二维各向异性谐振子的第三个独立守恒量及其对称性*

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二维各向异性谐振子和两分振子的能量是守恒的, 但三个守恒量中只有其中两个是独立的. 当频率比 ω_1/ω_2 为有理数时, 系统存在第三个独立的守恒量. 本文用扩展 Prelle-Singer 法得到五个典型谐振子的第三个独立守恒量, 并讨论了与守恒量相应的 Noether 对称性与 Lie 对称性.

关键词: 扩展 Prelle-Singer 法, 二维各向异性谐振子, 守恒量, 对称性

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1 引 言

力学系统的对称性与守恒量存在密切的关系, 寻求力学系统的守恒量, 特别是寻求典型力学系统的守恒量, 并解释所得守恒量的物理意义, 研究与守恒量相应的 Noether 对称性和 Lie 对称性并解释其意义, 已受到众多分析力学专家的关注. 各向同性和各向异性谐振子都是典型的力学模型, 而各向同性谐振子的守恒量与对称性已有文献报道 [1,2]. 事实上, 力学中常见的是各向异性谐振子, 其守恒量与对称性的研究尤为重要. 众所周知, 二维各向异性谐振子和两分振子的能量是守恒的, 但三个守恒量中只有其中的两个是独立的. 当频率比 $\omega_1/\omega_2 = p/q$ 为有理数时, 系统存在第三个独立的守恒量, 从而导致系统具有特殊的力学特性, 使运动轨道闭合. 因此, 寻找这第三个独立的守恒量显得十分重要. 寻找力学系统的守恒量有多种方法, 如 Noether 对称性法 [3−5]、Lie 对称性法 [6−8]、Mei 对称性法 [9,10]、Ermakov 方法 [11,12]、Poisson 括号法 [13−15] 和直接积分法 [16−18]. 而本文采用扩展 Prelle-Singer (P-S) 法求守恒量 [19−26], 此方法首先由 Prelle 和 Singer[19] 在 1983 年提出, 适用于求一阶微分方程的解和守恒量, 后来 Guha 等 [20] 将 P-S 法进行扩展, 并应用于求解二阶及二阶以上相互耦合的微分方程组的第一积分. 扩展 P-S 法的基本思路是先假设系统存在第一积分 I, 然后用几个积分乘子 (未知函数) 去乘以恒为零的 1- 形式微分式 $\mathrm{d}x - \dot{x}_1\mathrm{d}t$, $\mathrm{d}x_2 - \dot{x}_2\mathrm{d}t$, $\mathrm{d}\dot{x}_1 - \phi_1\mathrm{d}t$, 和 $\mathrm{d}\dot{x}_2 - \phi_2\mathrm{d}t$, 通过比较系数法求得积分乘子, 从而求得第一积分 I(守恒量).

本文首先用扩展 P-S 法求得频率比 $\dfrac{\omega_1}{\omega_2} = 2, 3, 4, \dfrac{3}{2}, \dfrac{4}{3}$ 五种典型情况下系统的第三个独立守恒量, 并讨论与第三个独立守恒量相应的 Noether 对称性和 Lie 对称性, 解释了系统存在第三个独立守恒量的原因.

2 二维各向异性谐振子的守恒量

在平面直角坐标系下, 二维各向异性谐振子的 Lagrange 函数可表示为

$$L = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 - \frac{1}{2}\omega_1^2 x_1^2 - \frac{1}{2}\omega_2^2 x_2^2, \qquad (1)$$

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其中 x_1, x_2 为广义坐标, $\omega_1/\omega_2 = p/q$, p, q 为不可约的正整数. 很显然, 系统存在以下三个守恒量

$$I_1 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\omega_1^2 x_1^2, \tag{2a}$$

$$I_2 = \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}\omega_2^2 x_2^2, \tag{2b}$$

$$\begin{aligned} I_0 =& I_1 + I_2 \\ =& \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}\omega_1^2 x_1^2 + \frac{1}{2}\omega_2^2 x_2^2, \end{aligned} \tag{2c}$$

I_1, I_2 为系统分振子的能量, I_0 为系统的能量, 三个能量中只有其中两个能量是相互独立的. 对于频率比为有理数的二维各向异性谐振子, 其轨道是闭合的, 则一定存在第三个独立的守恒量 [1], 下面用扩展 P-S 法得到第三个独立守恒量.

