alpha_m - 2\beta_m) \\
& + b_m a_m^2 \sin(\beta_m - 2\alpha_m)] \cosh k_{m2m}^{-} Z \\
& + F_{n2n}^{-}[a_n b_n^2 \sin(\alpha_n - 2\beta_n) \\
& + b_n a_n^2 \sin(\beta_n - 2\alpha_n)] \cosh k_{n2n}^{-} Z \\
& + F_{2mm}^{+}[a_m^2 b_m \sin(2\alpha_m + \beta_m) \\
& + b_m^2 a_m \sin(2\beta_m + \alpha_m)] \cosh k_{2mm}^{+} Z \\
& + F_{2nn}^{+}[a_n^2 b_n \sin(2\alpha_n + \beta_n) \\
& + b_n^2 a_n \sin(2\beta_n + \alpha_n)] \cosh k_{2nn}^{+} Z \\
& + F_{m2n}^{+}[a_m a_n^2 \sin(\alpha_m + 2\alpha_n) \\
& + b_m b_n^2 \sin(\beta_m + 2\beta_n)] \cosh k_{m2n}^{+} Z \\
& + F_{m2n}^{-}[a_m a_n^2 \sin(\alpha_m - 2\alpha_n) \\
& + b_m b_n^2 \sin(\beta_m - 2\beta_n)] \cosh k_{m2n}^{-} Z \\
& + F_{2mn}^{+}[a_m^2 a_n \sin(2\alpha_m + \alpha_n)
\end{aligned}$$

$$+ b_m^2 b_n \sin(2\beta_m + \beta_n)]\cosh k_{2mn}^+ Z$$
$$+ F_{2mn}^-[a_m^2 a_n \sin(2\alpha_m - \alpha_n)$$
$$+ b_m^2 b_n \sin(2\beta_m - \beta_n)]\cosh k_{2mn}^- Z$$
$$+ F_{n2m}^+[a_n b_m^2 \sin(\alpha_n + 2\beta_m)$$
$$+ b_n a_m^2 \sin(\beta_n + 2\alpha_m)]\cosh k_{n2m}^+ Z$$
$$+ F_{n2m}^-[a_n b_m^2 \sin(\alpha_n - 2\beta_m)$$
$$+ b_n a_m^2 \sin(\beta_n - 2\alpha_m)]\cosh k_{n2m}^- Z$$
$$+ F_{2nm}^+[a_n^2 b_m \sin(2\alpha_n + \beta_m)$$
$$+ b_n^2 a_m \sin(2\beta_n + \alpha_m)]\cosh k_{2nm}^+ Z$$
$$+ F_{2nm}^-[a_n^2 b_m \sin(2\alpha_n - \beta_m)$$
$$+ b_n^2 a_m \sin(2\beta_n - \alpha_m)]\cosh k_{2nm}^- Z$$
$$+ F_{mnm}^+[a_m a_n b_m \sin(\alpha_m + \alpha_n - \beta_m)$$
$$+ b_m b_n a_m \sin(\beta_m + \beta_n - \alpha_m)]\cosh k_{mnm}^+ Z$$
$$+ F_{mnm}^-[a_m a_n b_m \sin(\alpha_m - \alpha_n - \beta_m)$$
$$+ b_m b_n a_m \sin(\beta_m - \beta_n - \alpha_m)]\cosh k_{mnm}^- Z$$
$$+ F_{mnn}^+[a_m a_n b_n \sin(\alpha_m + \alpha_n - \beta_n)$$
$$+ b_m b_n a_n \sin(\beta_m + \beta_n - \alpha_n)]\cosh k_{mnn}^+ Z$$
$$+ F_{mnn}^-[a_m a_n b_n \sin(\alpha_m - \alpha_n + \beta_n)$$
$$+ b_m b_n a_n \sin(\beta_m - \beta_n + \alpha_n)]\cosh k_{mnn}^- Z$$
$$+ F_{mmn}^+[a_m b_m b_n \sin(\alpha_m + \beta_m + \beta_n)$$
$$+ a_m b_m a_n \sin(\alpha_m + \beta_m + \alpha_n)]\cosh k_{mmn}^+ Z$$
$$+ F_{mmn}^-[a_m b_m b_n \sin(\alpha_m + \beta_m - \beta_n)$$
$$+ a_m b_m a_n \sin(\alpha_m + \beta_m - \alpha_n)]\cosh k_{mmn}^- Z$$
$$+ F_{nmn}^+[a_n b_m b_n \sin(\alpha_n + \beta_m + \beta_n)$$
$$+ a_n a_m b_n \sin(\alpha_n + \alpha_m + \beta_n)]\cosh k_{nmn}^+ Z$$
$$+ F_{nmn}^-[a_n b_m b_n \sin(\alpha_n - \beta_m + \beta_n)$$
$$+ a_n a_m b_n \sin(\alpha_n - \alpha_m + \beta_n)]\cosh k_{nmn}^- Z, \quad (53)$$

$$\omega_{21} = -\frac{\omega_0 a_m^2}{2\omega_m}\Gamma_{36}(F_{2m}, G_{2m}, k_{2m}, k_m, \omega_m)$$
$$-\frac{\omega_0 b_m^2}{2\omega_m}\Gamma_{37}(F_{mm}^+, G_{mm}^+, G_{mm}^-, k_{mm}^+,$$
$$k_m, \omega_m, \gamma_m^-, m_1)$$
$$-\frac{\omega_0 a_n^2}{2\omega_m}\Gamma_{38}(F_{mn}^+, F_{mn}^-, G_{mn}^+, G_{mn}^-,$$
$$k_{mn}^+, k_{mn}^-, k_n, k_m, \omega_n, \omega_m, \gamma_{mn}^+, \gamma_n^+) -$$
$$\frac{\omega_0 b_n^2}{2\omega_m}\Gamma_{38}(F_{nm}^+, -F_{nm}^-, G_{nm}^+, G_{nm}^-,$$
$$k_{nm}^+, k_{nm}^-, k_n, k_m, \omega_n, \omega_m, \gamma_{nm}^-, \gamma_n^+), \quad (54)$$

$$\omega_{22} = -\frac{\omega_0 a_m^2}{2\omega_m}\Gamma_{37}(F_{mm}^+, G_{mm}^+, G_{mm}^-,$$
$$k_{mm}^+, k_m, \omega_m, \gamma_m^-, m_1)$$
$$-\frac{\omega_0 b_m^2}{2\omega_m}\Gamma_{36}(F_{2m}, G_{2m}, k_{2m}, k_m, \omega_m)$$
$$-\frac{\omega_0 a_n^2}{2\omega_m}\Gamma_{38}(F_{nm}^+, -F_{nm}^-, G_{nm}^+, G_{nm}^-,$$
$$k_{nm}^+, k_{nm}^-, k_n, k_m, \omega_n, \omega_m, \gamma_{nm}^-, \gamma_n^+)$$
$$-\frac{\omega_0 b_n^2}{2\omega_m}\Gamma_{38}(F_{mn}^+, F_{mn}^-, G_{mn}^+, G_{mn}^-,$$
$$k_{mn}^+, k_{mn}^-, k_n, k_m, \omega_n, \omega_m, \gamma_{mn}^+, \gamma_n^+), \quad (55)$$

$$\omega_{23} = -\frac{\omega_0 a_m^2}{2\omega_n}\Gamma_{38}(F_{mn}^+, F_{mn}^-, G_{mn}^+, G_{mn}^-,$$
$$k_{mn}^+, k_{mn}^-, k_m, k_n, \omega_m, \omega_n, \gamma_{mn}^+, \gamma_m^+)$$
$$-\frac{\omega_0 b_m^2}{2\omega_n}\Gamma_{38}(F_{nm}^+, -F_{nm}^-, G_{nm}^+, G_{nm}^-,$$
$$k_{nm}^+, k_{nm}^-, k_m, k_n, \omega_m, \omega_n, \gamma_{nm}^-, \gamma_m^+)$$
$$-\frac{\omega_0 a_n^2}{2\omega_n}\Gamma_{36}(F_{2n}, G_{2n}, k_{2n}, k_n, \omega_n)$$
$$-\frac{\omega_0 b_n^2}{2\omega_n}\Gamma_{37}(F_{nn}^+, G_{nn}^+, G_{nn}^-,$$
$$k_{nn}^+, k_n, \omega_n, \gamma_n^-, n_1), \quad (56)$$

