

# 间歇同步分数阶统一混沌系统

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根据分数阶微分方程的性质, 研究了间歇控制分数阶系统的稳定性, 提出了间歇控制分数阶系统的一般理论并给出了数学证明. 根据该理论设计控制器实现了分数阶统一混沌系统的间歇同步, 数值仿真验证了该理论的正确性.

**关键词:** 分数阶, 统一混沌系统, 间歇同步, 稳定性

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## 1. 引言

分数阶微分方程几乎与整数阶微分方程具有相同的历史, 然而由于一直没有找到应用背景而未得到应有的重视. 直到1983年发现自然界中存在大量分数维现象且在整数阶和分数阶间存在自相似现象以来<sup>[1]</sup>, 分数阶微分方程相关理论得到了广泛研究<sup>[2]</sup>. 研究表明: 所有的物理现象都是以分数阶形式存在, 整数阶微分方程是分数阶微分方程的特例, 分数阶微分方程是整数阶微分方程的推广. 因而分数阶微分更具有普遍意义而成为当前的研究热点.

混沌同步由于在保密通信等领域的潜在应用价值而得到了广泛研究, 而这些研究更多的集中于对微分阶次为整数的整数阶混沌系统同步的研究. 研究表明一些典型混沌系统如统一混沌系统、Lorenz混沌系统、Chen混沌系统等当微分阶次为分数时也会出现混沌现象<sup>[3-6]</sup>. 整数阶微分是分数阶微分的特例, 因而研究分数阶混沌同步更具有普遍意义.

近年来分数阶混沌系统同步成为混沌领域的一个研究热点. 相继实现了耦合同步、反馈同步、自适应同步、投影同步、滑模控制同步、广义同步等<sup>[7-12]</sup>. 间歇同步由于在混沌保密通信中的突出特点而倍受青睐. 然而由于分数阶微分研究起步较晚

及相关理论的复杂性, 间歇同步分数阶混沌系统几乎还未见报道. 因而研究相关分数阶理论实现分数阶间歇同步具有积极意义.

## 2. 分数阶微分概述

分数阶微分方程是整数阶微分方程的推广, 在研究过程中对分数阶微分概念提出了多种定义, 其中常用的有 Riemann-Liouville (R-L) 定义、Caputo 定义. 令  $n$  为大于  $\alpha$  的最小整数,  $n-1 < \alpha < n$ ,  $\Gamma(\cdot)$  为伽马函数. R-L 分数阶微分定义数学表达式为<sup>[13]</sup>

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \times \frac{d^n}{dt^n} \left[ \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \right]. \quad (1)$$

Caputo 分数阶微分定义为<sup>[14]</sup>

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \times \int_a^t (t-\tau)^{-\alpha+n-1} f^{(n)}(\tau) d\tau. \quad (2)$$

Caputo 分数阶微分与 R-L 分数阶微分之间满足如下关系:

$${}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{(k-\alpha)},$$

本文以 Caputo 分数阶微分为基础进行研究.

对于一般的分数阶微分方程

$${}_a^C D_t^\alpha x(t) = Ax(t), \quad (3)$$

该方程的解为

$$x(t) = x(0) E_\alpha(At^\alpha). \quad (4)$$

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其中 Mittag-Leffler 函数  $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$ .

如果存在可逆矩阵  $P$ , 使得

$$PAP^{-1} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad (5)$$

则

$$x(t) = x(0)P^{-1} \times \begin{bmatrix} E_\alpha(\lambda_1 t^\alpha) & & & \\ & E_\alpha(\lambda_2 t^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(\lambda_n t^\alpha) \end{bmatrix} P. \quad (6)$$

引理<sup>[15]</sup> 对分数阶混沌系统(3), 如果系数矩阵  $A$  的任意特征值  $\lambda_i (i=1, 2, \dots, n)$  满足  $|\arg(\lambda_i)| \geq$

$\frac{\alpha\pi}{2}$ , 则分数阶系统(3)稳定.

根据该引理, 当  $|\arg(z)| \geq \frac{\alpha\pi}{2}$  时, Mittag-leffler 函数  $E_\alpha(z)$  稳定.

