sine-Gordon 型方程的无穷序列新精确解*

套格图桑†

(内蒙古师范大学数学科学学院,呼和浩特 010022)

(2010 年 7 月 27 日收到;2010 年 10 月 8 日收到修改稿)

为了获得 sine-Gordon 型方程的无穷序列精确解,给出三角函数型辅助方程和双曲函数型辅助方程及其 Bäcklund 变换和解的非线性叠加公式,借助符号计算系统 Mathematica,构造了 sine-Gordon 方程、mKdV-sine-Gordon 方程、(n+1)维双 sine-Gordon 方程和 sinh-Gordon 方程的无穷序列新精确解. 其中包括无穷序列三角函数解、无穷序列双曲函数解、无穷序列 Jacobi 椭圆函数解和无穷序列复合型解.

关键词: sine-Gordon 型方程, 解的非线性叠加公式, 辅助方程, 无穷序列精确解

PACS: 02.30.Ik, 02.30.Jr

1. 引 言

1967 年和 1974 年,Gardner, Greene, Kruskal 和 Miura 在求解著名的 KdV 方程时提出了一种解析方法,即反散射变换法[1,2]. 这是孤立子理论发展历史上的又一个里程碑. 这种一系列线性步骤求解非线性问题的技巧后来在 Lax, Ablowitz, 以及萨哈罗夫,沙巴特等人推广应用在非线性薛定谔方程和 sine-Gordon 方程等一系列的非线性发展方程求解问题上,从而反散射变换法发展成为非常有活力的一种崭新的求解方法[3]. 构造非线性发展方程的精确解,是孤立子理论中非常重要的研究课题之一. 1995 年,Wang 提出齐次平衡法[4],并获得了 sine-Gordon 方程等非线性发展方程的许多新精确解. 1996 年 Parkes 和 Duffy 提出双曲正切函数展开法[5],并构造了非线性发展方程的孤立波解. 2000 年,Fen 在双曲正切函数展开法和齐次平衡法为基础,把双曲正切函数展开法中的 $\tanh(\xi)$ 替换成 Riccati 方程(1)的五个解,给出推广的双曲正切函数展开法[6],构造了非线性发展方程的有限多个新精确解[7—21],

$$\frac{3z(\xi)}{\mathrm{d}\xi} = z'(\xi) = R + z^2(\xi). \tag{1}$$

在文献[6]的影响下,在非线性发展方程求解领域中提出各种辅助方程法,而且获得诸多新成果. 在辅助方程法当中三角函数型辅助方程法和双曲函数型辅助方程法的提出和应用[22—33]显得更重要. 这种方法,不仅解决 sine-Gordon 型方程的求解问题,而且也能解决其他非线性发展方程的求解问题. sine-Gordon 型方程,是代表许多物理问题的数学模型. 比如,非线性光学、等离子体物理学和超导物理等. sine-Gordon 型方程和 sinh-Gordon 型方程求解方法的研究引起数学物理学家的广泛关注,而且这方面已取得了诸多研究成果[22—24,34—41]. 文献[22,23]用三角函数型辅助方程,得到了 sine-Gordon 型方程(2),(3),(4)的有限多个新精确解,

$$u_{xt} = \sin(u), \tag{2}$$

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sin(u), \tag{3}$$

$$u_{xt} = p[\sin(u) + 2\lambda\sin(2u)], \tag{4}$$

$$u_{xt} = \sinh(u). \tag{5}$$

文献[34]获得了(n+1)维 sine-Gordon 方程的两种精确解,

$$-u_{tt} + \sum_{i=1}^{n} u_{x_i t_i} = \sin(u). \tag{6}$$

文献[35]用反散射变换法,讨论 sine-Gordon 方程(2)孤立子解的简洁表达式. 文献[36]讨论 sine-Gordon 方程(7)的解的存在唯一性,

* 国家自然科学基金(批准号:10461006),内蒙古自治区高等学校科学研究基金(批准号:NJZZ07031)和内蒙古自治区自然科学基金(批准号:2010MS0111)资助的课题.

† E-mail: tgts@imnu.edu.cn

$$u_{tt} - u_t - \Delta u + \beta\sin(u) = f. \tag{7}$$

文献[37]讨论(2+1)维 sine-Gordon 方程,

$$u_{tt} - u_{xx} - u_{yy} + \sin(u) = 0. \tag{8}$$

文献[38]证明了 sine-Gordon 方程孤波解的存在性,

$$u_{tt} + \varepsilon u_t - u_{xx} + f(\sin(u))) = g(x,t). \tag{9}$$

文献[39]构造了下列(n+1)维双 sine-Gordon 方程的精确解:

$$-u_{tt} + m\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} = 2\alpha\sin(\mu u) + \beta\sin(2\mu u), \tag{10}$$

这里$\mu \geqslant 1,\alpha,\beta,m$ 为常数.

当$\mu = 1,\alpha = \frac{1}{2},\beta = 0,m = 1$ 时,方程 (10) 变成为下列(n+1)维 sine-Gordon 方程:

$$u_{tt} - \sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2} + \sin(u) = 0. \tag{11}$$

当 $n = 1$ 时,方程 (10) 变成为文献[40,41]讨论的 sine-Gordon 方程,

$$u_{tt} - mu_{xx} + 2\alpha\sin(\mu u) + \beta\sin(2\mu u) = 0. \tag{12}$$

许多文献讨论了 sine-Gordon 方程解的存在唯一性和求解等问题. 从求解问题的研究结果来看,获得了 sine-Gordon 型方程的有限多个新精确解,未能获得无穷序列精确解. 理论上已经证实,非线性发展方程存在无穷多个解. 本文为了获得 sine-Gordon 型方程的无穷序列精确解,给出两类辅助方程(三角函数型辅助方程和双曲函数型辅助方程)及其 Bäcklund 变换和解的非线性叠加公式,在符号计算系统 Mathematica 的帮助下,获得了 sine-Gordon 方程、mKdV-sine-Gordon 方程、(n+1)维双 sine-Gordon 方程和 sinh-Gordon 方程的无穷序列新精确解. 其中包括无穷序列三角函数解、无穷序列双曲函数解、无穷序列 Jacobi 椭圆函数解和无穷序列复合型解.

2. Riccati 方程的 Bäcklund 变换和解的非线性叠加公式

2.1. Riccati 方程 (1) 的 Bäcklund 变换

本文为了获得 sine-Gordon 型方程的无穷序列精确解,首先给出 Riccati 方程的 Bäcklund 变换和解的非线性叠加公式,然后利用这些结果获得两类辅助方程相应的结论.

2.1.1. Riccati 方程 (1) 的解

在文献[7—21]中利用 Riccati 方程的下列五个解 (13)—(17) 来构造了非线性发展方程的新精确解. 这里我们也给出了该方程的其他解,

$$z_0(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi) \qquad (R < 0), \tag{13}$$

$$z_0(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi) \qquad (R < 0), \tag{14}$$

$$z_0(\xi) = \sqrt{R}\tan(\sqrt{R}\xi) \qquad (R > 0), \tag{15}$$

$$z_0(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi) \qquad (R > 0), \tag{16}$$

$$z_0(\xi) = -\frac{1}{\xi} \qquad (R = 0), \tag{17}$$

$$z_1(\xi) = \frac{BR + A\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-A + B\sqrt{-R}\tanh(\sqrt{-R}\xi)} \qquad (R < 0), \tag{18}$$

$$z_1(\xi) = \frac{-(r\sqrt{R} + CR)\cos(\sqrt{R}\xi) + (r - C\sqrt{R})\sqrt{R}\sin(\sqrt{R}\xi)}{(r - C\sqrt{R})\cos(\sqrt{R}\xi) + (r + C\sqrt{R})\sin(\sqrt{R}\xi)} \qquad (R > 0), \tag{19}$$

$$z_1(\xi) = \frac{-3BR + 4A\sqrt{R} - 5BR\sin(2\sqrt{R}\xi) - 5A\sqrt{R}\cos(2\sqrt{R}\xi)}{3A + 4B\sqrt{R} + 5A\sin(2\sqrt{R}\xi) - 5B\sqrt{R}\cos(2\sqrt{R}\xi)} \qquad (R > 0), \tag{20}$$

$$z_1(\xi) = \frac{-BR + A\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}{A + B\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]} \qquad (R > 0), \tag{21}$$

$$z_1(\xi) = \frac{\sqrt{R}[\cos(\sqrt{R}\xi) + \sin(\sqrt{R}\xi)]}{\cos(\sqrt{R}\xi) - \sin(\sqrt{R}\xi)} \qquad (R > 0), \tag{22}$$

$$z_1(\xi) = \frac{\sqrt{R}[-2AB\sqrt{R} + (A^2 - B^2R)[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]]}{A^2 - B^2R + 2AB\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]} \qquad (R > 0), \tag{23}$$

其中 r, A, B, C 是不全为零的任意常数, R 是 Riccati 方程来确定的任意常数.

