弱非线性耦合二维各向异性谐振子的一阶近似 Lie 对称性与近似守恒量*

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用近似 Lie 对称性理论研究弱非线性耦合二维各向异性谐振子的一阶近似 Lie 对称性与近似守恒量, 并以频率比为 2 : 1 的弱非线性耦合二维各向异性谐振子为例, 得到其 6 个一阶近似 Lie 对称性和一阶近似守恒量, 其中 1 个一阶近似守恒量实为系统的精确守恒量, 4 个一阶近似守恒量为平凡的一阶近似守恒量, 只有 1 个一阶近似守恒量为稳定的一阶近似守恒量.

关键词: 弱非线性耦合二维各向异性谐振子, 近似 Lie 对称性, 近似守恒量

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1 引 言

力学系统的对称性与守恒量之间存在着密切的联系, 关于力学系统对称性与守恒量的研究已渗透到现代数学、力学、物理学等各个领域. 寻求力学系统的对称性和守恒量已成为近代分析力学的一大热点问题 [1−8]. 但事实上, 许多力学系统会受到各种各样的微扰, 通过数学模型建立的运动微分方程总是近似的、非线性的, 因此研究受非线性微扰的耦合系统的近似对称性和近似守恒量对于研究力学系统的特性以及得到方程的近似解至关重要. 近年来关于常微分方程、偏微分方程近似对称性和近似守恒量的研究已取得不少的成果 [9−23]. 目前研究近似对称性和近似守恒量主要采用近似 Lie 对称性理论 [9] 和近似 Noether 对称性理论 [10]. 引进近似的群无限小变换, 微分方程在此变换下近似保持不变则为近似 Lie 对称性; 哈密顿作用量在此变换下近似保持不变则为近似 Noether 对称性, 所得的守恒量为近似守恒量. 弱非线性耦合二维各向异性谐振子是一典型而常见的力学模型, 在力学、分子与原子物理、声学等各个领域具有实际应用价值. 线性耦合的二维各向异性谐振子的守恒量可以通过改变坐标标度和旋转坐标轴等方法比较方便地求得 [6]. 弱非线性耦合项的存在, 给求解耦合谐振子的守恒量带来了困难. 本文用近似 Lie 对称性理论研究弱非线性耦合二维各向异性谐振子的一阶近似 Lie 对称性与近似守恒量, 并以频率比为 2 : 1 的弱非线性耦合二维各向异性谐振子为例, 得到了其 6 个一阶近似 Lie 对称性与近似守恒量, 其中 1 个一阶近似守恒量实为系统的精确守恒量, 4 个一阶近似守恒量为平凡的一阶近似守恒量, 只有 1 个一阶近似守恒量为稳定的一阶近似守恒量.

2 弱非线性耦合二维各向异性谐振子的一阶近似 Lie 对称性与近似守恒量

设弱非线性耦合二维各向异性谐振子的 Lagrange 函数可表示为

$$L=\frac{1}{2}\dot{x}_1^2+\frac{1}{2}\dot{x}_2^2-\frac{1}{2}\omega_1^2x_1^2-\frac{1}{2}\omega_2^2x_2^2+\delta x_1x_2^2, \quad (1)$$

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其中 $\omega_1 \neq \omega_2$, δ 为非线性耦合系数, 且 $|\delta| \ll 1$. 系统的运动微分方程为

$$\begin{aligned}\ddot{x}_1 &= -\omega_1^2 x_1 + \delta x_2^2 = g_1 = g_1(\delta^0) + g_1(\delta^1),\\ \ddot{x}_2 &= -\omega_2^2 x_2 + 2\delta x_1 x_2 = g_2\\ &= g_2(\delta^0) + g_2(\delta^1),\end{aligned} \tag{2}$$

其中 g_1, g_2 为广义加速度, $g_1(\delta^0)$, $g_1(\delta^1)$, $g_2(\delta^0)$, $g_2(\delta^1)$ 分别表示 g_1, g_2 中的 δ^0, δ^1 项 (下文表示类同).