$$\omega_{24} = -\frac{\omega_0 a_m^2}{2\omega_n}\Gamma_{38}(F_{nm}^+, -F_{nm}^-, G_{nm}^+, G_{nm}^-,$$
$$k_{nm}^+, k_{nm}^-, k_m, k_n, \omega_m, \omega_n, \gamma_{nm}^-, \gamma_m^+)$$
$$-\frac{\omega_0 b_m^2}{2\omega_n}\Gamma_{38}(F_{mn}^+, F_{mn}^-, G_{mn}^+, G_{mn}^-,$$
$$k_{mn}^+, k_{mn}^-, k_m, k_n, \omega_m, \omega_n, \gamma_{mn}^+, \gamma_m^+)$$
$$-\frac{\omega_0 a_n^2}{2\omega_n}\Gamma_{37}(F_{nn}^+, G_{nn}^+, G_{nn}^-,$$
$$k_{nn}^+, k_n, \omega_n, \gamma_n^-, n_1)$$
$$-\frac{\omega_0 b_n^2}{2\omega_n}\Gamma_{36}(F_{2n}, G_{2n}, k_{2n}, k_n, \omega_n), \quad (57)$$

其中,各传递函数和有关的波数如下(同样,对于具有同等表达形式的两个对称项,仅列出涉及下标 m 的项):

$$G_{m0} = \frac{\omega_2}{\omega_0} + a_m^2 \Lambda_{31}(F_{2m}, G_{2m}, k_{2m}, k_m, \omega_m)$$
$$+ b_m^2 \Lambda_{32}(F_{mm}^+, G_{mm}^+, G_{mm}^-, k_{mm}^+, k_m, \omega_m, m_1)$$
$$+ a_n^2 \Lambda_{33}(F_{mn}^+, F_{mn}^-, G_{mn}^+, G_{mn}^-,$$
$$k_{mn}^+, k_{mn}^-, \omega_m, \omega_n, \gamma_{mn}^+, \gamma_m^+, \gamma_n^+)$$

$$
\begin{aligned}
&+ b_{\mathrm{n}}^2\Lambda_{33}(F^+_{\mathrm{nm}}, -F^-_{\mathrm{nm}}, G^+_{\mathrm{nm}}, G^-_{\mathrm{nm}},\\
&k^+_{\mathrm{nm}}, k^-_{\mathrm{nm}}, \omega_{\mathrm{m}}, \omega_{\mathrm{n}}, \gamma^-_{\mathrm{nm}}, \gamma^+_{\mathrm{m}}, \gamma^+_{\mathrm{n}}),
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
G_{0\mathrm{m}} ={}& \frac{\omega_2}{\omega_0} + a_{\mathrm{m}}^2\Lambda_{32}(F^+_{\mathrm{mm}}, G^+_{\mathrm{mm}}, G^-_{\mathrm{mm}},\\
&k^+_{\mathrm{mm}}, k_{\mathrm{m}}, \omega_{\mathrm{m}}, m_1)\\
&+ b_{\mathrm{m}}^2\Lambda_{31}(F_{2\mathrm{m}}, G_{2\mathrm{m}}, k_{2\mathrm{m}}, k_{\mathrm{m}}, \omega_{\mathrm{m}})\\
&+ a_{\mathrm{n}}^2\Lambda_{33}(F^+_{\mathrm{nm}}, -F^-_{\mathrm{nm}}, G^+_{\mathrm{nm}}, G^-_{\mathrm{nm}},\\
&k^+_{\mathrm{nm}}, k^-_{\mathrm{nm}}, \omega_{\mathrm{m}}, \omega_{\mathrm{n}}, \gamma^-_{\mathrm{nm}}, \gamma^+_{\mathrm{m}}, \gamma^+_{\mathrm{n}})\\
&+ b_{\mathrm{n}}^2\Lambda_{33}(F^+_{\mathrm{mn}}, F^-_{\mathrm{mn}}, G^+_{\mathrm{mn}}, G^-_{\mathrm{mn}},\\
&k^+_{\mathrm{mn}}, k^-_{\mathrm{mn}}, \omega_{\mathrm{m}}, \omega_{\mathrm{n}}, \gamma^+_{\mathrm{mn}}, \gamma^+_{\mathrm{m}}, \gamma^+_{\mathrm{n}}),
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
G_{3\mathrm{m}} ={}& \frac{3}{2}F_{2\mathrm{m}}k_{2\mathrm{m}}\omega_{\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&- \frac{1}{4\omega_{\mathrm{m}}}F_{2\mathrm{m}}k_{2\mathrm{m}}^2\cosh k_{2\mathrm{m}}h + 3\omega_{\mathrm{m}}F_{3\mathrm{m}}\cosh 3k_{\mathrm{m}}h\\
&+ \frac{1}{2}G_{2\mathrm{m}}\omega_{\mathrm{m}}^2 + \frac{1}{8}\gamma^+_{\mathrm{m}},
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
G^-_{\mathrm{m}2\mathrm{m}} ={}& \frac{1}{2}F_{2\mathrm{m}}k_{2\mathrm{m}}\omega_{\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&- \frac{\gamma^-_{\mathrm{m}}}{\omega_{\mathrm{m}}}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h - \omega_{\mathrm{m}}F^-_{\mathrm{m}2\mathrm{m}}\cosh k^-_{\mathrm{m}2\mathrm{m}}h\\
&+ \frac{1}{2}(G_{2\mathrm{m}} + G^-_{\mathrm{mm}})\omega_{\mathrm{m}}^2\\
&+ \frac{1}{8}k_{\mathrm{m}}^2(3 - 8m_1^2),
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
G^+_{2\mathrm{mm}} ={}& \frac{3}{2}F_{2\mathrm{m}}k_{2\mathrm{m}}\omega_{\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&- \frac{\gamma^-_{\mathrm{m}}}{\omega_{\mathrm{m}}}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h + \frac{3}{2}F^+_{\mathrm{mm}}k^+_{\mathrm{mm}}\omega_{\mathrm{m}}\sinh k^+_{\mathrm{mm}}h\\
&- \frac{(k^+_{\mathrm{mm}})^2}{4\omega_{\mathrm{m}}}F^+_{\mathrm{mm}}\cosh k^+_{\mathrm{mm}}h\\
&+ 3\omega_{\mathrm{m}}F^+_{2\mathrm{mm}}\cosh k^+_{2\mathrm{mm}}h\\
&+ \frac{1}{2}(G_{2\mathrm{m}} + G^+_{\mathrm{mm}})\omega_{\mathrm{m}}^2\\
&+ \frac{1}{8}k_{\mathrm{m}}^2(11 - 8m_1^2),
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
G^\pm_{\mathrm{m}2\mathrm{n}} ={}& \Lambda_{34}(\pm F_{2\mathrm{n}}, F^\pm_{\mathrm{mn}}, G_{2\mathrm{n}}, G^\pm_{\mathrm{mn}}, \pm k_{2\mathrm{n}},\\
&k^\pm_{\mathrm{mn}}, k_{\mathrm{m}}, \pm k_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \pm\gamma^+_{\mathrm{mn}}, \gamma^+_{\mathrm{n}})\\
&+ (\omega_{\mathrm{m}} \pm 2\omega_{\mathrm{n}})F^\pm_{\mathrm{m}2\mathrm{n}}\cosh k^\pm_{\mathrm{m}2\mathrm{n}}h,
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
G^\pm_{2\mathrm{mn}} ={}& \Lambda_{34}(F_{2\mathrm{m}}, F^\pm_{\mathrm{mn}}, G_{2\mathrm{m}}, G^\pm_{\mathrm{mn}}, k_{2\mathrm{m}}, k^\pm_{\mathrm{mn}},\\
&\pm k_{\mathrm{n}}, k_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\gamma^+_{\mathrm{mn}}, \gamma^+_{\mathrm{m}})\\
&+ (2\omega_{\mathrm{m}} \pm \omega_{\mathrm{n}})F^\pm_{2\mathrm{mn}}\cosh k^\pm_{2\mathrm{mn}}h,
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
G^\pm_{\mathrm{n}2\mathrm{m}} ={}& \Lambda_{34}(F_{2\mathrm{m}}, \pm F^\pm_{\mathrm{nm}}, G_{2\mathrm{m}}, G^\pm_{\mathrm{nm}}, k_{2\mathrm{m}}, k^\pm_{\mathrm{nm}},\\
&\pm k_{\mathrm{n}}, k_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\gamma^-_{\mathrm{nm}}, \gamma^+_{\mathrm{m}})\\
&+ (\omega_{\mathrm{n}} \pm 2\omega_{\mathrm{m}})F^\pm_{\mathrm{n}2\mathrm{m}}\cosh k^\pm_{\mathrm{n}2\mathrm{m}}h,
\end{aligned}
\tag{65}
$$