2.1.2. 若 $z(\xi)$ 是 Riccati 方程 (1) 的解,则下面给出的 $\bar{z}(\xi)$ 也是 Riccati 方程 (1) 的解

$$\bar{z}(\xi) = \frac{p_0 + q_0 z(\xi) + m_0 z^2(\xi) + r_0 z'(\xi) + n_0 z^3(\xi) + l_0[z'(\xi)]^2}{A_0 + B_0 z(\xi) + D_0 z^2(\xi) + C_0 z'(\xi) + F_0 z^3(\xi) + K_0[z'(\xi)]^2}, \tag{24}$$

2.1.3. 若 $z(\xi)$ 是 Riccati 方程 (1) 的解,则下列 $\bar{z}(\xi)$ 也是 Riccati 方程 (1) 的解

$$\bar{z}(\xi) = \frac{-BR + Az(\xi)}{A + Bz(\xi)}. \tag{25}$$

Riccati 方程 (1) 的任意解与 Bäcklund 变换 (24), (25) 相结合,获得 Riccati 方程 (1) 的无穷序列新精确解. 下面列出解的四种叠加公式(其余公式这里不列出):

$$z_k(\xi) = \frac{p_0 + q_0 z_{k-1}(\xi) + m_0 z_{k-1}^2(\xi) + r_0 z_{k-1}'(\xi) + n_0 z_{k-1}^3(\xi) + l_0[z_{k-1}'(\xi)]^2}{A_0 + B_0 z_{k-1}(\xi) + D_0 z_{k-1}^{\ 2}(\xi) + C_0 z_{k-1}'(\xi) + F_0 z_{k-1}^{\ 3}(\xi) + K_0[z_{k-1}'(\xi)]^2},$$

$$z_0(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi) \quad (R < 0, k = 1, 2, \cdots); \tag{26}$$

$$z_k(\xi) = \frac{p_0 + q_0 z_{k-1}(\xi) + m_0 z_{k-1}^2(\xi) + r_0 z_{k-1}'(\xi) + n_0 z_{k-1}^3(\xi) + l_0[z_{k-1}'(\xi)]^2}{A_0 + B_0 z_{k-1}(\xi) + D_0 z_{k-1}^2(\xi) + C_0 z_{k-1}'(\xi) + F_0 z_{k-1}^3(\xi) + K_0[z_{k-1}'(\xi)]^2},$$

$$z_0(\xi) = \sqrt{R}\tan(\sqrt{R}\xi) \quad (R > 0, k = 1, 2, \cdots); \tag{27}$$

$$z_k(\xi) = \frac{-BR + Az_{k-1}(\xi)}{A + Bz_{k-1}(\xi)} \quad (k = 1, 2, \cdots),$$

$$z_0(\xi) = \frac{\sqrt{R}[-2AB\sqrt{R} + (A^2 - B^2R)[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]]}{A^2 - B^2R + 2AB\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]} \quad (R > 0). \tag{28}$$

其中 $p_0, q_0, r_0, m_0, n_0, l_0, A_0, B_0, C_0, D_0, F_0, K_0, A, B$ 是不全为零的任意常数,而且满足下列约束条件:

$$p_0 = R(-B_0 + m_0 + F_0 R),$$

$$q_0 = \frac{B_0 l_0^{\ 2} - (l_0^{\ 2} + K_0^{\ 2}R)[m_0 + r_0 + (F_0 + l_0)R]}{K_0 l_0},$$

$$A_0 = \frac{B_0 l_0 - l_0^{\ 2}R - l_0(m_0 + r_0 + F_0 R) - K_0 R(C_0 + K_0 R)}{K_0},$$

$$n_0 = \frac{1}{K_0}(F_0 l_0 - l_0^{\ 2} - K_0^{\ 2}R),$$

$$D_0 = -C_0 + \frac{1}{K_0}(F_0 l_0 - l_0^{\ 2}) + \frac{1}{l_0}(m_0 + r_0 + F_0 R)K_0 - K_0 R,$$

以下相同.

2.2. Riccati 方程解的非线性叠加公式

2.2.1. 若 $z_1(\xi), z_2(\xi)$ 是 Riccati 方程 (1) 的解,则下列 $\bar{z}(\xi)$ 也是 Riccati 方程 (1) 的解

$$\bar{z}(\xi) = \frac{\mathrm{i}R[\mathrm{i}m_1\sqrt{R} + (m_1 + \mathrm{i}D_1\sqrt{R} + C_1Rz_2(\xi) + [-C_1R + D_1z_2(\xi)]z_1(\xi)]}{-\sqrt{R^3}[D_1 + C_1z_2(\xi)] + [m_1\sqrt{R} + \mathrm{i}D_1R + C_1\sqrt{R^3} - \mathrm{i}m_1z_2(\xi)]z_1(\xi)} \quad (m_1D_1 < 0), \tag{29}$$

$$\bar{z}(\xi) = \frac{m_1 + D_1z_2(\xi) + \frac{1}{\sqrt{R}}[-\mathrm{i}C_1Rz_1(\xi) + \mathrm{i}[m_1 + C_1R + D_1z_1(\xi)]z_2(\xi)]}{D_1 + C_1z_2(\xi) - \frac{1}{\sqrt{R^3}}[m_1\sqrt{R} - \mathrm{i}D_1R + C_1\sqrt{R^3} + im_1z_2(\xi)]z_1(\xi)} \quad (m_1D_1 < 0), \tag{30}$$

根据 Riccati 方程 (1) 的已知解与解的非线性叠加公式 (29), (30), 获得 Riccati 方程 (13) 的下列解的非线性叠加公式(只列出一种公式):

$$z_k(\xi) = \frac{\mathrm{i}R[\mathrm{i}m_1\sqrt{R} + (m_1 + iD_1\sqrt{R} + C_1R)z_{k-1}(\xi) + [-C_1R + D_1z_{k-1}(\xi)]z_{k-2}(\xi)]}{-\sqrt{R^3}[D_1 + C_1z_{k-1}(\xi)] + [m_1\sqrt{R} + iD_1R + C_1\sqrt{R^3} - im_1z_{k-1}(\xi)]z_{k-2}(\xi)} \quad (m_1D_1 < 0),$$

$$z_0(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi)\quad z_1(\xi)=-\frac{1}{\xi}(k=2,3,\cdots). \tag{31}$$

$k=2$ 时,从解的非线性叠加公式(31),得到 Riccati 方程(1)下列复合型解:

$$z_{20}(\xi)=\frac{mv_1\left[\sqrt{\frac{m_1^2}{D_1^2}}(D_1-m_1\xi)+m_1\left[-1+\tanh\left(\frac{m_1}{D_1}\xi\right)\right]\right]}{D_1{}^3\sqrt{\left(\frac{m_1^2}{D_1^2}\right)^3}\xi+\left[D_1m_1^2-m_1{}^3\xi+D_1^3\left(\frac{m_1}{D_1}\right)^3\right]\tanh\left(\frac{m_1}{D_1}\xi\right)}\quad(m_1D_1<0), \tag{32}$$

$$z_{21}(\xi)=-\frac{m_1^2\left[-\frac{m_1^2C_1}{D_1^2}-D_1\sqrt{\frac{m_1^2}{D_1^2}}+m_1\left(1+\sqrt{\frac{m_1^2}{D_1^2}}\xi\right)-\sqrt{\frac{m_1^2}{D_1^2}}\left(D_1-\frac{C_1m_1^2}{D_1^2}\xi\right)\tanh\left(\frac{m_1}{D_1}\xi\right)\right]}{D_1^2\left[\sqrt{\left(\frac{m_1^2}{D_1^2}\right)^3}(-C_1+D_1\xi)+\Xi(\xi)\tanh\left(\frac{m_1}{D_1}\xi\right)\right]}\quad(m_1D_1<0). \tag{33}$$

这里 $\Xi(\xi)=m_1\left[\sqrt{\frac{m_1^2}{D_1^2}}+\frac{m_1^2}{D_1^2}\xi-\frac{1}{m_1}\sqrt{\left(\frac{m_1^2}{D_1^2}\right)^3}\left(D_1+C_1\sqrt{\frac{m_1^2}{D_1^2}}\right)\xi\right]$, m_1, D_1 满足 $m_1D_1<0$ 的任意常数.