### 3. 分数阶系统间歇控制理论

对于一般分数阶受控系统可用下式表述:

$${}_a^c D_t^\alpha X = A(X)X, \quad (7)$$

其中  $X = (x_1, x_2, \dots, x_n)$  为  $n$  维矢量. 使用间歇控制该系统则

$${}_a^c D_t^\alpha X = A(X)X + u(t), \quad (8)$$

其中

$$u(t) = \begin{cases} U & (nT < t < nT + T_c), \\ 0 & (nT + T_c < t < (n+1)T), \end{cases} \quad (9)$$

其中  $T$  为一个周期,  $T_c$  为一个周期中的控制时间,  $T - T_c$  为一个周期中的自由时间. 则有

$${}_a^c D_t^\alpha X = \begin{cases} A(X)X + U = A'(X)X & (nT < t < nT + T_c), \\ A(X)X & (nT + T_c < t < (n+1)T). \end{cases} \quad (10)$$

**定理 1** 令  $\lambda$  为  $A(X) (nT + T_c < t < (n+1)T$  其中  $n$  为任意正整数) 的实部最大的特征值,  $\lambda'$  为  $A'(X) (nT < t < nT + T_c)$  的实部最大的特征值, 当系统阶次  $0 < \alpha \leq 1$  时, 如果恒有  $\text{Re}(\lambda T_c^\alpha + \lambda'(T - T_c)^\alpha) < 0$ , 则分数阶受控系统(10)稳定.

**证明** 在任意  $t (0 < t < T_c)$  时刻, 一定存在可逆矩阵  $P(t)$  使得

$$P(t)A'(X(t))(P(t))^{-1} = \begin{bmatrix} \lambda_1(t) & & & \\ & \lambda_2(t) & & \\ & & \ddots & \\ & & & \lambda_n(t) \end{bmatrix}. \quad (11)$$

在足够小的  $\delta t \rightarrow 0$  时

$$\begin{aligned} \lim_{\delta t \rightarrow 0} X(t + \delta t) &= \lim_{\delta t \rightarrow 0} E_\alpha(A'(X(t))(\delta t)^\alpha)X(t) \\ &= \lim_{\delta t \rightarrow 0} (P(t))^{-1} \begin{bmatrix} E_\alpha(\lambda_1(t)(\delta t)^\alpha) & & & \\ & E_\alpha(\lambda_2(t)(\delta t)^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(\lambda_n(t)(\delta t)^\alpha) \end{bmatrix} P(t)X(t), \end{aligned} \quad (12)$$

$$\begin{aligned} \lim_{\delta t \rightarrow 0} (X(t + \delta t))^H &= \lim_{\delta t \rightarrow 0} (E_\alpha(A'(X(t))(\delta t)^\alpha)X(t))^H \\ &= \lim_{\delta t \rightarrow 0} (X(t))^H (P(t))^H \begin{bmatrix} E_\alpha(\bar{\lambda}_1(t)(\delta t)^\alpha) & & & \\ & E_\alpha(\bar{\lambda}_2(t)(\delta t)^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(\bar{\lambda}_n(t)(\delta t)^\alpha) \end{bmatrix} ((P(t))^{-1})^H, \end{aligned} \quad (13)$$

则有

$$\begin{aligned}
 & \lim_{\delta t \rightarrow 0} (\mathbf{X}(t + \delta t))^H \mathbf{X}(t + \delta t) \\
 &= \lim_{\delta t \rightarrow 0} (\mathbf{X}(t))^H \begin{bmatrix} E_\alpha(\bar{\lambda}_1(t)(\delta t)^\alpha) & & & \\ & E_\alpha(\bar{\lambda}_2(t)(\delta t)^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(\bar{\lambda}_n(t)(\delta t)^\alpha) \end{bmatrix} \\
 & \times \begin{bmatrix} E_\alpha(\lambda_1(t)(\delta t)^\alpha) & & & \\ & E_\alpha(\lambda_2(t)(\delta t)^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(\lambda_n(t)(\delta t)^\alpha) \end{bmatrix} \mathbf{X}(t) \\
 &= \lim_{\delta t \rightarrow 0} (\mathbf{X}(t))^H \\
 & \times \begin{bmatrix} E_\alpha(((\bar{\lambda}_1(t))^{\frac{1}{\alpha}}\delta t + (\lambda_1(t))^{\frac{1}{\alpha}}\delta t)^\alpha) & & & \\ & E_\alpha(((\bar{\lambda}_2(t))^{\frac{1}{\alpha}}\delta t + (\lambda_2(t))^{\frac{1}{\alpha}}\delta t)^\alpha) & & \\ & & \ddots & \\ & & & E_\alpha(((\bar{\lambda}_n(t))^{\frac{1}{\alpha}}\delta t + (\lambda_n(t))^{\frac{1}{\alpha}}\delta t)^\alpha) \end{bmatrix} \mathbf{X}(t) \\
 &\leq \max(E_\alpha(((\bar{\lambda}_1(t))^{\frac{1}{\alpha}}\delta t + (\lambda_1(t))^{\frac{1}{\alpha}}\delta t)^\alpha), E_\alpha(((\bar{\lambda}_2(t))^{\frac{1}{\alpha}}\delta t + (\lambda_2(t))^{\frac{1}{\alpha}}\delta t)^\alpha), \dots, E_\alpha(((\bar{\lambda}_n(t))^{\frac{1}{\alpha}}\delta t \\
 & + (\lambda_n(t))^{\frac{1}{\alpha}}\delta t)^\alpha)) (\mathbf{X}(t))^H \mathbf{X}(t). \tag{14}
 \end{aligned}$$