$$
\begin{aligned}
G^\pm_{2\mathrm{nm}} ={}& \Lambda_{34}(\pm F_{2\mathrm{n}}, \pm F^\pm_{\mathrm{nm}}, G_{2\mathrm{n}}, G^\pm_{\mathrm{nm}}, \pm k_{2\mathrm{n}},\\
&k^\pm_{\mathrm{nm}}, k_{\mathrm{m}}, \pm k_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \pm\gamma^-_{\mathrm{nm}}, \gamma^+_{\mathrm{n}})\\
&+ (2\omega_{\mathrm{n}} \pm \omega_{\mathrm{m}})F^\pm_{2\mathrm{nm}}\cosh k^\pm_{2\mathrm{nm}}h,
\end{aligned}
\tag{66}
$$

$$
\begin{aligned}
G^\pm_{\mathrm{mnm}} ={}& \Lambda_{35}(F^\pm_{\mathrm{mn}}, \mp F^\mp_{\mathrm{nm}}, G^-_{\mathrm{mm}}, G^\pm_{\mathrm{mn}}, G^\mp_{\mathrm{nm}}, k^\pm_{\mathrm{mn}},\\
&k^\mp_{\mathrm{nm}}, \pm k_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \pm\gamma^+_{\mathrm{mn}}, \pm\gamma^-_{\mathrm{mn}},\\
&\pm\gamma_{11}, \gamma^-_{\mathrm{m}}) \pm \omega_{\mathrm{n}}F^\pm_{\mathrm{mnm}}\cosh k^\pm_{\mathrm{mnm}}h,
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
G^\pm_{mnn} ={}& \Lambda_{35}(F^\pm_{\mathrm{mn}}, \mp F^\mp_{\mathrm{nm}}, G^-_{\mathrm{nn}}, G^\pm_{\mathrm{mn}}, G^\mp_{\mathrm{nm}},\\
&k^\pm_{\mathrm{mn}}, k^\mp_{\mathrm{nm}}, k_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\gamma^+_{\mathrm{nm}}, \pm\gamma^-_{\mathrm{nm}},\\
&\pm\gamma_{11}, \gamma^-_{\mathrm{n}}) + \omega_{\mathrm{m}}F^\pm_{\mathrm{mnn}}\cosh k^\pm_{\mathrm{mnn}}h,
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
G^\pm_{\mathrm{mmn}} ={}& \Lambda_{36}(F^+_{\mathrm{mm}}, F^\pm_{\mathrm{mn}}, \pm F^\pm_{\mathrm{nm}}, G^+_{\mathrm{mm}}, G^\pm_{\mathrm{mn}}, G^\pm_{\mathrm{nm}},\\
&k^+_{\mathrm{mm}}, k^\pm_{\mathrm{mn}}, k^\pm_{\mathrm{nm}}, k_{\mathrm{m}}, \pm k_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \pm\gamma^+_{\mathrm{mn}},\\
&\pm\gamma^-_{\mathrm{mn}}, \pm\gamma_{11}, \gamma^-_{\mathrm{m}})\\
&+ (2\omega_{\mathrm{m}} \pm \omega_{\mathrm{n}})F^\pm_{\mathrm{mmn}}\cosh k^\pm_{\mathrm{mmn}}h,
\end{aligned}
\tag{69}
$$

$$
\begin{aligned}
G^\pm_{\mathrm{nmn}} ={}& \Lambda_{36}(\pm F^+_{\mathrm{nn}}, F^\pm_{\mathrm{mn}}, \pm F^\pm_{\mathrm{nm}}, G^+_{\mathrm{nn}}, G^\pm_{\mathrm{mn}}, G^\pm_{\mathrm{nm}},\\
&\pm k^+_{\mathrm{nn}}, k^\pm_{\mathrm{mn}}, k^\pm_{\mathrm{nm}}, \pm k_{\mathrm{n}}, k_{\mathrm{m}}, \pm\omega_{\mathrm{n}}, \omega_{\mathrm{m}}, \pm\gamma^+_{\mathrm{nm}},\\
&\pm\gamma^-_{\mathrm{nm}}, \pm\gamma_{11}, \gamma^-_{\mathrm{n}})\\
&+ (2\omega_{\mathrm{n}} \pm \omega_{\mathrm{m}})F^\pm_{\mathrm{nmn}}\cosh k^\pm_{\mathrm{nmn}}h,
\end{aligned}
\tag{70}
$$

$$
F_{3\mathrm{m}} = \frac{A_{3\mathrm{m}}}{3k_{\mathrm{m}}\sinh 3k_{\mathrm{m}}h - 9\omega_{\mathrm{m}}^2\cosh 3k_{\mathrm{m}}h},
\tag{71}
$$

$$
F^-_{\mathrm{m}2\mathrm{m}} = \frac{A^-_{\mathrm{m}2\mathrm{m}}}{k^-_{\mathrm{m}2\mathrm{m}}\sinh k^-_{\mathrm{m}2\mathrm{m}}h - \omega_{\mathrm{m}}^2\cosh k^-_{\mathrm{m}2\mathrm{m}}h},
\tag{72}
$$

$$
F^+_{2\mathrm{mm}} = \frac{A^+_{2\mathrm{mm}}}{k^+_{2\mathrm{mm}}\sinh k^+_{2\mathrm{mm}}h - 9\omega_{\mathrm{m}}^2\cosh k^+_{2\mathrm{mm}}h},
\tag{73}
$$

$$
F^\pm_{\mathrm{m}2\mathrm{n}} = \frac{A^\pm_{\mathrm{m}2\mathrm{n}}}{k^\pm_{\mathrm{m}2\mathrm{n}}\sinh k^\pm_{\mathrm{m}2\mathrm{n}}h - (\omega_{\mathrm{m}} \pm 2\omega_{\mathrm{n}})^2\cosh k^\pm_{\mathrm{m}2\mathrm{n}}h},
\tag{74}
$$

$$
F^\pm_{2\mathrm{mn}} = \frac{A^\pm_{2\mathrm{mn}}}{k^\pm_{2\mathrm{mn}}\sinh k^\pm_{2\mathrm{mn}}h - (2\omega_{\mathrm{m}} \pm \omega_{\mathrm{n}})^2\cosh k^\pm_{2\mathrm{mn}}h},
\tag{75}
$$