2.2.2. 若 $z_1(\xi), z_2(\xi), z_3(\xi)$ 是 Riccati 方程(1)的三个解,则下面给出的 $\bar{z}(\xi)$ 也是 Riccati 方程(1)的解

$$\bar{z}(\xi)=\frac{R[-rz_1(\xi)+(p+r)z_2(\xi)-pz_3(\xi)]}{-rz_2(\xi)z_3(\xi)+z_1(\xi)[-pz_2(\xi)+(p+r)z_3(\xi)]}, \tag{34}$$

$$\bar{z}(\xi)=\frac{rz_2(\xi)z_3(\xi)-z_1(\xi)[qz_2(\xi)+(-q+r)z_3(\xi)]}{-rz_1(\xi)+(-q+r)z_2(\xi)+qz_3(\xi)}, \tag{35}$$

$$\bar{z}(\xi)=-\frac{R[-Nz_3(\xi)+[-L+mz_3(\xi)]z_2(\xi)+[L+N+nz_2(\xi)-(m+n)z_3(\xi)]z_1(\xi)]}{nRz_3(\xi)+[mR+Nz_2(\xi)+Lz_3(\xi)]z_1(\xi)-[(m+n)R+(L+N)z_3(\xi)]z_2(\xi)}. \tag{36}$$

这里 p, q, r, m, n, N, L 是不全为零的任意常数.

根据解的非线性叠加公式(34)—(36),获得 Riccati 方程(1)的无穷序列复合型精确解,下面列出解的几种非线性叠加公式:

$$z_k(\xi)=\frac{R[-rz_{k-3}(\xi)+(p+r)z_{k-2}(\xi)-pz_{k-1}(\xi)]}{-rz_{k-2}(\xi)z_{k-1}(\xi)+z_{k-3}(\xi)[-pz_{k-2}(\xi)+(p+r)z_{k-1}(\xi)]}\quad(k=3,4,\cdots),$$

$$z_0(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi),\ z_1(\xi)=\frac{BR+A\sqrt{-R}\tanh(\sqrt{-R}\xi)}{-A+B\sqrt{-R}\tanh(\sqrt{-R}\xi)},\quad z_2(\xi)=-\frac{1}{\xi}. \tag{37}$$

当 $k=3$ 时,从非线性叠加公式(37)得到 Riccati 方程(1)的如下复合型精确解:

$$z_{31}(\xi)=-\frac{R[-A+\sqrt{-R}(B+A\xi)\tanh(\sqrt{-R}\xi)+BR\xi\tanh^2(\sqrt{-R}\xi)]}{BR-\sqrt{-R}(-A+BR\xi)\tanh(\sqrt{-R}\xi)+AR\xi\tanh^2(\sqrt{-R}\xi)}, \tag{38}$$

$$z_k(\xi)=\frac{R[-rz_{k-3}(\xi)+(p+r)z_{k-2}(\xi)-pz_{k-1}(\xi)]}{-rz_{k-2}(\xi)z_{k-1}(\xi)+z_{k-3}(\xi)[-pz_{k-2}(\xi)+(p+r)z_{k-1}(\xi)]}\quad(k=3,4,\cdots),$$

$$z_0(\xi)=\frac{-BR+A\sqrt{R}[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]}{A+B\sqrt{R}[\sec(2\sqrt{R}\xi)+\tan(2\sqrt{R}\xi)]},$$

$$z_1(\xi)=\frac{\sqrt{R}[\cos(\sqrt{R}\xi)+\sin(\sqrt{R}\xi)]}{\cos(\sqrt{R}\xi)-\sin(\sqrt{R}\xi)},\quad z_2(\xi)=-\frac{1}{\xi}. \tag{39}$$

当 $k=3$ 时,从非线性叠加公式(39)得到 Riccati 方程(1)的如下复合型精确解:

$$z_{32}(\xi)=\frac{\sqrt{R}[A+BR\xi+\sqrt{R}(B+A\xi)\cos(2\sqrt{R}\xi)+(-A+BR\xi)\sin(2\sqrt{R}\xi)]}{(-A+BR\xi)\cos(2\sqrt{R}\xi)+\sqrt{R}[B-A\xi-(B+A\xi)\sin(2\sqrt{R}\xi)]}. \tag{40}$$

$$z_k(\xi)=\frac{R[-rz_{k-3}(\xi)+(p+r)z_{k-2}(\xi)-pz_{k-1}(\xi)]}{-rz_{k-2}(\xi)z_{k-1}(\xi)+z_{k-3}(\xi)[-pz_{k-2}(\xi)+(p+r)z_{k-1}(\xi)]}\quad(k=3,4,\cdots),$$

$$z_0(\xi) = \frac{\sqrt{R}[\cos(\sqrt{R}\xi) + \sin(\sqrt{R}\xi)]}{\cos(\sqrt{R}\xi) - \sin(\sqrt{R}\xi)},$$

$$z_1(\xi) = \frac{\sqrt{R}[-2AB\sqrt{R} + (A^2 - B^2R)[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]]}{A^2 - B^2R + 2AB\sqrt{R}[\sec(2\sqrt{R}\xi) + \tan(2\sqrt{R}\xi)]}, \quad z_2(\xi) = -\frac{1}{\xi}. \tag{41}$$

当 $k = 3$ 时,从非线性叠加公式 (41) 得到 Riccati 方程 (1) 的如下复合型精确解:

$$z_{33}(\xi) = -\frac{\sqrt{R}[\phi(\xi) + \sqrt{R}\varphi(\xi)\cos(2\sqrt{R}\xi) + \chi(\xi)\sin(2\sqrt{R}\xi)]}{-\chi(\xi)\cos(2\sqrt{R}\xi) + \sqrt{R}[\mu(\xi) + \varphi(\xi)\sin(2\sqrt{R}\xi)]}. \tag{42}$$

其中 $\phi(\xi) = -A^2 + B^2R - 2ABR\xi$, $\varphi(\xi) = -2AB + (-A^2 + B^2R)\xi$, $\chi(\xi) = A^2 - B^2R - 2ABR\xi$, $\mu(\xi) = 2AB + (-A^2 + B^2R)\xi$. 这里只列出几种 Bäcklund 变换和解的非线性叠加公式.

3. 两种椭圆辅助方程的 Bäcklund 变换

3.1. 两种椭圆辅助方程之间的 Bäcklund 变换

许多文献利用下列两种椭圆辅助方程,构造了非线性发展方程的有限多个 Jacobi 椭圆函数精确解:

$$\left(\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right)^2 = (z'(\xi))^2 = e + fz^2(\xi) + gz^4(\xi), \tag{43}$$

$$\left(\frac{\mathrm{d}z(\xi)}{\mathrm{d}\xi}\right)^2 = (z'(\xi))^2 = az(\xi) + bz^2(\xi) + cz^3(\xi). \tag{44}$$

第一种椭圆辅助方程 (43),通过下式变换:

$$z^2(\xi) = y(\xi), \tag{45}$$

转化为第二种椭圆辅助方程

$$[y'(\xi)]^2 = 4ey(\xi) + 4fy^2(\xi) + 4gy^3(\xi). \tag{46}$$

3.2. 第二种椭圆辅助方程的 Bäcklund 变换

若 $z_{n-1}(\xi)$ 是第二种椭圆方程 (44) 的解,则下列 $z_n(\xi)$ 也是方程 (44) 的解:

$$z_n(\xi) = \mp\frac{2a + (b \pm \sqrt{b^2 - 4ac})z_{n-1}(\xi)}{\pm b + \sqrt{b^2 - 4ac} \pm 2cz_{n-1}(\xi)}, \quad (n = 1,2,\cdots), \tag{47}$$

$$z_n(\xi) = \frac{a[-b + \sqrt{b^2 - 4ac} - 2cz_{n-1}(\xi)]}{c[2a + (b - \sqrt{b^2 - 4ac})z_{n-1}(\xi)]}, \quad (n = 1,2,\cdots) \tag{48}$$

$$z_n(\xi) = \frac{-ab^2 \pm a\sqrt{b^2(b^2 - 4ac)} - 4abcz_{n-1}(\xi) + [-b^2c \mp c\sqrt{b^2(b^2 - 4ac)}]z_{n-1}{}^2(\xi)}{2abc + 2b^2cz_{n-1}(\xi) + 2bc^2z_{n-1}^2(\xi)}, \quad (n = 1,2,\cdots), \tag{49}$$

$$z_n(\xi) = \frac{a[-\sqrt{3A_2} \mp 9z'_{n-1}(\xi)]}{\sqrt{3A_2}(b - \sqrt{b^2 - 3ac}) + 2\sqrt{3A_2}cz_{n-1}(\xi) \pm 3(b + \sqrt{b^2 - 3ac})z'_{n-1}(\xi)}, \quad (n = 1,2,\cdots), \tag{50}$$