故有

$$\begin{aligned}
 & \lim_{\delta t \rightarrow 0} \frac{(\mathbf{X}(t + \delta t))^H \mathbf{X}(t + \delta t)}{\mathbf{X}(t)^H \mathbf{X}(t)} \\
 &\leq \lim_{\delta t \rightarrow 0} \max(E_\alpha((\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}})\delta t)^\alpha), E_\alpha((\max((\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}})\delta t)^\alpha), \dots, \\
 & E_\alpha((\max(\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}})\delta t)^\alpha)) \\
 &\leq \lim_{\delta t \rightarrow 0} E_\alpha((\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}}, \dots, (\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}})(\delta t)^\alpha), \tag{15}
 \end{aligned}$$

则

$$\begin{aligned}
 \frac{(\mathbf{X}(T_c))^H \mathbf{X}(T_c)}{\mathbf{X}(t)^H \mathbf{X}(t)} &\leq E_\alpha(\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}}, \dots, \\
 & (\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}}) T_c)^\alpha). \tag{16}
 \end{aligned}$$

同理,在任意  $t(T_c < t < T)$  时刻,

$$\begin{aligned}
 \frac{(\mathbf{X}(T))^H \mathbf{X}(T)}{\mathbf{X}(T_c)^H \mathbf{X}(T_c)} &\leq E_\alpha(\max((\bar{\lambda}'_1(t))^{\frac{1}{\alpha}} + (\lambda'_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}'_2(t))^{\frac{1}{\alpha}} + (\lambda'_2(t))^{\frac{1}{\alpha}}, \dots, \\
 & (\bar{\lambda}'_n(t))^{\frac{1}{\alpha}} + (\lambda'_n(t))^{\frac{1}{\alpha}}) (T - T_c)^\alpha). \tag{17}
 \end{aligned}$$

当  $0 < \alpha < 1$  时,如果  $\text{Re}(\gamma) > 0$ ,有

$$(\gamma')^{\frac{1}{\alpha}} + (\bar{\gamma}')^{\frac{1}{\alpha}} < 2(\text{Re}(\gamma'))^{\frac{1}{\alpha}}. \tag{18}$$

根据分数阶混沌系统特性,  $T_c < t < T, \text{Re}(\lambda') > 0$ . 有

$$\frac{(\mathbf{X}(T))^H \mathbf{X}(T)}{\mathbf{X}(T_c)^H \mathbf{X}(T_c)} \leq E_\alpha(((2\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha). \tag{19}$$

因而有

$$\frac{(\mathbf{X}(T))^H \mathbf{X}(T)}{\mathbf{X}(0)^H \mathbf{X}(0)} \leq E_\alpha(\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}}, \dots, (\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}})(T - T_c)^\alpha + ((2\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha). \quad (20)$$

当

$$\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}}, \dots, (\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}})(T - T_c)^\alpha + ((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha < 0, \quad (21)$$

$$\frac{(\mathbf{X}(T))^H \mathbf{X}(T)}{\mathbf{X}(0)^H \mathbf{X}(0)} \leq E_\alpha(\max((\bar{\lambda}_1(t))^{\frac{1}{\alpha}} + (\lambda_1(t))^{\frac{1}{\alpha}}, (\bar{\lambda}_2(t))^{\frac{1}{\alpha}} + (\lambda_2(t))^{\frac{1}{\alpha}}, \dots, (\bar{\lambda}_n(t))^{\frac{1}{\alpha}} + (\lambda_n(t))^{\frac{1}{\alpha}})(T - T_c)^\alpha + ((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha). \quad (22)$$

如果恒有

$$((\bar{\lambda}(t))^{\frac{1}{\alpha}} + (\lambda(t))^{\frac{1}{\alpha}})T_c)^\alpha < -2((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha, \quad (23)$$