系统的运动微分方程可表示成

$$\begin{aligned} \ddot{x}_1 &= -\omega_1^2 x_1 = \phi_1, \\ \ddot{x}_2 &= -\omega_2^2 x_2 = \phi_2. \end{aligned} \tag{3}$$

设此力学系统存在第三个独立的守恒量 $I = I(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$, 则

$$\begin{aligned} \mathrm{d}I =& I_t \mathrm{d}t + I_{x_1}\mathrm{d}x_1 + I_{x_2}\mathrm{d}x_2 + I_{\dot{x}_1}\mathrm{d}\dot{x}_1 \\ &+ I_{\dot{x}_2}\mathrm{d}\dot{x}_2 = 0. \end{aligned} \tag{4}$$

利用 4 个独立且恒为 0 的 1- 形式微分式 $\mathrm{d}x - \dot{x}_1\mathrm{d}t$, $\mathrm{d}x_2 - \dot{x}_2\mathrm{d}t$, $\mathrm{d}\dot{x}_1 - \phi_1\mathrm{d}t$, $\mathrm{d}\dot{x}_2 - \phi_2\mathrm{d}t$, 用积分乘子 $S_1 = S_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$, $S_2 = S_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$, $R_1 = R_1(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$, $R_2 = R_2(t, x_1, x_2, \dot{x}_1, \dot{x}_2)$, 分别乘以上述 1- 形式微分式并求和, 则

$$\begin{aligned} & S_1(\mathrm{d}x_1 - \dot{x}_1\mathrm{d}t) + S_2(\mathrm{d}x_2 - \dot{x}_2\mathrm{d}t) \\ & + R_1(\mathrm{d}\dot{x}_1 - \phi_1\mathrm{d}t) + R_2(\mathrm{d}\dot{x}_2 - \phi_2\mathrm{d}t) = 0. \end{aligned} \tag{5}$$

令 (4) 式与 (5) 式相等, 并比较式中 $\mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}\dot{x}_1, \mathrm{d}\dot{x}_2$ 的系数得

$$\begin{aligned} I_t =& -(S_1\dot{x}_1 + S_2\dot{x}_2 + R_1\phi_1 + R_2\phi_2), \\ I_{x_1} =& S_1, \qquad I_{x_2} = S_2, \\ I_{\dot{x}_1} =& R_1, \qquad I_{\dot{x}_2} = R_2. \end{aligned} \tag{6}$$

由可积条件

$$\begin{aligned} & I_{tx_1} = I_{x_1 t}, \quad I_{tx_2} = I_{x_2 t}, \\ & I_{t\dot{x}_1} = I_{\dot{x}_1 t}, \quad I_{t\dot{x}_2} = I_{\dot{x}_2 t}, \\ & I_{x_1 x_2} = I_{x_2 x_1}, \quad I_{x_1\dot{x}_1} = I_{\dot{x}_1 x_1}, \\ & I_{x_1\dot{x}_2} = I_{\dot{x}_2 x_1}, \quad I_{x_2\dot{x}_1} = I_{\dot{x}_1 x_2}, \\ & I_{x_2\dot{x}_2} = I_{\dot{x}_2 x_2}, \quad I_{\dot{x}_1\dot{x}_2} = I_{\dot{x}_2\dot{x}_1}, \end{aligned}$$

并利用 (3) 式得

$$\dot{S}_1 = -\omega_1^2 R_1, \quad \dot{S}_2 = -\omega_2^2 R_2, \tag{7a}$$

$$\dot{R}_1 = -S_1, \quad \dot{R}_2 = -S_2. \tag{7b}$$

其中 $\dot{R}_1$ 表示 R_1 对时间的全导数, 即

$$\dot{R}_1 = \frac{\partial R_1}{\partial t} + \dot{x}_1\frac{\partial R_1}{\partial x_1} + \dot{x}_2\frac{\partial R_1}{\partial x_2} + \phi_1\frac{\partial R_1}{\partial \dot{x}_1} + \phi_2\frac{\partial R_1}{\partial \dot{x}_2}$$

(其他类同). 由 (7) 式可得关于 R_1, R_2 的二阶微分方程组

$$\begin{aligned} \ddot{R}_1 &= -\omega_1^2 R_1, \\ \ddot{R}_2 &= -\omega_2^2 R_2. \end{aligned} \tag{8}$$