$$
F^\pm_{\mathrm{n}2\mathrm{m}} = \frac{A^\pm_{\mathrm{n}2\mathrm{m}}}{k^\pm_{\mathrm{n}2\mathrm{m}}\sinh k^\pm_{\mathrm{n}2\mathrm{m}}h - (\omega_{\mathrm{n}} \pm 2\omega_{\mathrm{m}})^2\cosh k^\pm_{\mathrm{n}2\mathrm{m}}h},
\tag{76}
$$

$$
F^\pm_{2\mathrm{nm}} = \frac{A^\pm_{2\mathrm{nm}}}{k^\pm_{2\mathrm{nm}}\sinh k^\pm_{2\mathrm{nm}}h - (2\omega_{\mathrm{n}} \pm \omega_{\mathrm{m}})^2\cosh k^\pm_{2\mathrm{nm}}h},
\tag{77}
$$

$$
F^\pm_{\mathrm{mnm}} = \frac{A^\pm_{\mathrm{mnm}}}{k^\pm_{\mathrm{mnm}}\sinh k^\pm_{\mathrm{mnm}}h - \omega_{\mathrm{n}}^2\cosh k^\pm_{\mathrm{mnm}}h},
\tag{78}
$$

$$
F^\pm_{\mathrm{mnn}} = \frac{A^\pm_{\mathrm{mnn}}}{k^\pm_{\mathrm{mnn}}\sinh k^\pm_{\mathrm{mnn}}h - \omega_{\mathrm{m}}^2\cosh k^\pm_{\mathrm{mnn}}h},
\tag{79}
$$

$$F^{\pm}_{\mathrm{mmn}}=\frac{A^{\pm}_{\mathrm{mmn}}}{k^{\pm}_{\mathrm{mmn}}\sinh k^{\pm}_{\mathrm{mmn}}h-(2\omega_{\mathrm{m}}\pm\omega_{\mathrm{n}})^2\cosh k^{\pm}_{\mathrm{mmn}}h},\quad(80)$$

$$F^{\pm}_{\mathrm{nmn}}=\frac{A^{\pm}_{\mathrm{nmn}}}{k^{\pm}_{\mathrm{nmn}}\sinh k^{\pm}_{\mathrm{nmn}}h-(\omega_{\mathrm{m}}\pm2\omega_{\mathrm{n}})^2\cosh k^{\pm}_{\mathrm{nmn}}h},\quad(81)$$

$$k^{-}_{\mathrm{m2m}}=\sqrt{(k_{\mathrm{m}}m_1)^2+(3k_{\mathrm{m}}m_2)^2},\quad(82)$$

$$k^{+}_{\mathrm{2mm}}=\sqrt{(3k_{\mathrm{m}}m_1)^2+(k_{\mathrm{m}}m_2)^2},\quad(83)$$

$$k^{\pm}_{\mathrm{m2n}}=\sqrt{(k_{\mathrm{m}}m_1\pm2k_{\mathrm{n}}n_1)^2+(k_{\mathrm{m}}m_2\pm2k_{\mathrm{n}}n_2)^2},\quad(84)$$

$$k^{\pm}_{\mathrm{2mn}}=\sqrt{(2k_{\mathrm{m}}m_1\pm k_{\mathrm{n}}n_1)^2+(2k_{\mathrm{m}}m_2\pm k_{\mathrm{n}}n_2)^2},\quad(85)$$

$$k^{\pm}_{\mathrm{n2m}}=\sqrt{(\pm k_{\mathrm{n}}n_1+2k_{\mathrm{m}}m_1)^2+(\pm k_{\mathrm{n}}n_2-2k_{\mathrm{m}}m_2)^2},\quad(86)$$

$$k^{\pm}_{\mathrm{2nm}}=\sqrt{(\pm2k_{\mathrm{n}}n_1+k_{\mathrm{m}}m_1)^2+(\pm2k_{\mathrm{n}}n_2-k_{\mathrm{m}}m_2)^2},\quad(87)$$

$$k^{\pm}_{\mathrm{mnm}}=\sqrt{(k_{\mathrm{n}}n_1)^2+(2k_{\mathrm{m}}m_2\pm k_{\mathrm{n}}n_2)^2},\quad(88)$$

$$k^{\pm}_{\mathrm{mnn}}=\sqrt{(k_{\mathrm{m}}m_1)^2+(k_{\mathrm{m}}m_2\pm2k_{\mathrm{n}}n_2)^2},\quad(89)$$

$$k^{\pm}_{\mathrm{mmn}}=\sqrt{(2k_{\mathrm{m}}m_1\pm k_{\mathrm{n}}n_1)^2+(k_{\mathrm{n}}n_2)^2},\quad(90)$$

$$k^{\pm}_{\mathrm{nmn}}=\sqrt{(k_{\mathrm{m}}m_1\pm2k_{\mathrm{n}}n_1)^2+(k_{\mathrm{m}}m_2)^2}.\quad(91)$$

$$\begin{aligned}&\Gamma_{36}(F_{2\mathrm{m}},G_{2\mathrm{m}},k_{2\mathrm{m}},k_{\mathrm{m}},\omega_{\mathrm{m}})\\&=\omega^2_{\mathrm{m}}F_{2\mathrm{m}}k_{2\mathrm{m}}\sinh k_{2\mathrm{m}}h-k^2_{2\mathrm{m}}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h\\&\quad+\frac{k^4_{\mathrm{m}}-5k^2_{\mathrm{m}}\omega^4_{\mathrm{m}}}{4\omega^3_{\mathrm{m}}}+\frac{G_{2\mathrm{m}}(k^2_{\mathrm{m}}-\omega^4_{\mathrm{m}})}{2\omega_{\mathrm{m}}},\end{aligned}\quad(92)$$

$$\begin{aligned}&\Gamma_{37}(F^{+}_{\mathrm{mm}},G^{+}_{\mathrm{mm}},G^{-}_{\mathrm{mm}},k^{+}_{\mathrm{mm}},k_{\mathrm{m}},\omega_{\mathrm{m}},\gamma^{-}_{\mathrm{m}},m_1)\\&=\omega^2_{\mathrm{m}}F^{+}_{\mathrm{mm}}k^{+}_{\mathrm{mm}}\sinh k^{+}_{\mathrm{mm}}h\\&\quad-(k^{+}_{\mathrm{mm}})^2F^{+}_{\mathrm{mm}}\cosh k^{+}_{\mathrm{mm}}h+\frac{(\gamma^{-}_{\mathrm{m}})^2}{2\omega^3_{\mathrm{m}}}\\&\quad+\frac{1}{2}k^2_{\mathrm{m}}\omega_{\mathrm{m}}(3-8m^2_1)\\&\quad+\frac{(G^{+}_{\mathrm{mm}}-G^{-}_{\mathrm{mm}})(k^2_{\mathrm{m}}-\omega^4_{\mathrm{m}})}{2\omega_{\mathrm{m}}},\end{aligned}\quad(93)$$

$$\begin{aligned}&\Gamma_{38}(F^{+}_{\mathrm{mn}},F^{-}_{\mathrm{mn}},G^{+}_{\mathrm{mn}},G^{-}_{\mathrm{mn}},k^{+}_{\mathrm{mn}},\\&k^{-}_{\mathrm{mn}},k_{\mathrm{m}},k_{\mathrm{n}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{n}})\\&=\Gamma^{-}_{32}(F^{+}_{\mathrm{mn}},k^{+}_{\mathrm{mn}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{n}})\\&\quad+\Gamma^{-}_{32}(F^{-}_{\mathrm{mn}},k^{-}_{\mathrm{mn}},\omega_{\mathrm{m}},-\omega_{\mathrm{n}},-\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{n}})\\&\quad+\frac{(\gamma^{+}_{\mathrm{mn}})^2}{2\omega_{\mathrm{m}}\omega^2_{\mathrm{n}}}-\frac{\gamma^{+}_{\mathrm{mn}}(\omega^2_{\mathrm{m}}+\omega^2_{\mathrm{n}})}{\omega_{\mathrm{n}}}\\&\quad+\frac{(G^{+}_{\mathrm{mn}}-G^{-}_{\mathrm{mn}})(k^2_{\mathrm{n}}-\omega^4_{\mathrm{n}})}{2\omega_{\mathrm{n}}}\\&\quad+\frac{k^2_{\mathrm{m}}\omega^2_{\mathrm{n}}-2k^2_{\mathrm{n}}\omega^2_{\mathrm{m}}}{2\omega_{\mathrm{m}}},\end{aligned}\quad(94)$$