即满足  $\left| \arg(\lambda(t) + \text{Re}(\lambda')) \left( \frac{T - T_c}{T_c} \right)^\alpha \right| \geq \frac{\alpha\pi}{2}$  时, 有

$$\frac{(\mathbf{X}(nT))^H \mathbf{X}(nT)}{\mathbf{X}((n-1)T)^H \mathbf{X}((n-1)T)} < E_\alpha((\bar{\lambda}(t))^{\frac{1}{\alpha}} + (\lambda(t))^{\frac{1}{\alpha}})T_c)^\alpha + 2((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha, \quad (24)$$

其中  $n \in Z^+$ . 故

$$0 \leq \frac{(\mathbf{X}(nT))^H \mathbf{X}(nT)}{\mathbf{X}(0)^H \mathbf{X}(0)} < E_\alpha(n((\bar{\lambda}(t))^{\frac{1}{\alpha}} + (\lambda(t))^{\frac{1}{\alpha}})T_c)^\alpha + 2n((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha). \quad (25)$$

由于

$$\lim_{n \rightarrow \infty} E_\alpha(n((\bar{\lambda}(t))^{\frac{1}{\alpha}} + (\lambda(t))^{\frac{1}{\alpha}})T_c)^\alpha + 2n((\text{Re}(\lambda'))^{\frac{1}{\alpha}})(T - T_c)^\alpha = 0, \quad (26)$$

因此

$$\lim_{n \rightarrow \infty} (\mathbf{X}(nT))^H \mathbf{X}(nT) = 0. \quad (27)$$

从而

$$\lim_{n \rightarrow \infty} (\mathbf{X}((n-1)T))^H \mathbf{X}((n-1)T) = 0. \quad (28)$$

根据非线性系统的有界性, 有

$$\lim_{n \rightarrow \infty} (\mathbf{X}(t))^H \mathbf{X}(t) = 0 \quad ((n-1)T_c < t < nT). \quad (29)$$

从而

$$\lim_{t \rightarrow \infty} \mathbf{X}(t) = 0. \quad (30)$$

从上述证明过程可以清楚: 只要恒满足

$$\left| \arg\left(\lambda(t) + \text{Re}(\lambda') \left( \frac{T - T_c}{T_c} \right)^\alpha\right) \right| \geq \frac{\alpha\pi}{2},$$

即  $\lambda(t)$  位于如图 1 所示的稳定区域, 则分数阶受控系统(7)稳定.

显然, 当  $\lambda(t)$  位于直线  $AB$  以左, 即  $\text{Re}(\lambda(t))$

$< -\text{Re}(\lambda') \left( \frac{T - T_c}{T_c} \right)^\alpha$  时,  $\lambda(t)$  位于稳定区域, 分数

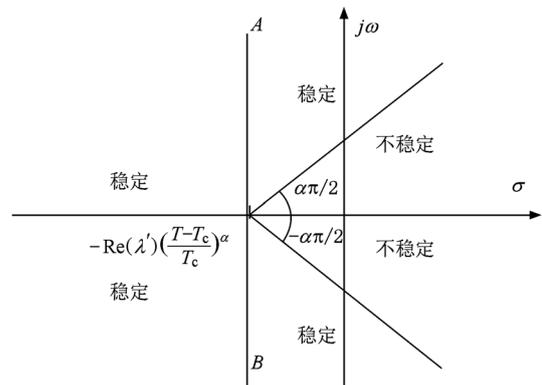


图 1 分数阶受控系统(7)稳定区域

阶受控系统稳定. 定理 1 证毕.

#### 4. 间歇同步分数阶统一混沌系统

分数阶统一混沌系统的数学模型为

$$\begin{bmatrix} \frac{d^\alpha x_1}{dt^\alpha} \\ \frac{d^\alpha x_2}{dt^\alpha} \\ \frac{d^\alpha x_3}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} -(25a + 10) & (25a + 10) & \\ (28 - 35a) & (29a - 1) & -x_1 \\ & x_1 & (a + 8)/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \quad (31)$$

当参数  $\alpha \in [0, 0.8)$  时, 系统为分数阶 Lorenz 系统.  
当  $\alpha = 0.8$  时, 系统为分数阶 Lü 系统. 当  $\alpha \in (0.8,$

$1]$  时, 系统为分数阶 Chen 系统.  
设响应系统为

$$\begin{bmatrix} \frac{d^\alpha y_1}{dt^\alpha} \\ \frac{d^\alpha y_2}{dt^\alpha} \\ \frac{d^\alpha y_3}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} -(25a + 10) & (25a + 10) & \\ (28 - 35a) & (29a - 1) & -y_1 \\ & y_1 & (a + 8)/3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad (32)$$