引进近似的群无限小变换

$$\begin{aligned}&t^* = t + \varepsilon\tau(t, x_s, \dot{x}_s, \delta),\\ &x_s^*(t^*) = x_s(t) + \varepsilon\xi_s(t, x_s, \dot{x}_s, \delta) \quad (s = 1, 2),\end{aligned} \tag{3}$$

其中 ε 为无限小参数, τ, ξ_s 为无限小变换的生成元. (3) 式的无限小生成元向量为

$$\boldsymbol{X}^{(0)} = \tau\frac{\partial}{\partial t} + \sum_{s=1}^{2}\xi_s\frac{\partial}{\partial x_s}, \tag{4}$$

(4) 式的一次扩展为

$$\boldsymbol{X}^{(1)} = \boldsymbol{X}^{(0)} + \sum_{s=1}^{2}(\dot{\xi}_s - \dot{x}_s\dot{\tau})\frac{\partial}{\partial \dot{x}_s}, \tag{5}$$

二次扩展为

$$\boldsymbol{X}^{(2)} = \boldsymbol{X}^{(1)} + \sum_{s=1}^{2}(\ddot{\xi}_s - \dot{x}_s\ddot{\tau} - 2\ddot{x}_s\dot{\tau})\frac{\partial}{\partial \ddot{x}_s}, \tag{6}$$

(3)—(6) 式中

$$\begin{aligned}\tau &= \tau_0 + \delta\tau_1,\\ \xi_s &= \xi_{s0} + \delta\xi_{s1} \quad (s = 1, 2),\end{aligned} \tag{7}$$

并有

$$\begin{aligned}\dot{\tau} &= \dot{\tau}_0 + \delta\dot{\tau}_1,\\ \ddot{\tau} &= \ddot{\tau}_0 + \delta\ddot{\tau}_1,\end{aligned} \tag{8a}$$

$$\begin{aligned}\dot{\xi}_s &= \dot{\xi}_{s0} + \delta\dot{\xi}_{s1},\\ \ddot{\xi}_s &= \ddot{\xi}_{s0} + \delta\ddot{\xi}_{s1} \quad (s = 1, 2).\end{aligned} \tag{8b}$$

对于自治系统, 可设 τ, ξ_s 不显含时间 t, 则

$$\begin{aligned}\dot{\tau}_i &= \tau_{ix_1}\dot{x}_1 + \tau_{ix_2}\dot{x}_2 + g_1\tau_{i\dot{x}_1} + g_2\tau_{i\dot{x}_2}\\ &= \dot{\tau}_i(\delta^0) + \dot{\tau}_i(\delta^1) \quad (i = 0, 1),\end{aligned} \tag{9a}$$

$$\begin{aligned}\dot{\xi}_{si} &= \xi_{six_1}\dot{x}_1 + \xi_{six_2}\dot{x}_2 + g_1\xi_{si\dot{x}_1} + g_2\xi_{si\dot{x}_2}\\ &= \dot{\xi}_{si}(\delta^0) + \dot{\xi}_{si}(\delta^1) \quad (s = 1, 2; i = 0, 1),\end{aligned} \tag{9b}$$

$$\begin{aligned}\ddot{\tau}_i &= \dot{x}_1^2\tau_{ix_1x_1} + \dot{x}_2\dot{x}_1\tau_{ix_1x_2} + g_1\dot{x}_1\tau_{ix_1\dot{x}_1}\\ &\quad + g_2\dot{x}_1\tau_{ix_1\dot{x}_2} + g_1\tau_{ix_1} + \dot{x}_1\dot{x}_2\tau_{ix_2x_1}\\ &\quad + \dot{x}_2^2\tau_{ix_2x_2} + g_1\dot{x}_2\tau_{ix_2\dot{x}_1} + g_2\dot{x}_2\tau_{ix_2\dot{x}_2}\\ &\quad + g_2\tau_{ix_2} + g_1\dot{x}_1\tau_{i\dot{x}_1x_1} + g_1\dot{x}_2\tau_{i\dot{x}_1x_2}\\ &\quad + g_1^2\tau_{i\dot{x}_1\dot{x}_1} + g_1g_2\tau_{i\dot{x}_1\dot{x}_2} + \dot{g}_1\tau_{i\dot{x}_1}\\ &\quad + g_2\dot{x}_1\tau_{i\dot{x}_2x_1} + g_2\dot{x}_2\tau_{i\dot{x}_2x_2} + g_1g_2\tau_{i\dot{x}_2\dot{x}_1}\\ &\quad + g_2^2\tau_{i\dot{x}_2\dot{x}_2} + \dot{g}_2\tau_{i\dot{x}_2}\\ &= \ddot{\tau}_i(\delta^0) + \ddot{\tau}_i(\delta^1) + \ddot{\tau}_i(\delta^2) \quad (i = 0, 1),\end{aligned} \tag{9c}$$