其中,所引入函数的具体表达式或新引入的函数为

$$\begin{aligned}&\Lambda_{31}(F_{2\mathrm{m}},G_{2\mathrm{m}},k_{2\mathrm{m}},k_{\mathrm{m}},\omega_{\mathrm{m}})\\&=\frac{1}{2}F_{2\mathrm{m}}k_{2\mathrm{m}}\omega_{\mathrm{m}}\sinh k_{2\mathrm{m}}h\\&\quad-\frac{1}{4\omega_{\mathrm{m}}}F_{2\mathrm{m}}k^2_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h\\&\quad+\frac{1}{2}G_{2\mathrm{m}}\omega^2_{\mathrm{m}}-\frac{5}{8}k^2_{\mathrm{m}},\end{aligned}\quad(95)$$

$$\begin{aligned}&\Lambda_{32}(F^{+}_{\mathrm{mm}},G^{+}_{\mathrm{mm}},G^{-}_{\mathrm{mm}},k^{+}_{\mathrm{mm}},k_{\mathrm{m}},\omega_{\mathrm{m}},m_1)\\&=\frac{1}{2}F^{+}_{\mathrm{mm}}k^{+}_{\mathrm{mm}}\omega_{\mathrm{m}}\sinh k^{+}_{\mathrm{mm}}h\\&\quad-\frac{1}{4\omega_{\mathrm{m}}}F^{+}_{\mathrm{mm}}(k^{+}_{\mathrm{mm}})^2\cosh k^{+}_{\mathrm{mm}}h\\&\quad+\frac{1}{2}(G^{+}_{\mathrm{mm}}+G^{-}_{\mathrm{mm}})\omega^2_{\mathrm{m}}+\frac{1}{4}k^2_{\mathrm{m}}(3-8m^2_1),\end{aligned}\quad(96)$$

$$\begin{aligned}&\Lambda_{33}(F^{+}_{\mathrm{mn}},F^{-}_{\mathrm{mn}},G^{+}_{\mathrm{mn}},G^{-}_{\mathrm{mn}},k^{+}_{\mathrm{mn}},\\&k^{-}_{\mathrm{mn}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{m}},\gamma^{+}_{\mathrm{n}})\\&=\frac{1}{2}F^{+}_{\mathrm{mn}}k^{+}_{\mathrm{mn}}\omega_{\mathrm{m}}\sinh k^{+}_{\mathrm{mn}}h-\frac{\gamma^{+}_{\mathrm{mn}}+\gamma^{+}_{\mathrm{n}}}{2\omega_{\mathrm{n}}}F^{+}_{\mathrm{mn}}\cosh k^{+}_{\mathrm{mn}}h\\&\quad+\frac{1}{2}F^{-}_{\mathrm{mn}}k^{-}_{\mathrm{mn}}\omega_{\mathrm{m}}\sinh k^{-}_{\mathrm{mn}}h-\frac{\gamma^{+}_{\mathrm{mn}}-\gamma^{+}_{\mathrm{n}}}{2\omega_{\mathrm{n}}}F^{-}_{\mathrm{mn}}\cosh k^{-}_{\mathrm{mn}}h\\&\quad+\frac{1}{2}(G^{+}_{\mathrm{mn}}+G^{-}_{\mathrm{mn}})\omega^2_{\mathrm{n}}-\frac{\gamma^{+}_{\mathrm{mn}}}{2}\left(\frac{\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}+\frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}}\right)\\&\quad+\frac{1}{4}(\gamma^{+}_{\mathrm{m}}-2\gamma^{+}_{\mathrm{n}}),\end{aligned}\quad(97)$$

$$\begin{aligned}&\Lambda_{34}(F_{2\mathrm{n}},F^{\pm}_{\mathrm{mn}},G_{2\mathrm{n}},G^{\pm}_{\mathrm{mn}},k_{2\mathrm{n}},\\&k^{\pm}_{\mathrm{mn}},k_{\mathrm{m}},k_{\mathrm{n}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{n}})\\&=\frac{(2\omega_{\mathrm{n}}+\omega_{\mathrm{m}})}{2}F_{2\mathrm{n}}k_{2\mathrm{n}}\sinh k_{2\mathrm{n}}h-\frac{\gamma^{+}_{\mathrm{mn}}}{\omega_{\mathrm{m}}}F_{2\mathrm{n}}\cosh k_{2\mathrm{n}}h\\&\quad+\frac{(\omega_{\mathrm{m}}+2\omega_{\mathrm{n}})}{2}F^{\pm}_{\mathrm{mn}}k^{\pm}_{\mathrm{mn}}\sinh k^{\pm}_{\mathrm{mn}}h\\&\quad-\frac{\gamma^{+}_{\mathrm{mn}}+\gamma^{+}_{\mathrm{n}}}{2\omega_{\mathrm{n}}}F^{\pm}_{\mathrm{mn}}\cosh k^{\pm}_{\mathrm{mn}}h\\&\quad+\frac{1}{2}(G_{2\mathrm{n}}\omega^2_{\mathrm{m}}+G^{\pm}_{\mathrm{mn}}\omega^2_{\mathrm{n}})+\frac{1}{8}(k^2_{\mathrm{m}}+2k^2_{\mathrm{n}})\\&\quad-\frac{\gamma^{+}_{\mathrm{mn}}}{4}\left(\frac{\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}+\frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}}\right)+\frac{1}{4}\left(\frac{k^2_{\mathrm{n}}\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}+\frac{k^2_{\mathrm{m}}\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}}\right),\end{aligned}\quad(98)$$

$$\begin{aligned}&\Lambda_{35}(F^{\pm}_{\mathrm{mn}},\mp F^{\mp}_{\mathrm{nm}},G^{-}_{\mathrm{mm}},G^{\pm}_{\mathrm{mn}},G^{\mp}_{\mathrm{nm}},k^{\pm}_{\mathrm{mn}},\\&k^{\mp}_{\mathrm{nm}},k_{\mathrm{n}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{-}_{\mathrm{mn}},\gamma_{11},\gamma^{-}_{\mathrm{m}})\\&=\frac{1}{2}F^{\pm}_{\mathrm{mn}}k^{\pm}_{\mathrm{mn}}\omega_{\mathrm{n}}\sinh k^{\pm}_{\mathrm{mn}}h-\frac{\gamma^{-}_{\mathrm{m}}+\gamma^{-}_{\mathrm{mn}}}{2\omega_{\mathrm{n}}}F^{\pm}_{\mathrm{mn}}\cosh k^{\pm}_{\mathrm{mn}}h\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}F^{\mp}_{\mathrm{nm}}k^{\mp}_{\mathrm{nm}}\omega_{\mathrm{n}}\sinh k^{\mp}_{\mathrm{nm}}h-\frac{\gamma^{-}_{\mathrm{m}}-\gamma^{+}_{\mathrm{mn}}}{2\omega_{\mathrm{m}}}F^{\mp}_{\mathrm{nm}}\cosh k^{\mp}_{\mathrm{nm}}h\\
&+\frac{1}{2}G^{-}_{\mathrm{mm}}\omega^{2}_{\mathrm{n}}+\frac{1}{2}(G^{\pm}_{\mathrm{mn}}+G^{\mp}_{\mathrm{nm}})\omega^{2}_{\mathrm{m}}\\
&+\frac{1}{4}(k^{2}_{\mathrm{n}}-2\gamma^{-}_{\mathrm{m}})-\frac{\gamma_{11}}{2}\left(\frac{\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}+\frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}}\right),\qquad(99)
\end{aligned}
$$