其同步误差为

$$\begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \\ \frac{d^\alpha e_3}{dt^\alpha} \end{bmatrix} = \begin{cases} \begin{bmatrix} -(25a + 10) & (25a + 10) \\ (28 - 35a - x_3) & (29a - 1) \\ x_2 & y_1 & (a + 8)/3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} - \begin{bmatrix} U_1(t) \\ U_2(t) \\ U_3(t) \end{bmatrix} = \mathbf{A}' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} & (nT < t < nT + T_c), \\ \begin{bmatrix} -(25a + 10) & (25a + 10) \\ (28 - 35a - x_3) & (29a - 1) \\ x_2 & y_1 & (a + 8)/3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{A} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} & (nT + T_c < t < (n + 1)T), \end{cases} \quad (33)$$

其中  $\mathbf{U} = [U_1(t), U_2(t), U_3(t)]^T$  为待设计的间歇控制器.

当  $nT + T_c < t < (n + 1)T$  时,

$$\begin{aligned} & [e_1, e_2, e_3] \mathbf{A} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} / ([e_1, e_2, e_3][e_1, e_2, e_3]^T) \\ &= (- (25a + 10)e_1^2 + (25a + 10 + 28 - 35a - x_3)e_1e_2 + (29a - 1)e_2^2 + x_2e_1e_3 \\ &+ (a + 8)e_3^2/3) / (e_1^2 + e_2^2 + e_3^2) \\ &\leq (- (25a + 10)e_1^2 + |25a + 10 + 28 - 35a| \frac{e_1^2 + e_2^2}{2} + |x_3| \frac{e_1^2 + e_2^2}{2} + (29a - 1)e_2^2 \\ &+ |x_2| \frac{e_1^2 + e_3^2}{2} + (a + 8)e_3^2/3) / (e_1^2 + e_2^2 + e_3^2) \\ &= ((- (28.5a + 9) + (|x_3| + |x_3|)/2)e_1^2 + (25.5a + 18 - 3.5a \\ &+ |x_3|/2)e_2^2 + (a + 8 + 1.5|x_2|)e_3^2/3) / (e_1^2 + e_2^2 + e_3^2) \\ &\leq \max((- (28.5a + 9) + (|x_3| + |x_3|)/2), (25.5a + 18 - 3.5a \\ &+ |x_3|/2), (a + 8 + 1.5|x_2|/3)). \end{aligned} \quad (34)$$

故

$$\operatorname{Re}(\lambda) \leq \max((-(28.5a + 9) + (\max(x_2) + \max(x_3))/2), (25.5a + 18 - 3.5a + \max(x_3)/2), (a + 8 + 1.5\max(x_2))/3)). \quad (35)$$

**定理 2** 令  $\lambda_m = \max((-(28.5a + 9) + (\max(x_2) + \max(x_3))/2), (25.5a + 18 - 3.5a + \max(x_3)/2), (a + 8 + 1.5\max(x_2))/3))$ , 如果设计的控制器满足

$$\begin{aligned} U_1(t) &= (38 - 10a - x_3)e_2 + x_2e_3 + k_1e_1, \\ U_2(t) &= k_2e_2, \\ U_3(t) &= k_3e_3, \end{aligned} \quad (36)$$

其中,  $k_1 > -(25a + 10) + \lambda_m \left(\frac{T - T_c}{T_c}\right)^\alpha$ ,  $k_2 > 29a - 1 + \lambda_m \left(\frac{T - T_c}{T_c}\right)^\alpha$ ,  $k_3 > (a + 8)/3 + \lambda_m \left(\frac{T - T_c}{T_c}\right)^\alpha$  时, 分数阶误差系统在间歇控制下能够稳定, 也即分数阶响应系统(32)在间歇控制器下能与分数阶响应系统(31)实现同步.