对于自治系统, 可假设 S_1, S_2, R_1, R_2 不显含时间 t. 由 (8) 式可解得 R_1, R_2, 将 R_1, R_2 代入 (7b) 式可解得 S_1, S_2, 将每组积分乘子 R_1, R_2, S_1, S_2 代入守恒量的一般表达式

$$\begin{aligned} I = \int & S_1(\mathrm{d}x_1 - \dot{x}_1\mathrm{d}t) + S_2(\mathrm{d}x_2 - \dot{x}_2\mathrm{d}t) \\ & + R_1(\mathrm{d}\dot{x}_1 - \phi_1\mathrm{d}t) + R_2(\mathrm{d}\dot{x}_2 - \phi_2\mathrm{d}t), \end{aligned} \tag{9}$$

可求得守恒量.

因篇幅有限, 本文只讨论频率比 $\frac{\omega_1}{\omega_2} = 2, 3, 4, \frac{3}{2}, \frac{4}{3}$ 五种典型情况下系统的第三个独立守恒量 $I_3^{p/q}$, 上标 p/q 表示频率比, 下标 3 表示其为系统的第三个独立守恒量. 其他频率比下的守恒量也可类似求得. 下面列出五种典型频率比下的积分乘子和守恒量.

当 $\frac{\omega_1}{\omega_2} = 2$ 时,

$$\begin{aligned} R_1 &= 2\omega_2 x_2\dot{x}_2, \\ R_2 &= -2\omega_1 x_1\dot{x}_2 + 2\omega_2 x_2\dot{x}_1, \\ S_1 &= -\omega_1\dot{x}_2^2 + \omega_1\omega_2^2 x_2^2, \\ S_2 &= 2\omega_2\dot{x}_2\dot{x}_1 + 2\omega_1\omega_2^2 x_1 x_2, \end{aligned} \tag{10}$$

$$I_3^{2/1} = \omega_1\omega_2^2 x_1 x_2^2 - \omega_1 x_1\dot{x}_2^2 + 2\omega_2 x_2\dot{x}_1\dot{x}_2. \tag{11}$$

当 $\frac{\omega_1}{\omega_2} = 3$ 时,

$$\begin{aligned} R_1 =& 3\omega_2^2 x_2^2\dot{x}_2 - \dot{x}_2^3, \\ R_2 =& -6\omega_1\omega_2 x_1 x_2\dot{x}_2 + 3\omega_2^2 x_2^2\dot{x}_1 - 3\dot{x}_1\dot{x}_2^2, \\ S_1 =& \omega_1\omega_2^3 x_2^3 - 3\omega_1\omega_2 x_2\dot{x}_2^2, \\ S_2 =& 3\omega_1\omega_2^3 x_1 x_2^2 - 3\omega_1\omega_2 x_1\dot{x}_2^2 \\ & + 6\omega_2^2 x_2\dot{x}_1\dot{x}_2, \end{aligned} \tag{12}$$

$$\begin{aligned} I_3^{3/1} =& \omega_1\omega_2^3 x_1 x_2^3 - 3\omega_1\omega_2 x_1 x_2\dot{x}_2^2 + 3\omega_2^2 x_2^2\dot{x}_1\dot{x}_2 \\ & - \dot{x}_1\dot{x}_2^3. \end{aligned} \tag{13}$$

当 $\dfrac{\omega_1}{\omega_2}=4$ 时,

$$\begin{aligned}
R_1 =&4\omega_2^3x_2^3\dot{x}_2-4\omega_2x_2\dot{x}_2^3,\\
R_2 =&-12\omega_1\omega_2^2x_1x_2^2\dot{x}_2+4\omega_2^3x_2^3\dot{x}_1\\
&-12\omega_2x_2\dot{x}_1\dot{x}_2^2+4\omega_1x_1\dot{x}_2^3,\\
S_1 =&\omega_1\omega_2^4x_2^4-6\omega_1\omega_2^2x_2^2\dot{x}_2^2+\omega_1\dot{x}_2^4,\\
S_2 =&4\omega_1\omega_2^4x_1x_2^3-12\omega_1\omega_2^2x_1x_2\dot{x}_2^2\\
&+12\omega_2^3x_2^2\dot{x}_1\dot{x}_2-4\omega_2\dot{x}_1\dot{x}_2^3,
\end{aligned}\tag{14}$$

$$\begin{aligned}
I_3^{4/1} =&\omega_1\omega_2^4x_1x_2^4-6\omega_1\omega_2^2x_1x_2^2\dot{x}_2^2+4\omega_2^3x_2^3\dot{x}_1\dot{x}_2\\
&-4\omega_2x_2\dot{x}_1\dot{x}_2^3+\omega_1x_1\dot{x}_2^4.
\end{aligned}\tag{15}$$