$$
\begin{aligned}
&\Lambda_{36}(F^{+}_{\mathrm{mm}},F^{\pm}_{\mathrm{mn}},\pm F^{\pm}_{\mathrm{nm}},G^{+}_{\mathrm{mm}},G^{\pm}_{\mathrm{mn}},G^{\pm}_{\mathrm{nm}},k^{+}_{\mathrm{mm}},\\
&k^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{nm}},k_{\mathrm{m}},\pm k_{\mathrm{n}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}},\gamma^{-}_{\mathrm{mn}},\gamma_{11},\gamma^{-}_{\mathrm{m}})\\
={}&\frac{2\omega_{\mathrm{m}}+\omega_{\mathrm{n}}}{2}F^{+}_{\mathrm{mm}}k^{+}_{\mathrm{mm}}\sinh k^{+}_{\mathrm{mm}}h-\frac{\gamma_{11}}{\omega_{\mathrm{n}}}F^{+}_{\mathrm{mm}}\cosh k^{+}_{\mathrm{mm}}h\\
&+\frac{2\omega_{\mathrm{m}}+\omega_{\mathrm{n}}}{2}F^{\pm}_{\mathrm{mn}}k^{\pm}_{\mathrm{mn}}\sinh k^{\pm}_{\mathrm{mn}}h\\
&-\frac{\gamma^{-}_{\mathrm{m}}+\gamma^{-}_{\mathrm{mn}}}{2\omega_{\mathrm{m}}}F^{\pm}_{\mathrm{mn}}\cosh k^{\pm}_{\mathrm{mn}}h\\
&+\frac{2\omega_{\mathrm{m}}+\omega_{\mathrm{n}}}{2}F^{\pm}_{\mathrm{nm}}k^{\pm}_{\mathrm{nm}}\sinh k^{\pm}_{\mathrm{nm}}h\\
&-\frac{\gamma^{-}_{\mathrm{m}}+\gamma^{+}_{\mathrm{mn}}}{2\omega_{\mathrm{m}}}F^{\pm}_{\mathrm{nm}}\cosh k^{\pm}_{\mathrm{nm}}h+\frac{1}{2}G^{+}_{\mathrm{mm}}\omega^{2}_{\mathrm{n}}\\
&+\frac{1}{2}(G^{\pm}_{\mathrm{mn}}+G^{\pm}_{\mathrm{nm}})\omega^{2}_{\mathrm{m}}+\frac{1}{4}(k^{2}_{\mathrm{n}}+2k^{2}_{\mathrm{m}})\\
&-\frac{1}{2}(\gamma^{-}_{\mathrm{m}}-k^{2}_{\mathrm{m}})-\frac{\gamma_{11}}{2}(\frac{\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}\\
&+\frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}})+\frac{1}{2}\left(\frac{k^{2}_{\mathrm{n}}\omega_{\mathrm{m}}}{\omega_{\mathrm{n}}}+\frac{k^{2}_{\mathrm{m}}\omega_{\mathrm{n}}}{\omega_{\mathrm{m}}}\right),\qquad(100)
\end{aligned}
$$

$$
\begin{aligned}
A_{3\mathrm{m}}={}&5\omega^{2}_{\mathrm{m}}F_{2\mathrm{m}}k_{2\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&-2k^{2}_{2\mathrm{m}}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h\\
&+\frac{k^{4}_{\mathrm{m}}-k^{2}_{\mathrm{m}}\omega^{4}_{\mathrm{m}}}{4\omega^{3}_{\mathrm{m}}}-\frac{G_{2\mathrm{m}}(k^{2}_{\mathrm{m}}-\omega^{4}_{\mathrm{m}})}{2\omega_{\mathrm{m}}},\qquad(101)
\end{aligned}
$$

$$
\begin{aligned}
A^{-}_{\mathrm{m2m}}={}&-\omega^{2}_{\mathrm{m}}F_{2\mathrm{m}}k_{2\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&+k^{2}_{2\mathrm{m}}m^{2}_{2}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h\\
&+\frac{(4m^{2}_{2}-1)k^{4}_{\mathrm{m}}+(5-8m^{2}_{2})k^{2}_{\mathrm{m}}\omega^{4}_{\mathrm{m}}}{4\omega^{3}_{\mathrm{m}}}\\
&-\frac{(G_{2\mathrm{m}}-G^{-}_{\mathrm{mm}})(k^{2}_{\mathrm{m}}-\omega^{4}_{\mathrm{m}})}{2\omega_{\mathrm{m}}},\qquad(102)
\end{aligned}
$$