$$z_n(\xi) = \frac{a[-\sqrt{3B_2} \pm 9z'_{n-1}(\xi)]}{\sqrt{3B_2}(b + \sqrt{b^2 - 3ac}) + 2\sqrt{3B_2}cz_{n-1}(\xi) \pm 3(-b + \sqrt{b^2 - 3ac})z'_{n-1}(\xi)}, (n = 1,2,\cdots), \tag{51}$$

$$z_n(\xi) = \frac{\sqrt{3A_2}(-b + \sqrt{b^2 - 3ac}) - 2\sqrt{3A_2}cz_{n-1}(\xi) \pm 3(b + \sqrt{b^2 - 3ac})z'_{n-1}(\xi)}{c[\sqrt{3A_2} \mp 9z'_{n-1}(\xi)]}, \quad (n = 1,2,\cdots), \tag{52}$$

$$z_n(\xi) = \frac{-\sqrt{3B_2}(b + \sqrt{b^2 - 3ac}) - 2\sqrt{3B_2}cz_{n-1}(\xi) \pm 3(-b + \sqrt{b^2 - 3ac})z'_{n-1}(\xi)}{c[\sqrt{3B_2} \pm 9z'_{n-1}(\xi)]}, \quad (n = 1,2,\cdots), \tag{53}$$

其中 $A_2 = \sqrt{\frac{1}{c^2}[2b^3 - 9abc + 2(b^2 - 3ac)^{3/2}]}, B_2 = \sqrt{\frac{1}{c^2}[2b^3 - 9abc - 2(b^2 - 3ac)^{3/2}]}$.

3.3. 第一种椭圆辅助方程的 Bäcklund 变换

根据第二种椭圆辅助方程(44),(46)的 Bäcklund 变换和变换(45),可以获得第一种椭圆辅助方程(43)相应的 Bäcklund 变换. 下面列出两种 Bäcklund 变换(其余情况不列出).

若 $z_{n-1}(\xi)$ 是第一种椭圆方程(43)的解,则下列 $z_n(\xi)$ 也是方程(43)的解:

$$z_n^2(\xi) = \mp \frac{8e + (4f \pm \sqrt{16f^2 - 64eg})z_{n-1}^2(\xi)}{\pm 4f + \sqrt{16f^2 - 64eg} \pm 8gz_{n-1}^2(\xi)}, \quad (n = 1,2,\cdots), \tag{54}$$

$$z_n^2(\xi) = \frac{e[-\sqrt{3A_2} \mp 9(z'_{n-1}(\xi))^2]}{\sqrt{3A_2}(f - \sqrt{f^2 - 3eg}) + 2\sqrt{3A_2}gz_{n-1}{}^2(\xi) \pm 3(f + \sqrt{f^2 - 3eg})(z'_{n-1}(\xi))^2},$$
$$(n = 1,2,\cdots), \tag{55}$$

其中 $A_2 = \sqrt{\frac{1}{16g^2}[128f^3 - 432efg + 2(16f^2 - 48eg)^{3/2}]}$.

4. 两类辅助方程的 Bäcklund 变换和解的非线性叠加公式

4.1. 三角函数型辅助方程的 Bäcklund 变换和解的非线性叠加公式

下面引入两种三角函数型辅助方程以及它们的 Bäcklund 变换和解的非线性叠加公式,

$$\left(\frac{du(\xi)}{d\xi}\right)^2 = a\sin^2(\mu u(\xi)) + b\cos(\mu u(\xi)) + c, \tag{56}$$

$$\frac{du(\xi)}{d\xi} = a\sin(\mu u(\xi)) + b\cos(\mu u(\xi)) + c, \tag{57}$$

经变换 $\sin(\mu u(\xi)) = \frac{2\tan\left(\frac{\mu u(\xi)}{2}\right)}{1 + \tan^2\left(\frac{\mu u(\xi)}{2}\right)}, \cos(\mu u(\xi)) = \frac{1 - \tan^2\left(\frac{\mu u(\xi)}{2}\right)}{1 + \tan^2\left(\frac{\mu u(\xi)}{2}\right)}, \tan\left(\frac{\mu u(\xi)}{2}\right) = v(\xi)$ 方程(56),(57)变成下列形式的第一种椭圆辅助方程:

$$\left(\frac{dv(\xi)}{d\xi}\right)^2 = \frac{(c - b)\mu^2}{4}v^4(\xi) + \frac{(4a + 2c)\mu^2}{4}v^2(\xi) + \frac{(b + c)\mu^2}{4} \tag{58}$$

和 Riccati 方程

$$\frac{dv(\xi)}{d\xi} = \frac{(c - b)\mu}{2}v^2(\xi) + a\mu v(\xi) + \frac{(b + c)\mu}{2}. \tag{59}$$

方程(58)通过下式变换:

$$v^2(\xi) = y(\xi), \tag{60}$$

转化为下列方程:

$$\left(\frac{dy(\xi)}{d\xi}\right)^2 = (b + c)\mu^2 y(\xi) + (4a + 2c)\mu^2 y^2(\xi) + (c - b)\mu^2 y^3(\xi). \tag{61}$$

根据第二种椭圆辅助方程的 Bäcklund 变换(47)—(53)和变换(60),可以获得三角函数型辅助方程(56)相应的 Bäcklund 变换(这里列出解的两种非线性叠加公式),

$$u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n{}^2(\xi) = y_n(\xi),$$

$$y_n(\xi) = \mp \frac{2(b + c) + [(4a + 2c) \pm \sqrt{(4a + 2c)^2 - 4(c^2 - b^2)}]y_{n-1}(\xi)}{\pm(4a + 2c) + \sqrt{(4a + 2c)^2 - 4(c^2 - b^2)} \pm 2(c - b)y_{n-1}(\xi)}, \quad (n = 1,2,\cdots). \tag{62}$$

$$u_n(\xi)=\frac{2}{\mu}\arctan v_n(\xi), v_n{}^2(\xi)=y_n(\xi),$$

$$y_n(\xi)=\frac{-\sqrt{3B_2}[C_2+2(c-b)y_{n-1}(\xi)]\pm3[-(4a+2c)+\sqrt{(4a+2c)^2-3(c^2-b^2)}]y'_{n-1}(\xi)}{(c-b)[\sqrt{3B_2}\pm9y_{n-1}{}'(\xi)]},$$

$$(n=1,2,\cdots),$$

$$C_2=[(4a+2c)+\sqrt{(4a+2c)^2-3(c^2-b^2)}].\tag{63}$$

其中

$$B_2=\sqrt{\frac{1}{(c-b)^2\mu^4}[2(4a+2c)^3\mu^6-9(4a+2c)(c^2-b^2)\mu^6-2[(4a+2c)^2\mu^4-3(c^2-b^2)\mu^4]^{3/2}]}.$$

方程 (59) 通过下式变换:

$$v(\xi)=\frac{2y(\xi)}{(c-b)\mu}-\frac{a}{c-b},\tag{64}$$

转化为下列 Riccati 方程:

$$\frac{\mathrm{d}y(\xi)}{\mathrm{d}\xi}=y^2(\xi)+\frac{(c^2-a^2-b^2)\mu^2}{4}.\tag{65}$$

根据以上得到的 Riccati 方程 (1) 的 Bäcklund 变换和解的非线性叠加公式,获得 Riccati 方程 (65) 以及三角函数型辅助方程 (57) 的 Bäcklund 变换和解的非线性叠加公式(限于篇幅这里没列出).