当 $\dfrac{\omega_1}{\omega_2}=\dfrac{3}{2}$ 时,

$$\begin{aligned}
R_1 =&6\omega_1\omega_2^2x_1x_2^2\dot{x}_2-2\omega_2^3x_2^3\dot{x}_1\\
&+6\omega_2x_2\dot{x}_1\dot{x}_2^2-2\omega_1x_1\dot{x}_2^3,\\
R_2 =&6\omega_1\omega_2^2x_1x_2^2\dot{x}_1-6\omega_1^2\omega_2x_1^2x_2\dot{x}_2\\
&+6\omega_2x_2\dot{x}_1^2\dot{x}_2-6\omega_1x_1\dot{x}_1\dot{x}_2^2,\\
S_1 =&2\omega_1^2\omega_2^3x_1x_2^3+6\omega_1\omega_2^2x_2^2\dot{x}_1\dot{x}_2\\
&-6\omega_1^2\omega_2x_1x_2\dot{x}_2^2-2\omega_1\dot{x}_1\dot{x}_2^3,\\
S_2 =&3\omega_1^2\omega_2^3x_1^2x_2^2+12\omega_1\omega_2^2x_1x_2\dot{x}_1\dot{x}_2\\
&-3\omega_1^2\omega_2x_1^2\dot{x}_2^2-3\omega_2^3x_2^2\dot{x}_1^2+3\omega_2\dot{x}_1^2\dot{x}_2^2,
\end{aligned}\tag{16}$$

$$\begin{aligned}
I_3^{3/2} =&\omega_1^2\omega_2^3x_1^2x_2^3+6\omega_1\omega_2^2x_1x_2^2\dot{x}_1\dot{x}_2\\
&-3\omega_1^2\omega_2x_1^2x_2\dot{x}_2^2-\omega_2^3x_2^3\dot{x}_1^2\\
&+3\omega_2x_2\dot{x}_1^2\dot{x}_2^2-2\omega_1x_1\dot{x}_1\dot{x}_2^3.
\end{aligned}\tag{17}$$

当 $\dfrac{\omega_1}{\omega_2}=\dfrac{4}{3}$ 时,

$$\begin{aligned}
R_1 =&-6\omega_1\omega_2^4x_1x_2^4\dot{x}_1+36\omega_1\omega_2^2x_1x_2^2\dot{x}_1\dot{x}_2^2\\
&-6\omega_1x_1\dot{x}_1\dot{x}_2^4+12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_2\\
&-12\omega_2^3x_2^3\dot{x}_1^2\dot{x}_2-12\omega_1^2\omega_2x_1^2x_2\dot{x}_2^3\\
&+12\omega_2x_2\dot{x}_1^2\dot{x}_2^3,\\
R_2 =&-12\omega_1^3\omega_2^2x_1^3x_2^2\dot{x}_2+36\omega_1\omega_2^2x_1x_2^2\dot{x}_1^2\dot{x}_2\\
&+4\omega_1^3x_1^3\dot{x}_2^3-12\omega_1x_1\dot{x}_1^2\dot{x}_2^3\\
&+12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_1-4\omega_2^3x_2^3\dot{x}_1^3\\
&-36\omega_1^2\omega_2x_1^2x_2\dot{x}_1\dot{x}_2^2+12\omega_2x_2\dot{x}_1^3\dot{x}_2^2,\\
S_1 =&3\omega_1^3\omega_2^4x_1^2x_2^4-3\omega_1\omega_2^4x_2^4\dot{x}_1^2-18\omega_1^3\omega_2^2x_1^2x_2^2\dot{x}_2^2\\
&+18\omega_1\omega_2^2x_2^2\dot{x}_1^2\dot{x}_2^2+3\omega_1^3x_1^2\dot{x}_2^4\\
&-3\omega_1\dot{x}_1^2\dot{x}_2^4+24\omega_1^2\omega_2^3x_1x_2^3\dot{x}_1\dot{x}_2\\
&-24\omega_1^2\omega_2x_1x_2\dot{x}_1\dot{x}_2^3,\\
S_2 =&4\omega_1^3\omega_2^4x_1^3x_2^3-12\omega_1\omega_2^4x_1x_2^3\dot{x}_1^2\\
&-12\omega_1^3\omega_2^2x_1^3x_2\dot{x}_2^2+36\omega_1\omega_2^2x_1x_2\dot{x}_1^2\dot{x}_2^2\\
&+36\omega_1^2\omega_2^3x_1^2x_2^2\dot{x}_1\dot{x}_2-12\omega_2^3x_2^2\dot{x}_1^3\dot{x}_2\\
&-12\omega_1^2\omega_2x_1^2\dot{x}_1\dot{x}_2^3+4\omega_2\dot{x}_1^3\dot{x}_2^3,
\end{aligned}\tag{18}$$