$$\begin{aligned}\ddot{\xi}_{si} &= \dot{x}_1^2\xi_{six_1x_1} + \dot{x}_2\dot{x}_1\xi_{six_1x_2} + g_1\dot{x}_1\xi_{six_1\dot{x}_1}\\ &\quad + g_2\dot{x}_1\xi_{six_1\dot{x}_2} + g_1\xi_{six_1} + \dot{x}_1\dot{x}_2\xi_{six_2x_1}\\ &\quad + \dot{x}_2^2\xi_{six_2x_2} + g_1\dot{x}_2\xi_{six_2\dot{x}_1} + g_2\dot{x}_2\xi_{six_2\dot{x}_2}\\ &\quad + g_2\xi_{six_2} + g_1\dot{x}_1\xi_{si\dot{x}_1x_1} + g_1\dot{x}_2\xi_{si\dot{x}_1x_2}\\ &\quad + g_1^2\xi_{si\dot{x}_1\dot{x}_1} + g_1g_2\xi_{si\dot{x}_1\dot{x}_2} + \dot{g}_1\xi_{si\dot{x}_1}\\ &\quad + g_2\dot{x}_1\xi_{si\dot{x}_2x_1} + g_2\dot{x}_2\xi_{si\dot{x}_2x_2} + g_1g_2\xi_{si\dot{x}_2\dot{x}_1}\\ &\quad + g_2^2\xi_{si\dot{x}_2\dot{x}_2} + \dot{g}_2\xi_{si\dot{x}_2}\\ &= \ddot{\xi}_{si}(\delta^0) + \ddot{\xi}_{si}(\delta^1) + \ddot{\xi}_{si}(\delta^2)\\ &\quad (s = 1, 2; \quad i = 0, 1).\end{aligned} \tag{9d}$$

(9a) 式中的 τ_{ix_1} 表示 τ_i 对 x_1 的偏导, 其他表示类同.

根据近似 Lie 对称性理论 [9], 运动微分方程 (2) 式的一阶近似 Lie 对称性是指 (2) 式在近似的群无限小变换 (3) 式下近似保持不变, 即

$$X^{(2)}(\ddot{x}_1 - g_1) = O(\delta^2), \tag{10a}$$

$$X^{(2)}(\ddot{x}_2 - g_2) = O(\delta^2). \tag{10b}$$

利用 (6) 式, (10a) 和 (10b) 式可表示为

$$\ddot{\xi}_1 - \dot{x}_1\ddot{\tau} - 2g_1\dot{\tau} + \omega_1^2\xi_1 - 2\delta\xi_2 x_2 = O(\delta^2), \tag{11a}$$

$$\begin{aligned}&\ddot{\xi}_2 - \dot{x}_2\ddot{\tau} - 2g_2\dot{\tau} + \omega_2^2\xi_2 - 2\delta\xi_1 x_2\\ &- 2\delta\xi_2 x_1 = O(\delta^2).\end{aligned} \tag{11b}$$

将 (2) 和 (9) 式代入 (11) 式并展开, 令 δ^0, δ^1 项的系数为 0, 可得到 4 个关于 τ_0, τ_1, ξ_{10}, ξ_{11}, ξ_{20}, ξ_{21} 的二阶微分方程组. 令 (11a) 式中 δ^0 项的系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{10}(\delta^0) - \dot{x}_1\ddot{\tau}_0(\delta^0) + 2\omega_1^2 x_1\dot{\tau}_0(\delta^0)\\ &+ \omega_1^2\xi_{10} = 0,\end{aligned} \tag{12}$$

令 (11a) 式中 δ^1 项系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{10}(\delta^1) + \ddot{\xi}_{11}(\delta^0) - \dot{x}_1\ddot{\tau}_0(\delta^1) - \dot{x}_1\ddot{\tau}_1(\delta^0)\\ &+ 2\omega_1^2 x_1\dot{\tau}_0(\delta^1) + 2\omega_1^2 x_1\dot{\tau}_1(\delta^0) - 2x_2^2\dot{\tau}_0(\delta^0)\\ &+ \omega_1^2\xi_{11} - 2\xi_{20}x_2 = 0,\end{aligned} \tag{13}$$

