变质量非完整系统的 Lagrange 对称性与守恒量*

张斌 方建会† 张克军

(中国石油大学 (华东) 物理科学与技术学院, 青岛 266555)

(2011 年 3 月 18 日收到; 2011 年 4 月 25 日收到修改稿)

本文研究变质量非完整系统的 Lagrange 对称性, 给出变质量非完整系统 Lagrange 对称性的判据, 得到变质量非完整系统 Lagrange 对称性导致的守恒量及其存在的条件, 并举例说明结果的应用.

关键词: 变质量, 非完整系统, Lagrange 对称性, 守恒量

PACS: 11.30.−j, 11.10.Ef, 02.40.−k

1 引 言

对称性原理是物理学中更高层次的法则, 分析力学中的近代对称性主要有 Noether 对称性, Lie 对称性和 Mei 对称性. 近年来, 关于约束力学系统三种对称性及其导致守恒量的研究取得了一系列重要成果 [1−7]. Currie 和 Saletan 在 1966 年研究了单自由度 Lagrange 函数等价问题 [8]. 1981 年 Hojman 和 Harleston 将这种 Lagrange 函数等价问题的研究对象推广到多自由度系统 [9]. 1999 年赵跃宇和梅凤翔将这种性质称为 Lagrange 对称性 [10], 并研究了完整非保守系统的这种对称性. 2008—2009 年梅凤翔和吴惠彬相继研究了非完整力学系统 [11]、相对运动力学系统 [12] 和准坐标下完整系统 [13] 的 Lagrange 对称性. Wu 等研究了非 Chetaev 型非完整系统的 Lagrange 对称性 [14,15]. Xia 等将 Lagrange 对称性理论进一步拓展到非完整可控力学系统领域 [16]. 本文对变质量非完整系统的 Lagrange 对称性进行研究, 给出变质量非完整系统 Lagrange 对称性的判据, 得到其导致守恒量的条件以及守恒量的存在形式.

2 系统的 Lagrange 对称性

设力学系统由 N 个变质量质点组成, 第 i 个质点在 t 时刻的质量为 $m_i(i=1,\cdots,N)$, 在 $t+\mathrm{d}t$ 时刻质量为 $m_i+\mathrm{d}m_i$. 其中 m_i 为 $t, \boldsymbol{q}$ 和 $\dot{\boldsymbol{q}}$ 的函数, 即

$$m_i=m_i(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad (i=1,\cdots,N). \tag{1}$$

系统受有 g 个非完整约束

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad (\beta=1,\cdots,g), \tag{2}$$

约束方程 f_β 和变分 δq_s 满足 Appell-Chetaev 条件

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s=0,\quad (\beta=1,\cdots g),$$

则系统的微分方程可表示为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+P_s+\Lambda_s,\quad (s=1,\cdots,n), \tag{3}$$

其中 Q_s, Λ_s 和 P_s 分别为广义非势力, 广义约束反力, 由变化质量产生的广义反推力. Λ_s 和 P_s 的形式为

$$\Lambda_s=\Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\Lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})\frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{4}$$

$$\begin{aligned}P_s=&\sum_{i=1}^{N}\Big[\frac{\mathrm{d}m_i}{\mathrm{d}t}(\boldsymbol{u}_i+\dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s}\\&-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial \dot{q}_s}\Big)\Big],\end{aligned} \tag{5}$$

$\boldsymbol{u}_i$ 为 $\mathrm{d}m_i$ 相对于 m_i 的速度.

对于上述系统的两组动力学函数 L, Q_r, Λ_r, P_r 和 $\bar{L}$, $\bar{Q}_r$, $\bar{\Lambda}_r$, $\bar{P}_r$, 令 $\partial^2 L/\partial\dot{q}_r\partial\dot{q}_k=W_{rk}$, $\partial^2\bar{L}/\partial\dot{q}_r\partial\dot{q}_k=\bar{W}_{rk}$, $(\bar{W}_{rk})^{-1}=\bar{U}^{kr}$, $(W_{rk})^{-1}=$

* 中央高校基本科研业务费专项资金 (批准号: 09CX04018A)、山东省自然科学基金 (批准号: ZR2011AM012) 和中国石油大学 (华东) 研究生创新工程项目 (批准号: CXYB11-12) 资助的课题.