$$\begin{aligned}
I_3^{4/3} =&\omega_1^3\omega_2^4x_1^3x_2^4-3\omega_1\omega_2^4x_1x_2^4\dot{x}_1^2-6\omega_1^3\omega_2^2x_1^3x_2^2\dot{x}_2^2\\
&+18\omega_1\omega_2^2x_1x_2^2\dot{x}_1^2\dot{x}_2^2+\omega_1^3x_1^3\dot{x}_2^4\\
&-3\omega_1x_1\dot{x}_1^2\dot{x}_2^4+12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_1\dot{x}_2\\
&-4\omega_2^3x_2^3\dot{x}_1^3\dot{x}_2-12\omega_1^2\omega_2x_1^2x_2\dot{x}_1\dot{x}_2^3\\
&+4\omega_2x_2\dot{x}_1^3\dot{x}_2^3.
\end{aligned}\tag{19}$$

3 系统的 Noether 对称性与 Lie 对称性

下面讨论系统第三个独立守恒量 $I_3^{p/q}$ 的对称性. 引进群的无限小变换

$$\begin{aligned}
&t^*=t+\varepsilon\tau_{p/q}(t,x_1,x_2,\dot{x}_1,\dot{x}_2),\\
&x_1^*=x_1+\varepsilon\xi_1^{p/q}(t,x_1,x_2,\dot{x}_1,\dot{x}_2),\\
&x_2^*=x_2+\varepsilon\xi_2^{p/q}(t,x_1,x_2,\dot{x}_1,\dot{x}_2).
\end{aligned}\tag{20}$$

其无限小生成元向量为

$$X^{(0)}=\tau_{p/q}\frac{\partial}{\partial t}+\xi_1^{p/q}\frac{\partial}{\partial x_1}+\xi_2^{p/q}\frac{\partial}{\partial x_2},\tag{21}$$

(21) 式的一次扩展为

$$\begin{aligned}
X^{(1)}=&X^{(0)}+(\dot{\xi}_1^{p/q}-\dot{x}_1\dot{\tau}_{p/q})\frac{\partial}{\partial\dot{x}_1}\\
&+(\dot{\xi}_2^{p/q}-\dot{x}_2\dot{\tau}_{p/q})\frac{\partial}{\partial\dot{x}_2},
\end{aligned}\tag{22}$$

二次扩展为

$$\begin{aligned}
X^{(2)}=&X^{(1)}+(\ddot{\xi}_1^{p/q}-\dot{x}_1\ddot{\tau}_{p/q}-2\ddot{x}_1\dot{\tau}_{p/q})\frac{\partial}{\partial\ddot{x}_1}\\
&+(\ddot{\xi}_2^{p/q}-\dot{x}_2\ddot{\tau}_{p/q}-2\ddot{x}_2\dot{\tau}_{p/q})\frac{\partial}{\partial\ddot{x}_2},
\end{aligned}\tag{23}$$

其中 ε 为无限小参数, $\tau_{p/q}$, $\xi_1^{p/q}$, $\xi_2^{p/q}$ 是与频率比为 p/q 的守恒量相应的无限小变换的生成元.

由 Lagrange 系统的 Noether 逆定理 [3] 可知: 如果已知 Lagrange 系统的第一积分 (守恒量) $I_3^{p/q}$, 那么可由守恒量 $I_3^{p/q}$ 找到相应的生成元 $\tau_{p/q}$, $\xi_1^{p/q}$, $\xi_2^{p/q}$, 使无限小变换 (20) 式为系统的 Noether 对称变换 (或 Noether 准对称变换), 即系统具有 Noether 对称性 (或 Noether 准对称性).