令 (11b) 式中 δ^0 项的系数为 0, 可得

$$\ddot{\xi}_{20}(\delta^0) - \dot{x}_2\ddot{\tau}_0(\delta^0) + 2\omega_2^2 x_2\dot{\tau}_0(\delta^0) + \omega_2^2\xi_{20} = 0, \tag{14}$$

令 (11b) 式中 δ^1 项的系数为 0, 可得

$$\begin{aligned}&\ddot{\xi}_{20}(\delta^1) + \ddot{\xi}_{21}(\delta^0) - \dot{x}_2\ddot{\tau}_0(\delta^1) - \dot{x}_2\ddot{\tau}_1(\delta^0)\\&+ 2\omega_2^2 x_2\dot{\tau}_0(\delta^1) + 2\omega_2^2 x_2\dot{\tau}_1(\delta^0) - 4x_1x_2\dot{\tau}_0(\delta^0)\\&+ \omega_2^2\xi_{21} - 2x_2\xi_{10} - 2x_1\xi_{20} = 0.\end{aligned} \tag{15}$$

将 (9a)—(9d) 式代入 (12)—(15) 式, 可解得 $\tau_0, \tau_1, \xi_{10}, \xi_{11}, \xi_{20}, \xi_{21}$.

若 $\tau_1 = \xi_{11} = \xi_{21} = 0$, 则相应的对称性为精确的 Lie 对称性, 所得的守恒量为精确守恒量; 若 $\tau_0 = \xi_{10} = \xi_{20} = 0$, 则相应的对称性为平凡的一阶近似 Lie 对称性, 所得的守恒量为平凡的一阶近似守恒量; 若 $\tau_0, \xi_{10}, \xi_{20}$ 不全为 0, 同时 $\tau_1, \xi_{11}, \xi_{21}$ 也不全为 0, 则相应的对称性为稳定的一阶近似 Lie 对称性, 所得的守恒量为稳定的一阶近似守恒量.

下面讨论系统的一阶近似守恒量. 若存在规范函数

$$G = G(x_s, \dot{x}_s, \delta) = G_0 + \delta G_1 \quad (s = 1, 2) \tag{16}$$

满足

$$\begin{aligned}&\frac{\partial L}{\partial t}\tau + \sum_{s=1}^{2}\frac{\partial L}{\partial x_s}\xi_s + \sum_{s=1}^{2}\frac{\partial L}{\partial \dot{x}_s}\dot{\xi}_s\\&+ \Big(L - \sum_{s=1}^{2}\frac{\partial L}{\partial \dot{x}_s}\dot{x}_s\Big)\dot{\tau} = -\dot{G},\end{aligned} \tag{17}$$

则系统存在一阶近似守恒量 $I = I_0 + \delta I_1$,

$$I = L\tau + \sum_{s=1}^{2}\frac{\partial L}{\partial \dot{x}_s}(\xi_s - \dot{x}_s\tau) + G \tag{18}$$

满足 $\dfrac{\mathrm{d}I}{\mathrm{d}t} = O(\delta^2)$.

对于自治系统, $\dfrac{\partial L}{\partial t} = 0$. 将 (1), (2), (9a), (9b) 式代入 (17) 式, 并比较等式两边 δ^0, δ^1 项的系数, 可得关于 G_0, G_1 的两个方程:

$$\begin{aligned}&g_1(\delta^0)\xi_{10} + g_2(\delta^0)\xi_{20} + \dot{x}_1\dot{\xi}_{10}(\delta^0)\\&+ \dot{x}_2\dot{\xi}_{20}(\delta^0) - H(\delta^0)\dot{\tau}_0(\delta^0)\\&= -\dot{G}_0(\delta^0),\end{aligned} \tag{19a}$$

$$\begin{aligned}&g_1(\delta^1)\xi_{10} + g_1(\delta^0)\xi_{11} + g_2(\delta^1)\xi_{20}\\&+ g_2(\delta^0)\xi_{21} + \dot{x}_1\dot{\xi}_{10}(\delta^1) + \dot{x}_1\dot{\xi}_{11}(\delta^0)\\&+ \dot{x}_2\dot{\xi}_{20}(\delta^1) + \dot{x}_2\dot{\xi}_{21}(\delta^0) - H(\delta^1)\dot{\tau}_0(\delta^0)\\&- H(\delta^0)\dot{\tau}_0(\delta^1) - H(\delta^0)\dot{\tau}_1(\delta^0)\\&= -\dot{G}_0(\delta^1) - \dot{G}_1(\delta^0),\end{aligned} \tag{19b}$$