† E-mail: fangjh@upc.edu.cn

$U^{kr}, \bar{W}_{sk}U^{kr} = A_s^r$,

$$\begin{aligned} L_r &= \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k}\ddot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k \\ &\quad + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} - \frac{\partial L}{\partial q_r} - Q_r - \Lambda_r - P_r \\ &= W_{rk}\ddot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} \\ &\quad - \frac{\partial L}{\partial q_r} - Q_r - \Lambda_r - P_r, \end{aligned} \tag{6}$$

$$\begin{aligned} \bar{L}_r &= \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial \dot{q}_k}\ddot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_k}\dot{q}_k \\ &\quad + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial t} - \frac{\partial \bar{L}}{\partial q_r} - \bar{Q}_r - \bar{\Lambda}_r - \bar{P}_r \\ &= \bar{W}_{rk}\ddot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_k}\dot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial t} \\ &\quad - \frac{\partial \bar{L}}{\partial q_r} - \bar{Q}_r - \bar{\Lambda}_r - \bar{P}_r. \end{aligned} \tag{7}$$

根据 Lagrange 对称性理论, 如果由

$$L_r = 0, \tag{8}$$

可得

$$\bar{L}_r = 0. \tag{9}$$

反之亦然, 则表明系统具有 Lagrange 对称性.

考虑到 (9) 式, (7) 式可写为

$$\begin{aligned} \bar{W}_{rk}\ddot{q}_k &= \bar{Q}_r + \bar{\Lambda}_r + \bar{P}_r + \frac{\partial \bar{L}}{\partial q_r} \\ &\quad - \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial t}, \end{aligned} \tag{10}$$

(10) 式进一步可表示为

$$\begin{aligned} \ddot{q}_k &= \bar{U}^{kr}(\bar{Q}_r + \bar{\Lambda}_r + \bar{P}_r + \frac{\partial \bar{L}}{\partial q_r} \\ &\quad - \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial t}). \end{aligned} \tag{11}$$

由 (6) 式, (8) 式和 (11) 式, 得

$$\begin{aligned} & W_{rk}\bar{U}^{ks}\Big(\bar{Q}_s + \bar{\Lambda}_s + \bar{P}_s + \frac{\partial \bar{L}}{\partial q_s} \\ & - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t}\Big) \\ &= Q_r + \Lambda_r + P_r + \frac{\partial L}{\partial q_r} - \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k \\ & - \frac{\partial^2 L}{\partial \dot{q}_r \partial t}. \end{aligned} \tag{12}$$

注意到 $\bar{W}_{rs}U^{sk} = A_r^k$, (12) 式可写为

$$\begin{aligned} & \bar{Q}_s + \bar{\Lambda}_s + \bar{P}_s + \frac{\partial \bar{L}}{\partial q_s} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t} \\ &= A_s^r(Q_r + \Lambda_r + P_r + \frac{\partial L}{\partial q_r} \\ & - \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k - \frac{\partial^2 L}{\partial \dot{q}_r \partial t}). \end{aligned} \tag{13}$$

于是可得如下判据.

判据 对于变质量非完整系统 (3), 如果两组动力学函数 L, Q_r, Λ_r, P_r 和 $\bar{L}, \bar{Q}_s, \bar{\Lambda}_s, \bar{P}_s$ 满足方程 (13), 则系统具有 Lagrange 对称性.

3 Lagrange 对称性导致的守恒量

下面研究变质量非完整系统 Lagrange 对称性导致的守恒量.

将 (12) 式代入 (6) 式, 得

$$\begin{aligned} L_r &= W_{rk}\ddot{q}_k - W_{rk}\bar{U}^{ks}\Big(\bar{Q}_s + \bar{\Lambda}_s + \bar{P}_s \\ &\quad + \frac{\partial \bar{L}}{\partial q_s} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t}\Big) \\ &= W_{rk}\bar{U}^{ks}\bar{W}_{sk}\ddot{q}_k - W_{rk}\bar{U}^{ks}\Big(\bar{Q}_s + \bar{\Lambda}_s \\ &\quad + \bar{P}_s + \frac{\partial \bar{L}}{\partial q_s} - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k - \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t}\Big) \\ &= W_{rk}\bar{U}^{ks}\Big(\bar{W}_{sk}\ddot{q}_k - \bar{Q}_s - \bar{\Lambda}_s - \bar{P}_s \\ &\quad - \frac{\partial \bar{L}}{\partial q_s} + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_k}\dot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial t}\Big) \\ &= W_{rk}\bar{U}^{ks}\bar{L}_s. \end{aligned} \tag{14}$$