对于给定的守恒量和 Lagrange 函数, 由下面的 (24) 和 (25) 式可确定生成元 $\tau_{p/q},\xi_1^{p/q},\xi_2^{p/q}$, 能使无限小变换 (20) 式为系统的 Noether 对称变换, 系统具有 Noether 对称性.

$$\xi_1^{p/q}-\dot{x}_1\tau_{p/q}=\tilde{h}_{11}\frac{\partial I_3^{p/q}}{\partial\dot{x}_1}+\tilde{h}_{12}\frac{\partial I_3^{p/q}}{\partial\dot{x}_2},$$

$$\xi_2^{p/q}-\dot{x}_2\tau_{p/q}=\tilde{h}_{21}\frac{\partial I_3^{p/q}}{\partial \dot{x}_1}+\tilde{h}_{22}\frac{\partial I_3^{p/q}}{\partial \dot{x}_2}, \tag{24}$$

$$\begin{aligned}\tau_{p/q}=&\frac{1}{L}\Big[I_{\mathrm{p/q}}-(\xi_1^{p/q}-\dot{x}_1\tau_{p/q})\frac{\partial L}{\partial \dot{x}_1}\\&-(\xi_2^{p/q}-\dot{x}_2\tau_{p/q})\frac{\partial L}{\partial \dot{x}_2}\Big],\end{aligned} \tag{25}$$

其中 $\tilde{h}_{sk}$ $(s,k=1,2)$ 为 Lagrange 函数的 Hess 矩阵的逆矩阵元.

如果由下面的 (26) 式确定 $\tau_{p/q}$,

$$\begin{aligned}\tau_{p/q}=&\frac{1}{L}\left[I_{p/q}-(\xi_1^{p/q}-\dot{x}_1\tau_{p/q})\frac{\partial L}{\partial \dot{x}_1}\right.\\&\left.-(\xi_2^{p/q}-\dot{x}_2\tau_{p/q})\frac{\partial L}{\partial \dot{x}_2}-G_{p/q}\right],\end{aligned} \tag{26}$$

其中 $G_{p/q}=G_{p/q}(t,x_1,x_2,\dot{x}_1,\dot{x}_2)$ 为规范函数, 且规范函数满足 Noether 等式 (27) 式

$$\begin{aligned}&\frac{\partial L}{\partial t}\tau_{p/q}+\sum_{i=1}^{2}\frac{\partial L}{\partial x_i}\xi_i^{p/q}+\sum_{i=1}^{2}\frac{\partial L}{\partial \dot{x}_i}\dot{\xi}_i^{p/q}\\&+\left(L-\sum_{i=1}^{2}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i\right)\dot{\tau}_{p/q}=-\dot{G}_{p/q}.\end{aligned} \tag{27}$$

则由 (24) 和 (26) 式确定的生成元 $\tau_{p/q},\xi_1^{p/q},\xi_2^{p/q}$, 使无限小变换 (20) 式为系统的 Noether 准对称变换, 系统具有 Noether 准对称性.

根据 Lie 对称性理论 [3], 如果由 (24), (25) 式或 (24), (26) 式确定的生成元 $\tau_{p/q}$, $\xi_1^{p/q}$, $\xi_2^{p/q}$, 满足下面的 Lie 对称性确定方程:

$$\begin{aligned}&\ddot{\xi}_1^{p/q}-\dot{x}_1\ddot{\tau}_{p/q}-2\phi_1\dot{\tau}_{p/q}=X^{(1)}(\phi_1),\\&\ddot{\xi}_2^{p/q}-\dot{x}_2\ddot{\tau}_{p/q}-2\phi_2\dot{\tau}_{p/q}=X^{(1)}(\phi_2),\end{aligned} \tag{28}$$

则说明与守恒量 $I_3^{p/q}$ 相应的无限小变换 (20) 式是 Lie 对称变换, 系统具有 Lie 对称性.

将 (1), (11), (13), (15), (17) 和 (19) 式分别代入 (24), (25) 式, 得不到 $\tau_{2/1}$, $\tau_{3/1}$, $\tau_{4/1}$, $\tau_{3/2}$, $\tau_{4/3}$ 的解析解, 则说明不存在 Noether 对称变换与守恒量 $I_3^{2/1}, I_3^{3/1}, I_3^{4/1}, I_3^{3/2}, I_3^{4/3}$ 相对应.