4.2. 双曲函数型辅助方程的 Bäcklund 变换和解的非线性叠加公式

下面引入两种双曲函数型辅助方程和它们的 Bäcklund 变换和解的非线性叠加公式:

$$\left(\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}\right)^2=a\sinh^2(\mu u(\xi))+b\cosh(\mu u(\xi))\sinh(\mu u(\xi))+c,\tag{66}$$

$$\frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}=a\sinh(\mu u(\xi))+b\cosh(\mu u(\xi))+c.\tag{67}$$

经变换 $\sinh(\mu u(\xi))=\frac{v^2(\xi)-1}{2v(\xi)}$, $\cosh(\mu u(\xi))=\frac{v^2(\xi)+1}{2v(\xi)}$, $\exp(\mu u(\xi))=v(\xi)$ 方程 (66), (67) 变成第一种椭圆辅助方程

$$\left(\frac{\mathrm{d}v(\xi)}{\mathrm{d}\xi}\right)^2=\frac{(a+b)\mu^2}{4}v^4(\xi)+\frac{(-2a+4c)\mu^2}{4}v^2(\xi)+\frac{(a-b)\mu^2}{4}\tag{68}$$

和 Riccati 方程

$$\frac{\mathrm{d}v(\xi)}{\mathrm{d}\xi}=\frac{(a+b)\mu}{2}v^2(\xi)+c\mu v(\xi)+\frac{(b-a)\mu}{2}.\tag{69}$$

方程 (68), (69) 通过变换

$$v^2(\xi)=y(\xi),\tag{70}$$

和

$$v(\xi)=\frac{2y(\xi)}{(a+b)\mu}-\frac{c}{a+b},\tag{71}$$

分别转化为第二种椭圆辅助方程

$$\left(\frac{\mathrm{d}y(\xi)}{\mathrm{d}\xi}\right)^2=(a+b)\mu^2y^3(\xi)+(-2a+4c)\mu^2y^2(\xi)+(a-b)\mu^2y(\xi)\tag{72}$$

和 Riccati 方程

$$\frac{\mathrm{d}y(\xi)}{\mathrm{d}\xi}=y^2(\xi)+\frac{(b^2-a^2-c^2)\mu^2}{4}.\tag{73}$$

这里 μ, a, b, c 是常数. 同理,根据以上得到的两种椭圆辅助方程和 Riccati 方程的 Bäcklund 变换以及解的非线性叠加公式,可以获得双曲函数型辅助方程 (66), (67) 相应的结果(这里不列出).

下面给出第二种椭圆方程 (72) 的几种解. 这些解在构造 sinh-Gordon 方程的无穷序列精确解时发挥重要作用.

当 $a=\frac{1}{\mu^2}, b=0, c=\frac{1-k^2}{\mu^2}$ 时,

$$y(\xi)=(\mathrm{ns}(\xi,k)\pm\mathrm{cs}(\xi,k))^2,$$

$$y(\xi)=(k\mathrm{sn}(\xi,k)\pm i\mathrm{dn}(\xi,k))^2,$$

$$y(\xi)=\frac{\mathrm{sn}^2(\xi,k)}{(1\pm\mathrm{cn}(\xi,k))^2};\tag{74}$$

当 $a=\frac{1-k^2}{\mu^2}, b=0, c=\frac{1}{\mu^2}$ 时,

$$y(\xi)=(\mathrm{nc}(\xi,k)\pm\mathrm{sc}(\xi,k))^2,$$

$$y(\xi)=\frac{\mathrm{cn}^2(\xi,k)}{(1\pm\mathrm{sn}(\xi,k))^2};\tag{75}$$

当 $a=\frac{k^2}{\mu^2}, b=0, c=\frac{-1+k^2}{\mu^2}$ 时,

$$y(\xi) = (\mathrm{sn}(\xi,k) \pm \mathrm{icn}(\xi,k))^2,$$

$$y(\xi) = \frac{\mathrm{dn}^2(\xi,k)}{(\sqrt{1-k^2}\mathrm{sn}(\xi,k) \pm \mathrm{cn}(\xi,k))^2},$$

$$y(\xi) = \frac{k^2\mathrm{sn}^2(\xi,k)}{(1 \pm \mathrm{dn}(\xi,k))^2}; \tag{76}$$

当 $a = \frac{-1+k^2}{\mu^2}, b = 0, c = \frac{k^2}{\mu^2}$ 时,

$$y(\xi) = \frac{\mathrm{dn}^2(\xi,k)}{(1 \pm k\mathrm{sn}(\xi,k))^2}. \tag{77}$$

5. sine-Gordon 型方程的无穷序列新精确解

下面利用两类辅助方程以及 Bäcklund 变换和解的非线性叠加公式,构造 sinh-Gordon 方程、sine-Gordon 方程、mKdV-sine-Gordon 方程和(n+1)维双 sine-Gordon 方程的无穷序列新精确解. 其中包括无穷序列三角函数解、无穷序列双曲函数解、无穷序列 Jacobi 椭圆函数解和无穷序列复合型解.

例 1 sinh-Gordon 方程的无穷序列精确解.

将 $u(x,t) = u(\xi), \xi = \lambda x + \omega t$ 代入 sinh-Gordon 方程 (5) 后得到

$$\lambda\omega u''(\xi) = \sinh(u(\xi)). \tag{78}$$

当 $\mu = \frac{1}{2}$ 时,把辅助方程 (66) 代入方程 (78),并令 $\sinh(u(\xi)), \cosh(u(\xi))$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$b = 0, a = \frac{4}{\lambda\omega}. \tag{79}$$

根据已经获得的解 (79) 和 (74),给出构造 sinh - Gordon 方程无穷序列 Jacobi 椭圆函数精确解的下列非线性叠加公式:

$$\exp\left(\frac{1}{2}u_n(\xi)\right) = v_n(\xi), \quad v_n{}^2(\xi) = y_n(\xi), \quad \left(\xi = \frac{1}{\omega}x + \omega t\right),$$

$$y_n(\xi) = \frac{-\left(\frac{1}{2}-k^2\right)^2 \pm K - (2-4k^2)y_{n-1}(\xi) + \left[-\left(\frac{1}{2}-k^2\right)^2 \mp K\right]y_{n-1}{}^2(\xi)}{(1-2k^2) + 2\left(\frac{1}{2}-k^2\right)^2 y_{n-1}(\xi) + 2\left(\frac{1}{2}-k^2\right)y_{n-1}{}^2(\xi)}, \quad (n = 1,2,\cdots),$$

$$y_0(\xi) = (\mathrm{ns}(\xi,k) \pm \mathrm{cs}(\xi,k))^2, \quad K = \sqrt{\left(\frac{1}{2}-k^2\right)^2\left[\left(\frac{1}{2}-k^2\right)^2 - 4\right]}; \tag{80}$$

$$\exp\left(\frac{1}{2}u_n(\xi)\right) = v_n(\xi), \quad v_n{}^2(\xi) = y_n(\xi), \quad \left(\xi = \frac{1}{\omega}x + \omega t\right),$$

$$y_n(\xi) = \frac{-\left(\frac{1}{2}-k^2\right)^2 \pm K - (2-4k^2)y_{n-1}(\xi) + \left[-\left(\frac{1}{2}-k^2\right)^2 \mp K\right]y_{n-1}{}^2(\xi)}{(1-2k^2) + 2\left(\frac{1}{2}-k^2\right)^2 y_{n-1}(\xi) + 2\left(\frac{1}{2}-k^2\right)y_{n-1}{}^2(\xi)}, \quad (n = 1,2,\cdots),$$

$$y_0(\xi) = (k\mathrm{sn}(\xi,k) \pm \mathrm{idn}(\xi,k))^2, \quad K = \sqrt{\left(\frac{1}{2}-k^2\right)^2\left[\left(\frac{1}{2}-k^2\right)^2 - 4\right]}; \tag{81}$$

$$\exp\left(\frac{1}{2}u_n(\xi)\right) = v_n(\xi), \quad v_n{}^2(\xi) = y_n(\xi), \quad \left(\xi = \frac{1}{\omega}x + \omega t\right);$$

$$y_n(\xi) = \frac{-\left(\frac{1}{2}-k^2\right)^2 \pm K - (2-4k^2)y_{n-1}(\xi) + \left[-\left(\frac{1}{2}-k^2\right)^2 \mp K\right]y_{n-1}{}^2(\xi)}{(1-2k^2) + 2\left(\frac{1}{2}-k^2\right)^2 y_{n-1}(\xi) + 2\left(\frac{1}{2}-k^2\right)y_{n-1}{}^2(\xi)}, \quad (n = 1,2,\cdots),$$

$$y_0(\xi) = \frac{\mathrm{sn}^2(\xi,k)}{[1 \pm \mathrm{cn}(\xi,k)]^2}, \quad K = \sqrt{\left(\frac{1}{2}-k^2\right)^2\left[\left(\frac{1}{2}-k^2\right)^2 - 4\right]}. \tag{82}$$