其中 H 为系统的 Hamilton 函数, $g_1(\delta^0)$, $g_2(\delta^0)$, $g_1(\delta^1)$, $g_2(\delta^1)$ 由 (2) 式给出. 由 (19) 式可解得规范函数. 将解得的 $\tau_0, \tau_1, \xi_{10}, \xi_{11}, \xi_{20}, \xi_{21}, G_0, G_1$ 及 (1) 式代入 (18) 式, 可得到系统的一阶近似守恒量.

3 频率比为 2 : 1 的弱非线性耦合谐振子的一阶近似 Lie 对称性与近似守恒量

为说明上述理论的应用, 下面讨论 $\omega_1/\omega_2 = 2/1$ 的弱非线性耦合谐振子的一阶 Lie 对称性与近似守恒量. 为方便, 令 $\omega_1 = 2, \omega_2 = 1$, 则系统的 Lagrange 函数为

$$L = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 - 2x_1^2 - \frac{1}{2}x_2^2 + \delta x_1 x_2^2, \tag{20}$$

Hamilton 函数为

$$H = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + 2x_1^2 + \frac{1}{2}x_2^2 - \delta x_1 x_2^2, \tag{21}$$

运动微分方程为

$$\begin{aligned}&\ddot{x}_1 = -4x_1 + \delta x_2^2 = g_1 = g_1(\delta^0) + g_1(\delta^1),\\&\ddot{x}_2 = -x_2 + 2\delta x_1 x_2 = g_2 = g_2(\delta^0) + g_2(\delta^1).\end{aligned} \tag{22}$$

将 $\omega_1 = 2, \omega_2 = 1$ 代入 (12)—(15) 式, 可解得如下 6 组生成元:

$$\begin{aligned}&\tau_0 = -1,\ \xi_{10} = \xi_{20} = 0,\\&\tau_1 = \xi_{11} = \xi_{21} = 0;\end{aligned} \tag{23a}$$

$$\begin{aligned}&\tau_0 = \xi_{10} = \xi_{20} = 0, \quad \tau_1 = -1,\\&\xi_{11} = \xi_{21} = 0;\end{aligned} \tag{23b}$$

$$\begin{aligned}&\tau_0 = \xi_{10} = \xi_{20} = 0, \quad \tau_1 = -1,\\&\xi_{11} = 0, \quad \xi_{21} = -\dot{x}_2;\end{aligned} \tag{23c}$$

$$\begin{aligned}&\tau_0 = \xi_{10} = \xi_{20} = 0, \quad \tau_1 = -1,\\&\xi_{11} = -\dot{x}_1, \quad \xi_{21} = 0;\end{aligned} \tag{23d}$$

$$\begin{aligned}&\tau_0 = \xi_{10} = \xi_{20} = 0, \quad \tau_1 = 0,\\&\xi_{11} = x_2\dot{x}_2, \quad \xi_{21} = x_2\dot{x}_1 - 2x_1\dot{x}_2;\end{aligned} \tag{23e}$$

$$\begin{aligned}&\tau_0 = 0, \quad \xi_{10} = x_2\dot{x}_2,\\&\xi_{20} = x_2\dot{x}_1 - 2x_1\dot{x}_2, \quad \tau_1 = 0,\\&\xi_{11} = \frac{-8x_1x_2\dot{x}_2 + 5x_2^2\dot{x}_1 - 3\dot{x}_1\dot{x}_2^2}{8},\\&\xi_{21} = \frac{1}{8}\big(-4x_1^2\dot{x}_2 - 8x_1x_2\dot{x}_1 + 3x_2^2\dot{x}_2\end{aligned}$$

$$- 3\dot{x}_1^2\dot{x}_2 + 3\dot{x}_2^3\big). \tag{23f}$$

将 (21)—(23) 式代入 (19) 式, 可解得与上述 6 组生成元相应的规范函数:

$$G_0 = 0, \quad G_1 = 0; \tag{24a}$$

$$G_0 = 0, \quad G_1 = 0; \tag{24b}$$

$$G_0 = 0, \quad G_1 = \frac{1}{2}\dot{x}_2^2 - \frac{1}{2}x_2^2; \tag{24c}$$

$$G_0 = 0, \quad G_1 = \frac{1}{2}\dot{x}_1^2 - 2x_1^2; \tag{24d}$$

$$G_0 = 0, \quad G_1 = x_1x_2^2 + x_1\dot{x}_2^2 - x_2\dot{x}_1\dot{x}_2; \tag{24e}$$

$$\begin{aligned} G_0 &= x_1x_2^2 + x_1\dot{x}_2^2 - x_2\dot{x}_1\dot{x}_2, \\ G_1 &= \frac{1}{32}(8x_1^2\dot{x}_2^2 - 8x_1^2x_2^2 + 32x_1x_2\dot{x}_1\dot{x}_2 - 5x_2^4 \\ &\quad - 10x_2^2\dot{x}_1^2 - 6x_2^2\dot{x}_2^2 + 18\dot{x}_1^2\dot{x}_2^2 - 9\dot{x}_2^4). \end{aligned} \tag{24f}$$

将 (20), (23), (24) 式代入 (18) 式, 得到 6 个一阶近似守恒量:

$$I^1 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + 2x_1^2 + x_2^2 - \delta x_1x_2^2, \tag{25a}$$

$$I^2 = \delta\Big(\frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + 2x_1^2 + x_2^2\Big) = \delta I_0^1, \tag{25b}$$

$$I^3 = \delta\Big(\frac{1}{2}\dot{x}_1^2 + 2x_1^2\Big) = \delta I_0^2, \tag{25c}$$

$$I^4 = \delta\Big(\frac{1}{2}\dot{x}_2^2 + \frac{1}{2}x_2^2\Big) = \delta I_0^3, \tag{25d}$$

$$I^5 = \delta(x_1x_2^2 - x_1\dot{x}_2^2 + x_2\dot{x}_1\dot{x}_2) = \delta I_0^4. \tag{25e}$$

$$\begin{aligned} I^6 =& x_1x_2^2 - x_1\dot{x}_2^2 + x_2\dot{x}_1\dot{x}_2 + \frac{\delta}{32}\big(10x_2^2\dot{x}_1^2 \\ &+ 6x_2^2\dot{x}_2^2 + 3\dot{x}_2^4 - 8x_1^2x_2^2 - 8x_1^2\dot{x}_2^2 \\ &- 32x_1x_2\dot{x}_1\dot{x}_2 - 5x_2^4 - 6\dot{x}_1^2\dot{x}_2^2\big). \end{aligned} \tag{25f}$$

其中

$$I_0^1 = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 + 2x_1^2 + x_2^2, \tag{26a}$$

$$I_0^2 = \frac{1}{2}\dot{x}_1^2 + 2x_1^2, \tag{26b}$$

$$I_0^3 = \frac{1}{2}\dot{x}_2^2 + \frac{1}{2}x_2^2, \tag{26c}$$

$$I_0^4 = x_1x_2^2 - x_1\dot{x}_2^2 + x_2\dot{x}_1\dot{x}_2. \tag{26d}$$

$I_0^1, I_0^2, I_0^3, I_0^4$ 为不受非线性耦合项作用的二维各向异性谐振子的四个守恒量, 均满足 $\frac{\mathrm{d}I_0^i}{\mathrm{d}t} = 0$ ($i = 1$, 2, 3, 4), 其中 I_0^1 为系统的总能量, I_0^2, I_0^3 为两个分振子的能量, 这四个量守恒量不相互独立, 存在 $I_0^1 = I_0^2 + I_0^3$ 的关系.

(25) 式表示的 6 个守恒量可分为三类: I^1 为系统的精确守恒量; 相应的对称性也为精确 Lie 对称性, 它是系统的总能量; I^2, I^3, I^4, I^5 是平凡的一阶近似守恒量, 相应的对称性也为平凡的一阶近似 Lie 对称性, 这 4 个一阶近似守恒量分别是无非线性耦合项的二维各向异性谐振子的 4 个守恒量 $I_0^1, I_0^2, I_0^3, I_0^4$ 乘以耦合系数 δ; I^6 为系统稳定的一阶近似守恒量, 相应的对称性也为稳定的一阶近似 Lie 对称性.