将 (1), (11), (13), (15), (17) 和 (19) 式分别代入 (24), (26) 式, 可解得

$$\begin{aligned}\tau_{2/1}=&0,\\\xi_1^{2/1}=&2\omega_2x_2\dot{x}_2,\\\xi_2^{2/1}=&-2\omega_1x_1\dot{x}_2+2\omega_2x_2\dot{x}_1,\\G_{2/1}=&\omega_1\omega_2^2x_1x_2^2+\omega_1x_1\dot{x}_2^2-2\omega_2x_2\dot{x}_1\dot{x}_2;\end{aligned} \tag{29}$$

$$\begin{aligned}\tau_{3/1}=&0,\\\xi_1^{3/1}=&3\omega_2^2x_2^2\dot{x}_2-\dot{x}_2^3,\\\xi_2^{3/1}=&-6\omega_1\omega_2x_1x_2\dot{x}_2+3\omega_2^2x_2^2\dot{x}_1-3\dot{x}_1\dot{x}_2^2,\\G_{3/1}=&\omega_1\omega_2^3x_1x_2^3+3\omega_1\omega_2x_1x_2\dot{x}_2^2\\&-3\omega_2^2x_2^2\dot{x}_1\dot{x}_2+3\dot{x}_1\dot{x}_2^3;\end{aligned} \tag{30}$$

$$\begin{aligned}\tau_{4/1}=&0,\\\xi_1^{4/1}=&4\omega_2^3x_2^3\dot{x}_2-4\omega_2x_2\dot{x}_2^3,\\\xi_2^{4/1}=&-12\omega_1\omega_2^2x_1x_2^2\dot{x}_2+4\omega_2^3x_2^3\dot{x}_1\\&-12\omega_2x_2\dot{x}_1\dot{x}_2^2+4\omega_1x_1\dot{x}_2^3,\\G_{4/1}=&\omega_1\omega_2^4x_1x_2^4+6\omega_1\omega_2^2x_1x_2^2\dot{x}_2^2\\&-4\omega_2^3x_2^3\dot{x}_1\dot{x}_2+12\omega_2x_2\dot{x}_1\dot{x}_2^3\\&-3\omega_1x_1\dot{x}_2^4;\end{aligned} \tag{31}$$

$$\begin{aligned}\tau_{3/2}=&0,\\\xi_1^{3/2}=&6\omega_1\omega_2^2x_1x_2^2\dot{x}_2-2\omega_2^3x_2^3\dot{x}_1\\&+6\omega_2x_2\dot{x}_1\dot{x}_2^2-2\omega_1x_1\dot{x}_2^3,\\\xi_2^{3/2}=&6\omega_1\omega_2^2x_1x_2^2\dot{x}_1-6\omega_1^2\omega_2x_1^2x_2\dot{x}_2\\&+6\omega_2x_2\dot{x}_1^2\dot{x}_2-6\omega_1x_1\dot{x}_1\dot{x}_2^2,\\G_{3/2}=&\omega_1^2\omega_2^3x_1^2x_2^3-6\omega_1\omega_2^2x_1x_2^2\dot{x}_1\dot{x}_2\\&+3\omega_1^2\omega_2x_1^2x_2\dot{x}_2^2+\omega_2^3x_2^3\dot{x}_1^2\\&-9\omega_2x_2\dot{x}_1^2\dot{x}_2^2+6\omega_1x_1\dot{x}_1\dot{x}_2^3;\end{aligned} \tag{32}$$