考虑到 $\bar{W}_{sk}U^{kr} = A_s^r$, (14) 式可化为如下形式:

$$\bar{L}_s = A_s^r L_r. \tag{15}$$

由于 $\bar{W}_{sk}U^{kr} = A_s^r$, 且 $(U^{kr})^{-1} = W_{rk}$, 即

$$\bar{W}_{sk} = A_s^r W_{rk}. \tag{16}$$

求 (16) 式关于 $\dot{q}_l$ 的偏导数, 得

$$\frac{\partial \bar{W}_{sk}}{\partial \dot{q}_l} = \frac{\partial A_s^r}{\partial \dot{q}_l}W_{rk} + A_s^r\frac{\partial W_{rk}}{\partial \dot{q}_l}, \tag{17}$$

(17) 式进一步可写为

$$\begin{aligned} \frac{\partial A_s^r}{\partial \dot{q}_l}W_{rk} &= \frac{\partial^3 \bar{L}}{\partial \dot{q}_s \partial \dot{q}_k \partial \dot{q}_l} - A_s^r\frac{\partial^3 L}{\partial \dot{q}_r \partial \dot{q}_k \partial \dot{q}_l} \\ &= \frac{\partial \bar{W}_{sl}}{\partial \dot{q}_k} - A_s^r\frac{\partial W_{rl}}{\partial \dot{q}_k}. \end{aligned} \tag{18}$$

由

$$A_s^r = \bar{W}_{sk}U^{kr} \text{和} U^{lr}W_{rl} = 1,$$

$$\frac{\partial (A_s^r W_{rl})}{\partial \dot{q}_k} = \frac{\partial A_s^r}{\partial \dot{q}_k}W_{rl} + A_s^r\frac{\partial W_{rl}}{\partial \dot{q}_k},$$

可得

$$\begin{aligned} \frac{\partial(\bar{W}_{sl}U^{lr}W_{rl})}{\partial \dot{q}_k} &= \frac{\partial \bar{W}_{sl}}{\partial \dot{q}_k} = \frac{\partial A_s^r}{\partial \dot{q}_k}W_{rl} \\ &\quad + A_s^r\frac{\partial W_{rl}}{\partial \dot{q}_k}. \end{aligned} \tag{19}$$

将 (18) 式代入 (19) 式, 得

$$\frac{\partial A_s^r}{\partial \dot{q}_k} W_{rl} = \frac{\partial \bar{W}_{sl}}{\partial \dot{q}_k} - A_s^r \frac{\partial W_{rl}}{\partial \dot{q}_k} = \frac{\partial A_s^r}{\partial \dot{q}_l} W_{rk}. \tag{20}$$

将 (13) 式对 $\dot{q}_l$ 求偏导数, 得

$$\begin{aligned}&\frac{\partial^3 \bar{L}}{\partial \dot{q}_s \partial q_k \partial \dot{q}_l}\dot{q}_k + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_l} + \frac{\partial^3 \bar{L}}{\partial \dot{q}_s \partial t \partial \dot{q}_l} \\ &- \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_l} - \frac{\partial \bar{Q}_s}{\partial \dot{q}_l} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_l} - \frac{\partial \bar{P}_s}{\partial \dot{q}_l} \\ =& \frac{\partial A_s^r}{\partial \dot{q}_l}\Big(\frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} \\ &- \frac{\partial L}{\partial q_r} - Q_r - \Lambda_r - P_r\Big) \\ &+ A_s^r\Big(\frac{\partial^3 L}{\partial \dot{q}_r \partial q_k \partial \dot{q}_l}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} \\ &+ \frac{\partial^3 L}{\partial \dot{q}_r \partial t \partial \dot{q}_l} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l} \\ &- \frac{\partial Q_r}{\partial \dot{q}_l} - \frac{\partial \Lambda_r}{\partial \dot{q}_l} - \frac{\partial P_r}{\partial \dot{q}_l}\Big).\end{aligned} \tag{21}$$