4 结 论

首先用近似 Lie 对称性理论研究了弱非线性耦合二维各向异性谐振子系统的一阶近似 Lie 对称性与近似守恒量. 通过研究系统的一阶近似 Lie 对称性与近似守恒量, 可以直接得到系统的三类对称性与守恒量, 分别为精确的 Lie 对称性与精确守恒量, 平凡的一阶近似 Lie 对称性与近似守恒量, 稳定的一阶近似 Lie 对称性与近似守恒量. 其次以频率比为 2 : 1 的弱非线性耦合二维各向异性谐振子为例, 得到其 6 个一阶近似 Lie 对称性与近似守恒量, 6 个近似守恒量中 1 个为精确守恒量, 4 个为平凡的一阶近似守恒量, 只有 1 个为稳定的一阶近似守恒量. 文中只研究了弱非线性耦合二维各向异性谐振子的一阶近似 Lie 对称性与近似守恒量, 更高阶的近似 Lie 对称性与近似守恒量或更多维系统的近似 Lie 对称性与近似守恒量也可以用类似的方法研究, 只是计算更加复杂.

[1] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) p90 (in Chinese) [梅凤翔 1999 李群和李代数对约束力学系统的应用 (北京: 科学出版社) 第 90 页]

[2] Zhao Y Y, Mei F X 1999 *Symmetries and Invariants of Mechanical System* (Beijing: Science Press) p1 (in Chinese) [赵跃宇, 梅凤翔 1999 力学系统的对称性与不变量 (北京: 科学出版社) 第 1 页]

[3] Lou Z M 2007 *Chin. Phys.* **16** 1182

[4] Fu J L, Chen L Q, Chen X W 2006 *Chin. Phys.* **15** 8

[5] Jia L Q, Xie J F, Luo S K 2008 *Chin. Phys.* B **17** 1560

[6] Lou Z M 2007 *Acta Phys. Sin.* **56** 2475 (in Chinese) [楼智美 2007 物理学报 **56** 2475]

[7] Fang J H 2010 *Chin. Phys.* B **19** 040301

[8] Zhang Y 2011 *Chin. Phys.* B **20** 034502

[9] Leach P G L, Moyo S, Cotsakis S, Lemmer R L 2001 *J. Nonlinear Math. Phys.* **8** 139

[10] Govinder K S, Heil T G, Uzer T 1998 *Phys. Lett.* A **240** 127
[11] Kara A H, Mahomed F M, Unal G 1999 *Int. J. Theoret. Phys.* **38** 2389
[12] Unal G 2000 *Phys. Lett.* A **269** 13
[13] Unal G 2001 *Nonlinear Dyn.* **26** 309
[14] Unal G, Gorali G 2002 *Nonlinear Dyn.* **28** 195
[15] Feroze T, Kara A H 2002 *Int. J. Non-linear Mech.* **37** 275
[16] Ibragimov N H, Unal G, Jogreus C 2004 *J. Math. Anal. Appl.* **297** 152
[17] Dolapci I T, Pakdemirli M 2004 *Int. J. Non-linear Mech.* **39** 1603
[18] Kara A H, Mahomed F M, Qadir A 2008 *Nonlinear Dyn.* **51** 183
[19] Pakdemirli M, Yurusoy M, Dolapci I T 2004 *Acta Appl. Math.* **80** 243
[20] Johnpillai A G, Kara A H, Mahomed F M 2006 *Int. J. Non-linear Mech.* **41** 830
[21] Grebenev V N, Oberlack M 2007 *J. Nonlinear Math. Phys.* **14** 157
[22] Johnpillai A G, Kara A H, Mahomed F M 2009 *J. Comput. Appl. Math.* **223** 508
[23] Lou Z M 2010 *Acta Phys. Sin.* **59** 6764 (in Chinese) [楼智美 2010 物理学报 **59** 6764]

The first-order approximate Lie symmetries and approximate conserved quantities of the weak nonlinear coupled two-dimensional anisotropic harmonic oscillator*

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Abstract

The first-order approximate Lie symmetries and approximate conserved quantities of the weak nonlinear coupled two-dimensional anisotropic harmonic oscillator are studied. When the ω_1/ω_2 is equal to 2/1, the system possesses six first-order approximate Lie symmetries and approximate conserved quantities, one of them is an exact conserved quantity, four of them are trivial conserved quantities, only one of them is a stable conserved quantity.

Keywords: weak nonlinear coupled two-dimensional anisotropic harmonic oscillator, approximate Lie symmetries, approximate conserved quantity

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