当 $\mu = \frac{1}{2}$ 时,把辅助方程 (67) 代入方程 (78),并令 $\sinh(u(\xi)), \cosh(u(\xi)), \sinh\left(\frac{u(\xi)}{2}\right)$, $\cosh\left(\frac{u(\xi)}{2}\right)$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$c=0, b=0, a=\mp\frac{2}{\sqrt{\lambda\omega}}, \tag{83}$$

$$c=0, a=0, b=\mp\frac{2}{\sqrt{\lambda\omega}}. \tag{84}$$

当 $c=0, b=0$ 时,用下列解的非线性叠加公式来构造 sinh-Gordon 方程的无穷序列精确解. 这里包括无穷序列复合型精确解:

$$\exp\left(\frac{1}{2}u_n(\xi)\right)=v_n(\xi), v_n(\xi)=\frac{4y_n(\xi)}{a}, \quad (\xi=\lambda x+\omega t),$$

$$y_n(\xi)=\frac{p_0+q_0y_{n-1}(\xi)+m_0y_{n-1}^2(\xi)+r_0y'_{n-1}(\xi)+n_0y_{n-1}{}^3(\xi)+l_0[y'_{n-1}(\xi)]^2}{A_0+B_0y_{n-1}(\xi)+D_0y_{n-1}{}^2(\xi)+C_0y'_{n-1}(\xi)+F_0y_{n-1}{}^3(\xi)+K_0[y'_{n-1}(\xi)]^2}, \quad (n=1,2,\cdots),$$

$$y_0(\xi)=-\sqrt{-R}\tanh(\sqrt{-R}\xi), \quad \left(R=-\frac{a^2}{16}<0, a=\mp\frac{2}{\sqrt{\lambda\omega}}\right); \tag{85}$$

$$\exp\left(\frac{1}{2}u_n(\xi)\right)=v_n(\xi), \quad v_n(\xi)=\frac{4y_n(\xi)}{a}, \quad \left(a=\mp\frac{2}{\sqrt{\lambda\omega}}, \xi=\lambda x+\omega t\right),$$

$$y_n(\xi)=\frac{p_0+q_0y_{n-1}(\xi)+m_0y_{n-1}{}^2(\xi)+r_0y'_{n-1}(\xi)+n_0y_{n-1}{}^3(\xi)+l_0[y'_{n-1}(\xi)]^2}{A_0+B_0y_{n-1}(\xi)+D_0y_{n-1}{}^2(\xi)+C_0y'_{n-1}(\xi)+F_0y_{n-1}{}^3(\xi)+K_0[y'_{n-1}(\xi)]^2}, \quad (n=1,2,\cdots),$$

$$y_0(\xi)=-\frac{R[-A+\sqrt{-R}(B+A\xi)\tanh(\sqrt{-R}\xi)+BR\xi\tanh^2(\sqrt{-R}\xi)]}{BR-\sqrt{-R}(-A+BR\xi)\tanh(\sqrt{-R}\xi)+AR\xi\tanh^2(\sqrt{-R}\xi)}, \quad \left(R=-\frac{a^2}{16}<0\right). \tag{86}$$

当 $c=0, a=0$ 时,用下列解的非线性叠加公式来构造 sinh - Gordon 方程的无穷序列精确解:

$$\exp\left(\frac{1}{2}u_n(\xi)\right)=v_n(\xi), v_n(\xi)=\frac{4y_n(\xi)}{b}, \quad (\xi=\lambda x+\omega t),$$

$$y_n(\xi)=\frac{p_0+q_0y_{n-1}(\xi)+m_0y_{n-1}{}^2(\xi)+r_0y'_{n-1}(\xi)+n_0y_{n-1}{}^3(\xi)+l_0[y'_{n-1}(\xi)]^2}{A_0+B_0y_{n-1}(\xi)+D_0y_{n-1}{}^2(\xi)+C_0y'_{n-1}(\xi)+F_0y_{n-1}^3(\xi)+K_0[y'_{n-1}(\xi)]^2}, \quad (n=1,2,\cdots),$$

$$y_0(\xi)=\sqrt{R}\tan(\sqrt{R}\xi), \quad \left(R=\frac{b^2}{16}>0, b=\mp\frac{2}{\sqrt{\lambda\omega}}\right); \tag{87}$$

$$\exp\left(\frac{1}{2}u_n(\xi)\right)=v_n(\xi), \quad v_n(\xi)=\frac{4y_n(\xi)}{b}, \quad \left(b=\mp\frac{2}{\sqrt{\lambda\omega}}, \xi=\lambda x+\omega t\right),$$

$$y_n(\xi)=\frac{p_0+q_0y_{n-1}(\xi)+m_0y_{n-1}{}^2(\xi)+r_0y'_{n-1}(\xi)+n_0y_{n-1}^3(\xi)+l_0[y'_{n-1}(\xi)]^2}{A_0+B_0y_{n-1}(\xi)+D_0y_{n-1}{}^2(\xi)+C_0y'_{n-1}(\xi)+F_0y_{n-1}^3(\xi)+K_0[y'_{n-1}(\xi)]^2}, \quad (n=1,2,\cdots),$$

$$y_0(\xi)=\frac{\sqrt{R}[A+BR\xi+\sqrt{R}(B+A\xi)\cos(2\sqrt{R}\xi)+(-A+BR\xi)\sin(2\sqrt{R}\xi)]}{(-A+BR\xi)\cos(2\sqrt{R}\xi)+\sqrt{R}[B-A\xi-(B+A\xi)\sin(2\sqrt{R}\xi)]},$$

$$\left(R=\frac{b^2}{16}>0\right). \tag{88}$$

例 2 sin-Gordon 方程的无穷序列精确解.

将 $u(x,t)=u(\xi), \xi=\lambda x+\omega t$ 代入 sin-Gordon 方程 (2) 后得到

$$\lambda\omega u''(\xi)=\sin(u(\xi)). \tag{89}$$

当 $\mu=\frac{1}{2}$ 时,把辅助方程 (56) 代入方程 (89),并令 $\sin\left(\frac{u(\xi)}{2}\right), \cos\left(\frac{u(\xi)}{2}\right)\sin\left(\frac{u(\xi)}{2}\right)$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$b=0, a=\frac{4}{\lambda\omega}. \tag{90}$$

当 $\mu=\frac{1}{2}$ 时,把辅助方程方程 (57) 代入方程(89),并 $\sin(u(\xi)), \cos(u(\xi)), \sin\left(\frac{u(\xi)}{2}\right), \cos\left(\frac{u(\xi)}{2}\right)$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$c = 0, b = 0, a = \mp \frac{2}{\sqrt{\lambda\omega}}, \tag{91}$$

$$c = 0, a = 0, b = \mp \frac{2i}{\sqrt{\lambda\omega}} \quad (\lambda\omega < 0). \tag{92}$$

例 3 mKdV sin-Gordon 方程的无穷序列精确解.

将 $u(x,t) = u(\xi), \xi = \lambda x + \omega t$ 代入 mKdV sin－Gordon 方程(3)后得到下列方程:

$$\lambda\omega u''(\xi) + \frac{3}{2}\lambda^4 u'(\xi)u''(\xi) + \lambda^4 u^{(4)}(\xi) = \sin(u(\xi)). \tag{93}$$

当 $\mu = \frac{1}{2}$ 时,把辅助方程(56)代入方程(93),并令 $\sin\left(\frac{u(\xi)}{2}\right)$, $\cos\left(\frac{u(\xi)}{2}\right)\sin\left(\frac{u(\xi)}{2}\right)$, $\cos(u(\xi))\sin\left(\frac{u(\xi)}{2}\right)$的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$b = 0, c = -\frac{a}{2} + \frac{8}{a\lambda^4} - \frac{2\omega}{\lambda^3}, \tag{94}$$

$$b = 0, c = \frac{-12\lambda\omega \pm \sqrt{960\lambda^4 + 144\lambda^2\omega^2}}{30\lambda^4},$$
$$a = \frac{-12\lambda\omega \mp \sqrt{960\lambda^4 + 144\lambda^2\omega^2}}{6\lambda^4}. \tag{95}$$

当 $\mu = \frac{1}{2}$ 时,把辅助方程(57)代入方程(93),并令 $\sin\left(\frac{ju(\xi)}{2}\right), \cos\left(\frac{ju(\xi)}{2}\right) (j = 1,2,3)$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$c = 0, b = 0,$$
$$a = \mp\sqrt{-\frac{2(\omega + \sqrt{4\lambda^2 + \omega^2})}{\lambda^3}}, \tag{96}$$

$$c = 0, b = 0,$$
$$a = \mp\sqrt{\frac{2(-\omega + \sqrt{4\lambda^2 + \omega^2})}{\lambda^3}}, \tag{97}$$

$$c = 0, a = 0,$$
$$b = \mp\sqrt{-\frac{2(\lambda\omega + \sqrt{-4\lambda^4 + \lambda^2\omega^2})}{\lambda^4}}, \tag{98}$$

$$c = 0, a = 0,$$
$$b = \mp\sqrt{\frac{2(-\lambda\omega + \sqrt{-4\lambda^4 + \lambda^2\omega^2})}{\lambda^4}}. \tag{99}$$

我们可以用构造 sinh-Gordon 方程的无穷序列精确解的方法来构造 sin-Gordon 方程和 mKdV sin-Gordon 方程的无穷序列新精确解(这里不一一讨论了).