$$
\begin{aligned}
A^{+}_{2\mathrm{mm}}={}&5\omega^{2}_{\mathrm{m}}F_{2\mathrm{m}}k_{2\mathrm{m}}\sinh k_{2\mathrm{m}}h\\
&+(3m^{2}_{2}-2)k^{2}_{2\mathrm{m}}F_{2\mathrm{m}}\cosh k_{2\mathrm{m}}h\\
&-\frac{(G_{2\mathrm{m}}+G^{+}_{\mathrm{mm}})(k^{2}_{\mathrm{m}}-\omega^{4}_{\mathrm{m}})}{2\omega_{\mathrm{m}}}\\
&+5\omega^{2}_{\mathrm{m}}F^{+}_{\mathrm{mm}}k^{+}_{\mathrm{mm}}\sinh k^{+}_{\mathrm{mm}}h-2(k^{+}_{\mathrm{mm}})^{2}F^{+}_{\mathrm{mm}}\cosh k^{+}_{\mathrm{mm}}h\\
&+\frac{(4m^{4}_{1}-1)k^{4}_{\mathrm{m}}+3(8m^{2}_{2}-1)k^{2}_{\mathrm{m}}\omega^{4}_{\mathrm{m}}}{4\omega^{3}_{\mathrm{m}}},\qquad(103)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{\mathrm{m2n}}={}&\Gamma_{31}(F_{2\mathrm{n}},k_{2\mathrm{n}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{+}_{\mathrm{nm}})\\
&+\Gamma^{+}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{n}})\\
&+\Gamma^{+}_{33}(G_{2\mathrm{n}},G^{\pm}_{\mathrm{mn}},\pm k_{\mathrm{n}},k_{\mathrm{m}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{+}_{\mathrm{mn}}),\qquad(104)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{2\mathrm{mn}}={}&\Gamma_{31}(F_{2\mathrm{m}},k_{2\mathrm{m}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{+}_{\mathrm{mn}})\\
&+\Gamma^{+}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{+}_{\mathrm{mn}},\gamma^{+}_{\mathrm{m}})\\
&+\Gamma^{-}_{33}(G_{2\mathrm{m}},G^{\pm}_{\mathrm{mn}},k_{\mathrm{m}},\pm k_{\mathrm{n}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{+}_{\mathrm{mn}}),\qquad(105)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{\mathrm{n2m}}={}&\pm\Gamma_{31}(F_{2\mathrm{m}},k_{2\mathrm{m}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{-}_{\mathrm{mn}})\\
&\pm\Gamma^{+}_{32}(\pm F^{\pm}_{\mathrm{nm}},k^{\pm}_{\mathrm{nm}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{-}_{\mathrm{nm}},\gamma^{+}_{\mathrm{m}})\\
&\pm\Gamma^{+}_{33}(G_{2\mathrm{m}},G^{\pm}_{\mathrm{nm}},k_{\mathrm{m}},\pm k_{\mathrm{n}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{-}_{\mathrm{mn}}),\qquad(106)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{2\mathrm{nm}}={}&\pm\Gamma_{31}(F_{2\mathrm{n}},k_{2\mathrm{n}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{-}_{\mathrm{nm}})\\
&\pm\Gamma^{+}_{32}(F^{\pm}_{\mathrm{nm}},\pm k^{\pm}_{\mathrm{nm}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{-}_{\mathrm{nm}},\gamma^{+}_{\mathrm{n}})\pm\\
&\Gamma^{-}_{33}(G_{2\mathrm{n}},G^{\pm}_{\mathrm{nm}},\pm k_{\mathrm{n}},k_{\mathrm{m}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{-}_{\mathrm{nm}}),\qquad(107)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{\mathrm{mnm}}={}&\Gamma^{-}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{+}_{\mathrm{mn}},\gamma^{-}_{\mathrm{m}})\\
&-\Gamma^{-}_{32}(\mp F^{\mp}_{\mathrm{nm}},k^{\mp}_{\mathrm{nm}},\omega_{\mathrm{m}},\mp\omega_{\mathrm{n}},\mp\gamma^{+}_{\mathrm{nm}},\gamma^{-}_{\mathrm{m}})\\
&+\Gamma_{34}(G^{-}_{\mathrm{mm}},G^{\pm}_{\mathrm{mn}},G^{\mp}_{\mathrm{nm}},k_{\mathrm{m}},\pm k_{\mathrm{n}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\\
&\pm\gamma_{11},\pm\gamma_{22},\pm\gamma^{+}_{\mathrm{mn}},\pm\gamma^{-}_{\mathrm{mn}},\gamma^{-}_{\mathrm{m}}),\qquad(108)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{mnn}={}&\Gamma^{-}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{+}_{\mathrm{mn}},\gamma^{-}_{\mathrm{n}})\\
&+\Gamma^{-}_{32}(\mp F^{\mp}_{\mathrm{nm}},k^{\mp}_{\mathrm{nm}},\mp\omega_{\mathrm{n}},\omega_{\mathrm{m}},\mp\gamma^{+}_{\mathrm{nm}},\gamma^{-}_{\mathrm{n}})\\
&+\Gamma_{34}(G^{-}_{\mathrm{nn}},G^{\pm}_{\mathrm{mn}},G^{\mp}_{\mathrm{nm}},\pm k_{\mathrm{n}},k_{\mathrm{m}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\\
&\pm\gamma_{11},\pm\gamma_{22},\pm\gamma^{+}_{\mathrm{nm}},\pm\gamma^{-}_{\mathrm{nm}},\gamma^{-}_{\mathrm{n}}),\qquad(109)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{\mathrm{mmn}}={}&\Gamma_{31}(F^{+}_{\mathrm{mm}},k^{+}_{\mathrm{mm}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma_{11})\\
&+\Gamma^{+}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{-}_{\mathrm{mn}},\gamma^{-}_{\mathrm{m}})\\
&+\Gamma^{+}_{32}(\pm F^{\pm}_{\mathrm{nm}},k^{\pm}_{\mathrm{nm}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\pm\gamma^{+}_{\mathrm{nm}},\gamma^{-}_{\mathrm{m}})\\
&+\Gamma_{35}(G^{+}_{\mathrm{mm}},G^{\pm}_{\mathrm{mn}},G^{\pm}_{\mathrm{nm}},k_{\mathrm{m}},\pm k_{\mathrm{n}},\omega_{\mathrm{m}},\pm\omega_{\mathrm{n}},\\
&\pm\gamma_{11},\pm\gamma^{+}_{\mathrm{mn}},\pm\gamma^{-}_{\mathrm{mn}},\gamma^{-}_{\mathrm{m}}),\qquad(110)
\end{aligned}
$$

$$
\begin{aligned}
A^{\pm}_{\mathrm{nmn}}={}&\pm\Gamma_{31}(F^{+}_{\mathrm{nn}},k^{+}_{\mathrm{nn}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma_{11})\\
&\pm\Gamma^{+}_{32}(F^{\pm}_{\mathrm{mn}},k^{\pm}_{\mathrm{mn}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{-}_{\mathrm{mn}},\gamma^{-}_{\mathrm{n}})\\
&\pm\Gamma^{+}_{32}(\pm F^{\pm}_{\mathrm{nm}},k^{\pm}_{\mathrm{nm}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\pm\gamma^{+}_{\mathrm{nm}},\gamma^{-}_{\mathrm{n}})\\
&\pm\Gamma_{35}(G^{+}_{\mathrm{nn}},G^{\pm}_{\mathrm{mn}},G^{\pm}_{\mathrm{nm}},\pm k_{\mathrm{n}},k_{\mathrm{m}},\pm\omega_{\mathrm{n}},\omega_{\mathrm{m}},\\
&\pm\gamma_{11},\pm\gamma^{+}_{\mathrm{nm}},\pm\gamma^{-}_{\mathrm{nm}},\gamma^{-}_{\mathrm{n}}),\qquad(111)
\end{aligned}
$$

其中,新引入函数的具体表达形式为

$$
\begin{aligned}
&\Gamma_{31}(F_{2\mathrm{m}},k_{2\mathrm{m}},\omega_{\mathrm{m}},\omega_{\mathrm{n}},\gamma^{+}_{\mathrm{mn}})\\
={}&(\omega^{2}_{\mathrm{n}}+2\omega_{\mathrm{m}}\omega_{\mathrm{n}}+2\omega^{2}_{\mathrm{m}})F_{2\mathrm{m}}k_{2\mathrm{m}}\sinh k_{2\mathrm{m}}h
\end{aligned}
$$

$$- \left(2\gamma_{mn}^{+} \frac{\omega_n + 2\omega_m}{\omega_n} + \frac{1}{2} k_{2m}^2 \right) F_{2m} \cosh k_{2m} h, \tag{112}$$

$$\Gamma_{32}^{\pm} (F_{mn}^{+}, k_{mn}^{+}, \omega_m, \omega_n, \gamma_{mn}^{+}, \gamma_m^{+})$$
$$= \left[\omega_m^2 + \frac{1}{2} (\omega_m + \omega_n)^2 \pm \omega_m (\omega_m + \omega_n) \right]$$
$$\times F_{mn}^{+} k_{mn}^{+} \sinh k_{mn}^{+} h$$
$$- \left[(\gamma_{mn}^{+} + \gamma_m^{+}) \left(\frac{\omega_n + \omega_m}{\omega_m} \pm 1 \right) \right.$$
$$\left. + \frac{1}{2} (k_{mn}^{+})^2 \right] F_{mn}^{+} \cosh k_{mn}^{+} h, \tag{113}$$

$$\Gamma_{33}^{\pm} (G_{2m}, G_{mn}^{+}, k_m, k_n, \omega_m, \omega_n, \gamma_{mn}^{+}, \gamma_m^{+})$$
$$= - \frac{1}{2} \left(G_{2m} \frac{k_n^2 - \omega_n^4}{\omega_n} + G_{mn}^{+} \frac{k_m^2 - \omega_m^4}{\omega_m} \right)$$
$$+ \frac{\gamma_{mn}^{+} (\gamma_{mn}^{+} + 2k_m^2)}{4\omega_m^2 \omega_n} + \frac{k_n^2 \omega_m^2 - 2\gamma_{mn}^{+} (\omega_m^2 + \omega_n^2)}{4\omega_n}$$
$$\pm \frac{1}{2} \left[\gamma_{mn}^{+} (\omega_m^2 + \omega_n^2) - (k_n^2 \omega_m^2 + k_m^2 \omega_n^2) \right]$$
$$\times \left(\frac{1}{\omega_m} + \frac{1}{\omega_n} \right), \tag{114}$$