**证明** 根据设计的控制器及分数阶误差系统(33), 当  $nT < t < nT + T_c$  时,

$$\begin{bmatrix} \frac{d^\alpha e_1}{dt^\alpha} \\ \frac{d^\alpha e_2}{dt^\alpha} \\ \frac{d^\alpha e_3}{dt^\alpha} \end{bmatrix} = \begin{bmatrix} -(25a + 10) - k_1 & (35a - 28 + x_3) & -x_2 \\ (28 - 35a - x_3) & (29a - 1) - k_2 & -y_1 \\ x_2 & y_1 & (a + 8)/3 - k_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{A}' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}. \quad (37)$$

$$\begin{aligned} & [e_1, e_2, e_3] \mathbf{A}' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} / ([e_1, e_2, e_3] [e_1, e_2, e_3]^T) \\ &= ((-(25a + 10) - k_1)e_1^2 + ((29a - 1) - k_2)e_2^2 + ((a + 8)/3 - k_3)e_3^2) / (e_1^2 + e_2^2 + e_3^2) \\ &\leq \max(-(25a + 10) - k_1, ((29a - 1) - k_2), ((a + 8)/3 - k_3)) = -\lambda_m \left(\frac{T - T_c}{T_c}\right)^\alpha. \end{aligned} \quad (38)$$

故

$$\lambda' \leq -\lambda_m \left(\frac{T - T_c}{T_c}\right)^\alpha. \quad (39)$$

根据定理 1, 分数阶误差系统稳定. 定理 2 证毕. 在控制器(36)作用下, 响应系统(33)与驱动系统(32)能实现间歇同步.

### 5. 数值仿真

基于改进的 Adams-Bashforth-Moulton 理论<sup>[16]</sup>, 文献[17]提出了分数阶混沌系统仿真算法. 本文利用该算法对驱动系统和响应系统进行仿真, 仿真时, 设阶次  $\alpha = 0.97$ , 初始值为  $x_1 = 0.123$ ,  $x_2 = 0.1451$ ,  $x_3 = 0.453$ ,  $y_1 = 0.23$ ,  $y_2 = 0.451$ ,  $y_3 = 0.42$ , 令  $T = 6$  s,  $T_c = 1.2$  s.

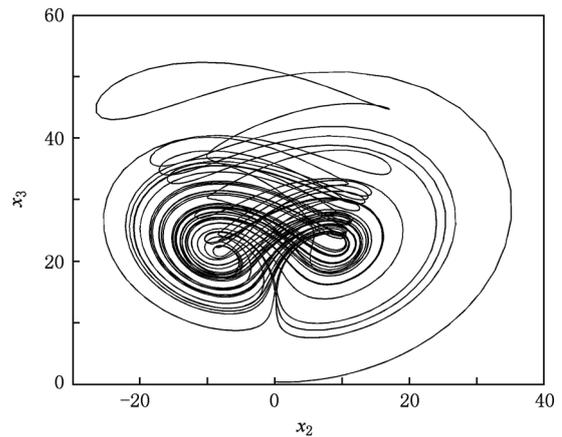


图 2  $\alpha = 0.97, a = 1$  时, 分数阶统一混沌系统吸引子图

由于驱动系统的未知参数  $a \in (0.8, 1)$  时, 分数阶统一混沌系统代表分数阶 Chen 系统. 取  $a = 1$  时混沌吸引子如图 2 所示, 由仿真结果得  $\max(x_2) =$

35.0535,  $\max(x_3) = 52.3487$ , 因而  $\text{Re}(\lambda) \leq 66.17435$ . 因而取  $k_1 = -17 > -(25 + 10) + 66.17435 \times \left(\frac{1}{4}\right)^{0.97} = -17.75$ ,  $k_2 = 46 > 29 - 1 + 66.17435 \times \left(\frac{1}{4}\right)^{0.97} = 45.24$ ,  $k_3 = 21 > (1 + 8)/3 + 66.17435 \times \left(\frac{1}{4}\right)^{0.97} = 20.246$ . 其间歇同步误差如图 3 所示.