例 4 (n + 1)维双 sine-Gordon 方程的无穷序列精确解.

将 $u(x_1, x_2, \cdots, x_n, t) = u(\xi), \xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t$ 代入(10)式后得到方程

$$\omega^2 u'' - m\sum_{j=1}^{n}\lambda_j^{\,2} u'' + 2\alpha\sin(\mu u) + \beta\sin(2\mu u) = 0. \tag{100}$$

把辅助方程(56)代入方程(100),并令 $\sin(\mu u(\xi)), \cos(\mu u(\xi))\sin(\mu u(\xi))$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$a = \frac{2\beta}{\mu(m\sum_{j=1}^{n}\lambda_j^{\,2} - \omega^2)},$$
$$b = -\frac{4\beta}{\mu(m\sum_{j=1}^{n}\lambda_j^{\,2} - \omega^2)}. \tag{101}$$

根据已经得到的解(101)和方程(56)的 Bäcklund 变换(102),(103),可以构造(n + 1)维双 sine-Gordon 方程的无穷序列 Jacobi 椭圆函数精确解.

$$u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n^{\,2}(\xi) = y_n(\xi),$$
$$y_n(\xi) = \mp\frac{2(b + c) + [(4a + 2c) \pm \sqrt{(4a + 2c)^2 - 4(c^2 - b^2)}]y_{n-1}(\xi)}{\pm(4a + 2c) + \sqrt{(4a + 2c)^2 - 4(c^2 - b^2)} \pm 2(c - b)y_{n-1}(\xi)}, \quad (n = 1, 2, \cdots),$$
$$y_0(\xi) = \mathrm{sn}^2(\xi, k), a = -\frac{2(1 + k^2)}{\mu^2}, b = \frac{2(1 - k^2)}{\mu^2}, c = \frac{2(1 + k^2)}{\mu^2}. \tag{102}$$

这里 $\alpha = \frac{(1 - k^2)\beta}{2(1 + k^2)}, \xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t, \omega = \mp\frac{\sqrt{(1 + k^2)m\sum_{j=1}^{n}\lambda_j^{\,2} + \beta\mu}}{\sqrt{1 + k^2}}, 0 \leqslant k \leqslant 1.$

$$
\begin{aligned}
&u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n{}^2(\xi) = y_n(\xi),\\
&y_n(\xi) = \frac{-\sqrt{3B_2}[C_2 + 2(c-b)y_{n-1}(\xi)] \pm 3[-(4a+2c) + \sqrt{(4a+2c)^2 - 3(c^2-b^2)}]y'_{n-1}(\xi)}{(c-b)[\sqrt{3B_2} \pm 9y'_{n-1}(\xi)]},\\
&\qquad (n = 1,2,\cdots),\\
&y_0(\xi) = \mathrm{cn}^2(\xi,k), a = \frac{2(-1+2k^2)}{\mu^2}, b = \frac{2}{\mu^2}, c = \frac{2(1-2k^2)}{\mu^2}.
\end{aligned} \tag{103}
$$

其中

$$
\begin{aligned}
&\alpha = \frac{\beta}{2(1-2k^2)},\\
&B_2 = \sqrt{\frac{1}{(c-b)^2\mu^4}[2(4a+2c)^3\mu^6 - 9(4a+2c)(c^2-b^2)\mu^6 - 2[(4a+2c)^2\mu^4 - 3(c^2-b^2)\mu^4]^{3/2}]},\\
&\xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t,\\
&\omega = \mp\frac{\sqrt{(-1+2k^2)m\sum_{j=1}^{n}\lambda_j{}^2 - \beta\mu}}{\sqrt{-1+2k^2}},\\
&C_2 = [(4a+2c) + \sqrt{(4a+2c)^2 - 3(c^2-b^2)}].
\end{aligned}
$$

把辅助方程(57)代入方程(100),并令 $\cos(\mu u(\xi)), \cos(2\mu\ u(\xi)), \sin(\mu\ u(\xi)), \sin(\mu\ u(\xi))\cos(\mu\ u(\xi))$ 的系数为零后得到一个非线性代数方程组,求出该方程组的下列解:

$$
a = 0, b = \mp\frac{\mathrm{i}\sqrt{2\beta}}{\sqrt{\mu(m\sum_{j=1}^{n}\lambda_j{}^2 - \omega^2)}}, c = \mp\frac{\mathrm{i}\sqrt{2}\alpha}{\sqrt{\beta\mu(m\sum_{j=1}^{n}\lambda_j{}^2 - \omega^2)}},\quad \left(\beta\mu(m\sum_{j=1}^{n}\lambda_j^2 - \omega^2) < 0\right). \tag{104}
$$

利用已获得的解(104)来构造(n+1)维双 sine-Gordon 方程的无穷序列精确解:

$$
\begin{aligned}
&u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n(\xi) = \frac{2y_n(\xi)}{(c-b)\mu},\quad \left(\xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t\right),\\
&y_n(\xi) = \frac{p_0 + q_0 y_{n-1}(\xi) + m_0 y_{n-1}{}^2(\xi) + r_0 y'_{n-1}(\xi) + n_0 y^3_{n-1}(\xi) + l_0[y'_{n-1}(\xi)]^2}{A_0 + B_0 y_{n-1}(\xi) + D_0 y_{n-1}{}^2(\xi) + C_0 y'_{n-1}(\xi) + F_0 y^3_{n-1}(\xi) + K_0[y'_{n-1}(\xi)]^2},\quad (n = 1,2,\cdots),\\
&y_0(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi),\quad \left(R = \frac{(c^2-b^2)\mu^2}{4} < 0\right);
\end{aligned} \tag{105}
$$

$$
\begin{aligned}
&u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n(\xi) = \frac{2y_n(\xi)}{(c-b)\mu},\quad \left(\xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t\right),\\
&y_n(\xi) = \frac{p_0 + q_0 y_{n-1}(\xi) + m_0 y_{n-1}{}^2(\xi) + r_0 y'_{n-1}(\xi) + n_0 y^3_{n-1}(\xi) + l_0[y'_{n-1}(\xi)]^2}{A_0 + B_0 y_{n-1}(\xi) + D_0 y_{n-1}{}^2(\xi) + C_0 y'_{n-1}(\xi) + F_0 y^3_{n-1}(\xi) + K_0[y'_{n-1}(\xi)]^2},\quad (n = 1,2,\cdots),\\
&y_0(\xi) = -\frac{R[-A + \sqrt{-R}(B + A\xi)\tanh(\sqrt{-R}\xi) + BR\xi\tanh^2(\sqrt{-R}\xi)]}{BR - \sqrt{-R}(-A + BR\xi)\tanh(\sqrt{-R}\xi) + AR\xi\tanh^2(\sqrt{-R}\xi)},\quad \left(R = \frac{(c^2-b^2)\mu^2}{4} < 0\right);
\end{aligned} \tag{106}
$$

$$
\begin{aligned}
&u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n(\xi) = \frac{2y_n(\xi)}{(c-b)\mu},\quad \left(\xi = \sum_{j=1}^{n}\lambda_j x_j + \omega t\right),\\
&y_n(\xi) = \frac{p_0 + q_0 y_{n-1}(\xi) + m_0 y_{n-1}{}^2(\xi) + r_0 y'_{n-1}(\xi) + n_0 y^3_{n-1}(\xi) + l_0[y'_{n-1}(\xi)]^2}{A_0 + B_0 y_{n-1}(\xi) + D_0 y_{n-1}{}^2(\xi) + C_0 y'_{n-1}(\xi) + F_0 y^3_{n-1}(\xi) + K_0[y'_{n-1}(\xi)]^2},\quad (n = 1,2,\cdots),\\
&y_0(\xi) = \sqrt{R}\tan(\sqrt{R}\xi)\left(R = \frac{(c^2-b^2)\mu^2}{4} > 0\right);
\end{aligned} \tag{107}
$$

$$
\begin{aligned}
&u_n(\xi) = \frac{2}{\mu}\arctan v_n(\xi), v_n(\xi) = \frac{2y_n(\xi)}{(c-b)\mu}, \quad \left(\xi = \sum\nolimits_{j=1}^{n}\lambda_j x_j + \omega t\right), \\
&y_n(\xi) = \frac{p_0 + q_0 y_{n-1}(\xi) + m_0 {y_{n-1}}^2(\xi) + r_0 y'_{n-1}(\xi) + n_0 y_{n-1}^3(\xi) + l_0[y'_{n-1}(\xi)]^2}{A_0 + B_0 y_{n-1}(\xi) + D_0 {y_{n-1}}^2(\xi) + C_0 y'_{n-1}(\xi) + F_0 y_{n-1}^3(\xi) + K_0[y'_{n-1}(\xi)]^2}, \quad (n = 1, 2, \cdots), \\
&y_0(\xi) = \frac{\sqrt{R}[A + BR\xi + \sqrt{R}(B + A\xi)\cos(2\sqrt{R}\xi) + (-A + BR\xi)\sin(2\sqrt{R}\xi)]}{(-A + BR\xi)\cos(2\sqrt{R}\xi) + \sqrt{R}[B - A\xi - (B + A\xi)\sin(2\sqrt{R}\xi)]}, \quad \left(R = \frac{(c^2 - b^2)\mu^2}{4} > 0\right).
\end{aligned} \tag{108}
$$

这里 b,c 是由 (104) 式来确定的常数.