$$\begin{aligned}\tau_{4/3}=&0,\\\xi_1^{4/3}=&-6\omega_1\omega_2^4x_1x_2^4\dot{x}_1+36\omega_1\omega_2^2x_1x_2^2\dot{x}_1\dot{x}_2^2\\&-6\omega_1x_1\dot{x}_1\dot{x}_2^4+12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_2\\&-12\omega_2^3x_2^3\dot{x}_1^2\dot{x}_2-12\omega_1^2\omega_2x_1^2x_2\dot{x}_2^3\\&+12\omega_2x_2\dot{x}_1^2\dot{x}_2^3,\\\xi_2^{4/3}=&-12\omega_1^3\omega_2^2x_1^3x_2^2\dot{x}_2+36\omega_1\omega_2^2x_1x_2^2\dot{x}_1^2\dot{x}_2\\&+4\omega_1^3x_1^3\dot{x}_2^3-12\omega_1x_1\dot{x}_1^2\dot{x}_2^3\\&+12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_1-4\omega_2^3x_2^3\dot{x}_1^3\\&-36\omega_1^2\omega_2x_1^2x_2\dot{x}_1\dot{x}_2^2+12\omega_2x_2\dot{x}_1^3\dot{x}_2^2,\\G_{4/3}=&\omega_1^3\omega_2^4x_1^3x_2^4+3\omega_1\omega_2^4x_1x_2^4\dot{x}_1^2\\&+6\omega_1^3\omega_2^2x_1^3x_2^2\dot{x}_2^2-54\omega_1\omega_2^2x_1x_2^2\dot{x}_1^2\dot{x}_2^2\\&-3\omega_1^3x_1^3\dot{x}_2^4+15\omega_1x_1\dot{x}_1^2\dot{x}_2^4\\&-12\omega_1^2\omega_2^3x_1^2x_2^3\dot{x}_1\dot{x}_2+12\omega_2^3x_2^3\dot{x}_1^3\dot{x}_2\\&+36\omega_1^2\omega_2x_1^2x_2\dot{x}_1\dot{x}_2^3-20\omega_2x_2\dot{x}_1^3\dot{x}_2^3.\end{aligned} \tag{33}$$

说明与守恒量 $I_3^{2/1}$, $I_3^{3/1}$, $I_3^{4/1}$, $I_3^{3/2}$, $I_3^{4/3}$ 相应的无限小变换是 Noether 准对称变换, 而不是 Noether 对称变换, 系统具有 Noether 准对称性.

下面讨论系统的 Lie 对称性. 将 (29)—(33) 式分别代入 Lie 对称性的确定方程 (28) 式, 并利用 (3)

式, 可以证明无限小生成元 (29)—(33) 式均满足确定方程 (28) 式, 说明与守恒量 $I_3^{2/1}$, $I_3^{3/1}$, $I_3^{4/1}$, $I_3^{3/2}$, $I_3^{4/3}$ 相对应的无限小变换均为 Lie 对称变换, 系统具有 Lie 对称性.

由 (29)—(33) 式知, 系统的运动微分方程 (3) 式对于无限小变换

$$t^* = t,$$
$$x_1^* = x_1 + \varepsilon\xi_1^{p/q}, \tag{34}$$
$$x_2^* = x_2 + \varepsilon\xi_2^{p/q}$$

是不变的, 这些变换时间均保持不变, 只有空间发生变换, 属于同一类空间变换.

众所周知, 对于频率比 ω_1/ω_2 为有理数的各向异性谐振子, 其轨道就是著名的 Lissajour 图形, 是闭合的. 由文献 [1] 可知, 要使二自由度系统的轨道闭合, 则必须存在 3 个独立的守恒量, 使系统成为完全简并系统. 这说明对于频率比 ω_1/ω_2 为有理数的各向异性谐振子, 除系统的能量和分振子能量守恒外, 一定存在第三个独立的守恒量, 就是本文找到的 $I_3^{p/q}$.

4 结 论

一般二维各向异性谐振子的守恒量的个数少于 $2n-1$ 个 (n 为自由度数), 系统不是完全简并系统, 其轨道也不闭合. 但是, 对于频率比 ω_1/ω_2 为有理数的二维各向异性谐振子, 存在第三个独立守恒量, 系统成为完全简并系统, 轨道是闭合的, 三个独立的守恒量可确定平面轨道的形状、大小和方位. 本文用扩展 P-S 法求得了频率比 ω_1/ω_2 为有理数的五个典型二维各向异性谐振子的第三个独立守恒量, 并讨论了与守恒量相应的 Noether 对称性与 Lie 对称性, 结果表明: 与第三个独立守恒量相应的无限小变换是 Noether 准对称变换, 也是 Lie 对称变换, 系统具有 Noether 准对称性和 Lie 对称性.

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The third independent conserved quantity and its symmetry of the two-dimensional anisotropic harmonic oscillator*

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Abstract

The energy and the two partial energies of two-dimensional anisotropic harmonic oscillator are conserved quantities, but only two of them are independent. The system possesses the third independent conserved quantity when the ω_1/ω_2 is a rational number. The extended Prelle-Singer method is used to find the third independent conserved quantity for the five typical two-dimensional anisotropic harmonic oscillators. The Noether symmetry and the Lie symmetry of the third independent conserved quantities are also discussed.

Keywords: extended P-S method, two-dimensional anisotropic harmonic oscillator, conserved quantity, symmetry

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