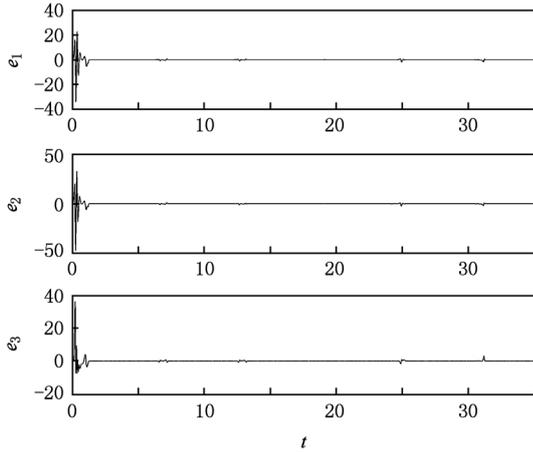


图 3  $\alpha=0.97, a=1$  时, 分数阶统一混沌系统间歇同步误差图

当  $a=0.8$  时, 分数阶统一混沌系统代表分数阶 Lü 系统, 混沌吸引子如图 4 所示, 仿真结果得  $\max(x_2) = 34.525, \max(x_3) = 51.4217$ , 因而  $\text{Re}(\lambda) \leq 67.98$ , 取  $k_1 = -12 > -(25 \times 0.8 + 10) + 67.98 \times \left(\frac{1}{4}\right)^{0.97} = -12.28$ ,  $k_2 = 40 > 29 \times 0.8 - 1 + 67.98 \times \left(\frac{1}{4}\right)^{0.97} = 39.92$ ,  $k_3 = 21 > (0.8 + 8)/3 + 67.98 \times$

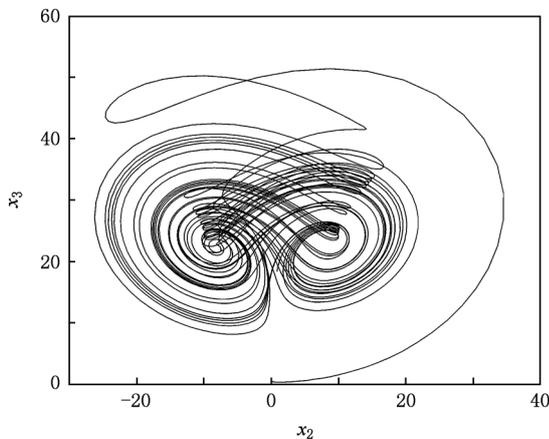


图 4  $\alpha=0.97, a=0.8$  时, 分数阶统一混沌系统吸引子图

$\left(\frac{1}{4}\right)^{0.97} = 20.65$ , 其仿真结果如图 5 所示.

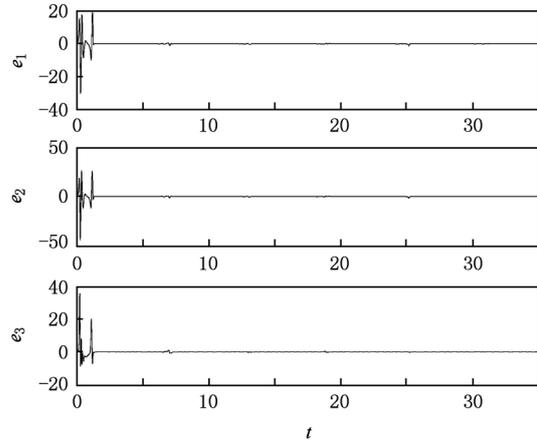


图 5  $\alpha=0.97, a=0.8$  时, 分数阶统一混沌系统间歇同步误差图

当  $a \in [0, 0.8]$  时, 分数阶统一混沌系统代表分数阶 Lorenz 系统,  $a=0.4$  时, 其混沌吸引子如图 6 所示. 由仿真结果得  $\max(x_2) = 32.2675, \max(x_3) = 50.8329$ , 因而  $\text{Re}(\lambda) \leq 50.35645$ .  $k_1 = 14 > -(25 \times 0.4 + 10) + 50.35645 \times \left(\frac{1}{4}\right)^{0.97} = 13.123$ ,  $k_2 = 24 > 29 \times 0.4 - 1 + 50.35645 \times \left(\frac{1}{4}\right)^{0.97} = 23.723$ ,  $k_3 = 16 > (0.4 + 8)/3 + 50.35645 \times \left(\frac{1}{4}\right)^{0.97} = 15.723$ . 其仿真结果如图 7 所示.

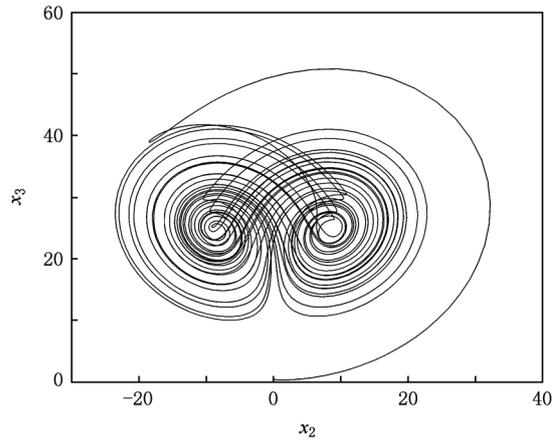
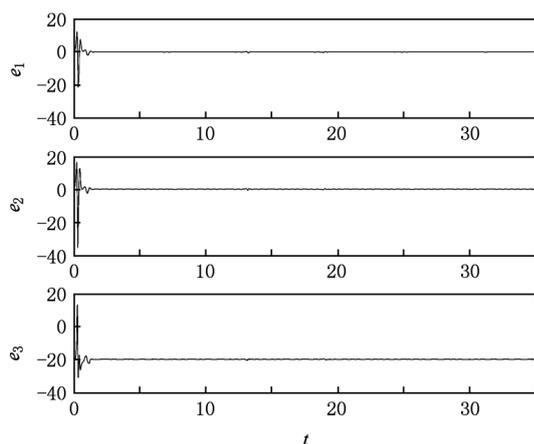