6. 结 论

sine-Gordon 型方程具有重要物理背景的数学模型,研究该模型具有重要意义. 文献[22—24, 34—41]用各种方法讨论了 sine-Gordon 型方程解的存在性和求解等问题. 在求解方面只获得了有限多个新精确解,没有获得无穷序列精确解. 理论上说:"非线性发展存在无穷多个解". 本文为了获得 sine-Gordon 型方程的无穷序列新精确解,引入双曲函数型辅助方程和三角函数型辅助方程,并给出这两类辅助方程的 Bäcklund 变换和解的非线性叠加公式,并在符号计算系统 Mathematica 的帮助下,构造了 sine-Gordon 方程、mKdV-sine-Gordon 方程、(n + 1) 维双 sine-Gordon 方程和 sinh-Gordon 方程的无穷序列新精确解. 其中包括无穷序列三角函数解、无穷序列双曲函数解、无穷序列 Jacobi 椭圆函数解和无穷序列复合型解. 无穷序列复合型解中包括双曲函数、三角函数和有理函数相结合的无穷序列新精确解.

[1] Gardner C S, Greene J M, Kruskal M D, Miura R M 1967 *Phys. Rev. Lett.* **19** 1095

[2] Gardner C S, Greene J M, Kruskal M D, Miura R M 1974 *Comn. Pure. Appi. Math.* **27** 97

[3] Miura R M 1976 *SIAM. Rev.* **18** 412

[4] Wang M L 1995 *Phys. Lett.* A **199** 169

[5] Parkes E J, Duffy B R 1996 *Comput. Phys. Commun.* **98** 288

[6] Fan E G 2000 *Phys. Lett.* A **277** 212

[7] Chen Y, Li B, Zhang H Q 2003 *Chin. Phys.* **12** 940

[8] Chen Y, Yan Z Y, Li B, Zhang H Q 2003 *Chin. Phys.* **12** 1

[9] Chen Y, Li B, Zhang H Q 2003 *Commun. Theor. Phys.* (Beijing) **40** 137

[10] Li D S, Zhang H Q 2003 *Commun. Theor. Phys.* (Beijing) **40** 143

[11] Zhen X D, Chen Y, Li B, Zhang H Q 2003 *Commun. Theor. Phys.* (Beijing) **39** 647

[12] Lü Z S, Zhang H Q 2003 *Commun. Theor. Phys.* (Beijing) **39** 405

[13] Bai C L 2003 *Commun. Theor. Phys.* (Beijing) **40** 147

[14] Li D S, Zhang H Q 2004 *Chin. Phys.* **13** 984

[15] Xie F D, Gao X S 2004 *Commun. Theor. Phys.* (Beijing) **41** 353

[16] Chen H T, Zhang H Q 2004 *Commun. Theor. Phys.* (Beijing) **42** 497

[17] Xie F D, Chen J, Lü Z S 2005 *Commun. Theor. Phys.* (Beijing) **43** 585

[18] Xie F D, Yuan Z T 2005 *Commun. Theor. Phys.* (Beijing) **43** 39

[19] Yong X L, Zhang H Q 2005 *Acta. Phys. Sin.* **54** 2514 (in Chinese) [雍雪林、张鸿庆 2005 物理学报 **54** 2514]

[20] Chen Y, Li B 2004 *Commun. Theor. Phys.* (Beijing) **41** 1

[21] Wang Z, Zhang H Q 2006 *Chin. Phys.* **15** 2210

[22] Sirendaoreji, Sun J 2002 *Phys. Lett.* A **298** 133

[23] Xie Y X, Tang J S 2005 *Chin. Phys.* **14** 1303

[24] Taogetusang, Sirendaoreji 2004 *Acta Sci. J. Nat. Univ. NeiMongol* **33** 361 (in Chinese) [套格图桑、斯仁道尔吉 2004 内蒙古师范大学学报 **33** 361]

[25] Taogetusang, Sirendaoerji 2007 *Acta. Phys. Sin.* **56** 627 (in Chinese) [套格图桑、斯仁道尔吉 2007 物理学报 **56** 627]

[26] Taogetusang, Sirendaoerji 2006 *Acta. Phys. Sin.* **55** 13 (in Chinese) [套格图桑、斯仁道尔吉 2006 物理学报 **55** 13]

[27] Mao J J, Yang J R 2005 *Acta. Phys. Sin.* **54** 4999 (in Chinese) [毛杰健、杨建荣 2005 物理学报 **54** 4999]

[28] Fu Z T, Liu S K, Liu S D 2003 *Commun. Theor. Phys.* (Beijing) **39** 27

[29] Yu J, Ke Y Q, Zhang W J 2004 *Commun. Theor. Phys.* (Beijing) **40** 493

[30] Li D S, Zhang H Q 2003 *Acta. Phys. Sin.* **52** 2373 (in Chinese) [李德生、张鸿庆 2006 物理学报 **52** 2373]

[31] Fu Z T, Liu S K, Liu S D 2004 *Acta. Phys. Sin.* **53** 343 (in Chinese) [付遵涛、刘式适、刘式达 2004 物理学报 **53** 343]

[32] Zhao C H, Sheng Z M 2004 *Acta. Phys. Sin.* **53** 1629 (in Chinese) [赵长海、盛正卯 2004 物理学报 **53** 1629]

[33] He F, Guo Q B, Liu L 2007 *Acta. Phys. Sin.* **56** 4326 (in Chinese) [贺 锋、郭启波、刘 辽 2007 物理学报 **56** 4326]

[34] Guo F K 1983 *Acta. Math. Sin.* **26** 745 (in Chinese) [郭福奎 1983 数学学报 **26** 745]

[35] Xu B Z, Fang X W 1992 *Appl. Math. Mech.* **13** 515 (in Chinese) [徐宝智、方小卫 1992 应用数学和力学 **13** 515]

[36] Zhu Z W, Lu Y 2000 *Chin. Quarterly J. Math. March* **15** 71

[37] Wang Y, Wang B, Qin M Z 2004 *Science. China. Series. A Math.* **47** 18

[38] Sheng P X 2005 *Acta. Math. Appl. Sin.* **28** 453 (in Chinese) [盛平兴 2005 应用数学学报 **28** 453]

[39] Li J B 2007 *Science. China.* A **50** 153

[40] Wazwaz A M 2006 *Phys. Lett.* A **350** 367

[41] Zhou J P, Yan Z L 2007 *Chin. J. Quantum Electronics* **24** 535 (in Chinese) [周建平、闫志莲 2007 量子电子学报 **24** 535]

New infinite sequences exact solutions to sine-Gordon-type equations*

Taogetusang†

(*College of Mathematical Science, Inner Mongolia Normal University, Huhhot 010022, China*)

(Received 27 July 2010; revised manuscript received 8 October 2010)

Abstract

In order to obtain infinite sequence exact solutions to the sine-Gordon-type equations, the auxiliary equations of triangular function type and hyperbolic function type, Bäcklund transformation and nonlinear superposition formula of the solutions are presented. And the method is used to construct new infinite sequence exact solutions to the sine-Gordon equation, mKdV-sine-Gordon equation, $(n+1)$-dimensional double sine-Gordon equation and the sinh-Gordon equation with the help of symbolic computation system Mathematica, which include infinite sequence triangular function solutions, the infinite sequence hyperbolic function solutions, infinite sequence Jacobi elliptic function solutions and infinite sequence computation solutions.

Keywords: sine-Gordon-type equations, solution of nonlinear superposition formula, auxiliary equation, infinite sequence exact solution

PACS: 02.30.Ik, 02.30.Jr

* Project supported by the Natural Natural cience Foundation of China (Grant No. 10461006), the Science Research Foundation of Institution of Higher Education of Inner Mongolia Autonomous Region, China (Grant No. NJZZ07031), and the Natural Science Foundation of Inner Mongolia Autonomous Region, China (Grant No. 2010MS0111).

† E-mail: tgts@imnu.edu.cn