$$\Gamma_{34} (G_{mm}^{-}, G_{mn}^{+}, G_{nm}^{-}, k_m, k_n, \omega_m, \omega_n,$$
$$\gamma_{11}, \gamma_{22}, \gamma_{mn}^{+}, \gamma_{mn}^{-}, \gamma_m^{-})$$
$$= (G_{mn}^{+} - G_{nm}^{-}) \frac{k_m^2 - \omega_m^4}{2\omega_m}$$
$$- G_{mm}^{-} \frac{k_n^2 - \omega_n^4}{2\omega_n} + \frac{\gamma_{mn}^{+} \gamma_{mn}^{-} + 2\gamma_m^{-} \gamma_{22}}{2\omega_m^2 \omega_n}$$
$$+ \frac{k_n^2 \omega_m^2 - 2\gamma_m^{-} \omega_n^2}{2\omega_n} - \frac{\gamma_{11} (\omega_m^2 + \omega_n^2)}{\omega_m}, \tag{115}$$

$$\Gamma_{35} (G_{mm}^{+}, G_{mn}^{+}, G_{nm}^{+}, k_m, k_n, \omega_m, \omega_n, \gamma_{11},$$
$$\gamma_{mn}^{+}, \gamma_{mn}^{-}, \gamma_m^{-})$$
$$= - \frac{1}{2} \left[G_{mm}^{+} \frac{k_n^2 - \omega_n^4}{\omega_n} + (G_{mn}^{+} + G_{nm}^{+}) \frac{k_m^2 - \omega_m^4}{\omega_m} \right]$$
$$+ \frac{\gamma_{mn}^{+} \gamma_{mn}^{-} + 2\gamma_m^{-} \gamma_{11}}{2\omega_m^2 \omega_n}$$
$$+ \frac{3k_n^2 \omega_m^2 - 2(\gamma_m^{-} - 2k_m^2) \omega_n^2 - 4\gamma_{11} (\omega_m^2 + \omega_n^2)}{2\omega_n}$$
$$+ \frac{(k_n^2 \omega_m^2 + k_m^2 \omega_n^2) - \gamma_{11} (\omega_m^2 + \omega_n^2)}{\omega_m}$$
$$+ 4k_m^2 \omega_m m_2^2. \tag{116}$$

其中，$\gamma_{11} = k_m k_n m_1 n_1$，$\gamma_{22} = k_m k_n m_2 n_2$.

显见，ζ_3 的前四项呈与 ζ_1 类似的表达形式，而 ζ_3 的其余众多项与 ϕ_3 的项一一对称对应，从中可抽取出下列丰富、完整的第三阶位相组合，表明对称性、位相决定第三阶波：同波同相和，$3\alpha_m$，$3\beta_m$，$3\alpha_n$，$3\beta_n$；同波异相倍和差，$\alpha_m - 2\beta_m$，$\beta_m - 2\alpha_m$，$\alpha_n - 2\beta_n$，$\beta_n - 2\alpha_n$，$2\alpha_m + \beta_m$，$2\beta_m + \alpha_m$，$2\alpha_n + \beta_n$，$2\beta_n + \alpha_n$；异波同函数相倍和差，$\alpha_m \pm 2\alpha_n$，$\beta_m \pm 2\beta_n$，$2\alpha_m \pm \alpha_n$，$2\beta_m \pm \beta_n$；异波异函数相倍和差，$\alpha_n \pm 2\beta_m$，$\beta_n \pm 2\alpha_m$，$2\alpha_n \pm \beta_m$，$2\beta_n \pm \alpha_m$；同波异相差与异波相和差（$\alpha_m - \beta_m$）$\pm \alpha_n$，（$\beta_m - \alpha_m$）$\pm \beta_n$，$\alpha_m \pm$（$\alpha_n - \beta_n$），$\beta_m \pm$（$\beta_n - \alpha_n$）；同波异相和与异波相和差（$\alpha_m + \beta_m$）$\pm \beta_n$，（$\alpha_m + \beta_m$）$\pm \alpha_n$，（$\alpha_n + \beta_n$）$\pm \beta_m$，（$\alpha_n + \beta_n$）$\pm \alpha_m$.

若要论 n 色 n 向波系统，则其某种高阶的位相组合也不难一一区分和得到. 以此刻画该波系统的基本构造.

依据上述三阶解，即可得到其一系列运动学、动力学变量表达式[12]，进而可最终推导出作用于直立防波堤上的三阶波浪力和波浪力矩表达式.

4. 普适法则

黄虎等[13]曾从部分反射的三阶单色单向短峰波系统中推断出适用于高阶的一般法则：倍频率通向短峰波. 现在，若将三阶双色双向波系统发展到高阶，其频率个数将作何变化呢？进一步，若将双色双向推广至 j 色 j 向，其频率个数又是怎样的呢？可以预先推断：j 色 j 向波必与单色单向波一脉相承，同样对前者可做如同后者[13]的如下频率个数的剖析.

前两阶频率为：$\omega_0 \neq 0$，$\omega_1 = 0$，第 3 阶频率 ω_2 存在 4 个值：ω_{21}，ω_{22}，ω_{23}，ω_{24}. 其原因在于：ω_1，ω_2 是通过消除各阶方程中出现的长期项而得到的，求解 ω_1 时，仅与(10)式右端项 F_2 中的 $-2\omega_0\omega_1\phi_{1tt}$ 项相关，而在求解 ω_2 时，则相关于(10)式右端项 F_3 中的 $-(2\omega_0\omega_2 + \omega_1^2)\phi_{1tt}$ 项和以 ϕ_1 中的 4 项为因子的项. 如此而论，偶数阶频率 ω_3，ω_5，…，将依次来源于 $-2\omega_0\omega_3\phi_{1tt}$，$-2\omega_0\omega_5\phi_{1tt}$，…，则 $\omega_3 = \omega_5 = 0$；奇数阶频率 ω_4，ω_6，，…，将依次来源于（注意：偶数阶频率已为零）$-(2\omega_0\omega_4 + \omega_2^2)\phi_{1tt}$，$-(2\omega_0\omega_6 + 2\omega_1\omega_5 + 2\omega_2\omega_4 + \omega_3^2)\phi_{1tt}$，…，以及相应各阶其他以 ϕ_1 中的 4 项为因子的项. 由于 ω_2，ω_4，ω_6，…，各自相关于求解方程中的 4 个不同长期项，则由消除长期项的条件，再纳入一阶频率 ω_0，则奇数阶频率个数构成一个等比数列：1，4，4^2，4^3，…. 如果扩展至高阶 j 色 j 向波，则不难推论出一项普适法则：偶数阶频率皆为零；奇数阶频率的个数则呈现一个等比数列：1，

$2j, (2j)^2, (2j)^3, \cdots$.

5. 结　　论

从单色单向短峰波理论发展到双色双向波，并不仅仅是添加一色一向. 这将带来高阶推导的繁冗性、内容要旨的丰富性和形式结构的对称性. 本文详尽刻画了双色双向波三阶解，从中提炼出其构造要素的基本位相组合，并意外发现了奇数阶频率个数构成一等比数列的多色多向波演变的普适法则. 这表明对称决定双色双向波相互作用恰为普适原理对称决定相互作用[14]的一个具体表现，同时也见证、预示着真实海洋波动的异常复杂性.

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Third-order theory for bichromatic bidirectional water waves partially reflected from vertical breakwaters*

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Abstract

To overcome the internal limitation of standing wave forces on vertical breakwaters and describe multichromatic multidirectional water waves in front of the breakwaters, a third-order theory, including the classical solutions of two-dimensional long-crested waves and monochromatic unidirectional short-crested waves, for bichromatic bidirectional water waves partially reflected from the breakwaters is developed. A universal law applicable to a higher-order multichromatic multidirectional wave system is proposed, that is, the even-order frequencies are zero, the numbers of the odd-order frequencies constitute a geometric progression.

Keywords: bichromatic bidirectional waves, a third-order theory, a universal law, partially reflected waves

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