图 6  $\alpha=0.97, a=0.4$  时, 分数阶统一混沌系统吸引子图

图 3, 5 和 7 都表明: 无论分数阶统一混沌系统代表分数阶 Chen 系统、分数阶 Lü 系统还是分数阶 Lorenz 系统, 在设计间歇控制器作用下, 响应分数

图7  $\alpha=0.97, a=0.4$  时, 分数阶统一混沌系统间歇同步误差图

阶统一混沌系统与驱动分数阶统一混沌系统能实现间歇同步. 仿真结果验证了所提出理论的正确性.

## 6. 结 论

根据分数阶系统稳定性理论, 从分数阶微分的性质出发, 提出了分数阶系统间歇控制的一般稳定性理论. 利用该理论实现了分数阶统一混沌系统间歇同步. 该理论不仅可用于间歇控制, 还可拓展到脉冲控制等领域. 因而该理论的提出不仅补充了分数阶微分相关理论, 也对分数阶系统控制方法进行拓展, 具有一定的理论意义和较广的应用价值.

- [1] Mandelbort B B 1983 *The fractal Geometry of Nature* (New York: Freeman)
- [2] Yan X M, Liu D 2010 *Acta Phys. Sin.* **59** 3403 (in Chinese) [阎晓妹、刘 丁 2010 物理学报 **59** 3403]
- [3] Huang L, Feng R, Wang M 2004 *Phys. Lett. A* **32** 271
- [4] Liu Y, Xie Y 2010 *Acta Phys. Sin.* **59** 2147 (in Chinese) [刘 勇、谢 勇 2010 物理学报 **59** 2147]
- [5] Chen X R, Liu C X, Wang F Q, Li Y X 2008 *Acta Phys. Sin.* **57** 1416 (in Chinese) [陈向荣、刘崇新、王发强、李永勋 2008 物理学报 **57** 1416]
- [6] Zhao P D, Zhang X D 2008 *Acta Phys. Sin.* **57** 2791 (in Chinese) [赵品栋、张晓丹 2008 物理学报 **57** 2791]
- [7] Tavazoei M S, Haeri M 2008 *Physica A* **387** 57
- [8] Hu J B, Han Y, Zhao L D 2009 *Acta Phys. Sin.* **58** 1441 (in Chinese) [胡建兵、韩 焱、赵灵冬 2009 物理学报 **58** 1441]
- [9] Li A, Chen G R 2004 *Chaos Soliton. Fract.* **22** 549
- [10] Zhang R X, Yang Y, Yang S P 2009 *Acta Phys. Sin.* **58** 6039 (in Chinese) [张若洵、杨 洋、杨世平 2009 物理学报 **58** 6039]
- [11] Wang J W, Zhang Y B 2009 *Phys. Lett. A* **374** 202
- [12] Deng W H 2007 *Phys. Rev. E* **75** 056201
- [13] Vedat Saat Erturk, Shaher Momani, Zaid Odibat 2008 *Commun. Nonlinear Sci. Num. Simulat.* **13** 1642
- [14] Ricardo Almeida, Delfim F M Torres 2011 *Commun. Nonlinear Sci. Num. Simulat.* **16** 1490
- [15] Matignon D 1996 *IEEE International Conference on Systems, Man and Cybernetics* Lille, France July 9–12, 1996
- [16] Yan J P, Li C P 2004 *Chaos, Soliton. Fract.* **22** 443
- [17] Deng W H 2007 *J. Comput. Phys.* **227** 1510

# Intermittent synchronizing fractional unified chaotic systems

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## Abstract

Stability about fractional chaotic system is studied and a theory about fractional chaotic system is proposed and proved under intermittent control in this paper. Based on the theorem, a controller is designed to realize the intermittent synchronizing fractional unified chaotic system. Numerical simulation demonstrates the effectiveness of the theorem.

**Keywords:** fractional, unified chaotic system, intermittent synchronization, stability

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