利用 (6) 式和 (8) 式, 得

$$\begin{aligned}-W_{rk}\ddot{q}_k =& \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k}\dot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} \\ &- \frac{\partial L}{\partial q_r} - Q_r - \Lambda_r - P_r,\end{aligned} \tag{22}$$

将 (22) 式代入 (21) 式, 得

$$\begin{aligned}&\frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_l} + \frac{\partial \bar{W}_{sl}}{\partial t} - \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_l} \\ &- \frac{\partial \bar{Q}_s}{\partial \dot{q}_l} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_l} - \frac{\partial \bar{P}_s}{\partial \dot{q}_l} \\ =& -\frac{\partial \bar{W}_{sl}}{\partial q_l}\dot{q}_l + A_s^r \frac{\partial W_{rl}}{\partial q_l}\dot{q}_l - \frac{\partial A_s^r}{\partial \dot{q}_l} W_{rl}\ddot{q}_l \\ &+ A_s^r\Big(\frac{\partial W_{rl}}{\partial t} + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l} \\ &- \frac{\partial Q_r}{\partial \dot{q}_l} - \frac{\partial \Lambda_r}{\partial \dot{q}_l} - \frac{\partial P_r}{\partial \dot{q}_l}\Big).\end{aligned} \tag{23}$$

求 (16) 式关于 q_l 的偏导数, 得

$$A_s^r \frac{\partial W_{rk}}{\partial q_l}\dot{q}_l - \frac{\partial \bar{W}_{sk}}{\partial q_l}\dot{q}_l = -\frac{\partial A_s^r}{\partial q_l} W_{rk}\dot{q}_l. \tag{24}$$

将 (24) 式代入 (23) 式, 得

$$\begin{aligned}&\frac{\partial \bar{W}_{sl}}{\partial t} + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_l} - \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_l} \\ &- \frac{\partial \bar{Q}_s}{\partial \dot{q}_l} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_l} - \frac{\partial \bar{P}_s}{\partial \dot{q}_l} \\ =& -\frac{\partial A_s^r}{\partial \dot{q}_l} W_{rl}\ddot{q}_l - \frac{\partial A_s^r}{\partial q_l} W_{rl}\dot{q}_l \\ &+ A_s^r\Big(\frac{\partial W_{rl}}{\partial t} + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l} \\ &- \frac{\partial Q_r}{\partial \dot{q}_l} - \frac{\partial \Lambda_r}{\partial \dot{q}_l} - \frac{\partial P_r}{\partial \dot{q}_l}\Big),\end{aligned}$$

即

$$\begin{aligned}&\frac{\partial \bar{W}_{sl}}{\partial t} + \frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_l} - \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_l} \\ &- \frac{\partial \bar{Q}_s}{\partial \dot{q}_l} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_l} - \frac{\partial \bar{P}_s}{\partial \dot{q}_l} \\ =& -\frac{\mathrm{d}A_s^r}{\mathrm{d}t} W_{rl} + \frac{\partial A_s^r}{\partial t} W_{rl} \\ &+ A_s^r\Big(\frac{\partial W_{rl}}{\partial t} + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} \\ &- \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l} - \frac{\partial Q_r}{\partial \dot{q}_l} - \frac{\partial \Lambda_r}{\partial \dot{q}_l} - \frac{\partial P_r}{\partial \dot{q}_l}\Big).\end{aligned} \tag{25}$$

由 (20) 式可得

$$\frac{\partial A_s^r}{\partial t} W_{rl} = \frac{\partial \bar{W}_{sl}}{\partial t} - A_s^r \frac{\partial W_{rl}}{\partial t}. \tag{26}$$

考虑到 (26) 式, (25) 式可写为

$$\begin{aligned}&\frac{\partial^2 \bar{L}}{\partial \dot{q}_s \partial q_l} - \frac{\partial^2 \bar{L}}{\partial q_s \partial \dot{q}_l} - \frac{\partial \bar{Q}_s}{\partial \dot{q}_l} - \frac{\partial \bar{\Lambda}_s}{\partial \dot{q}_l} - \frac{\partial \bar{P}_s}{\partial \dot{q}_l} \\ =& -\frac{\mathrm{d}A_s^r}{\mathrm{d}t} W_{rl} + A_s^r\Big(\frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} \\ &- \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l} - \frac{\partial Q_r}{\partial \dot{q}_l} - \frac{\partial \Lambda_r}{\partial \dot{q}_l} - \frac{\partial P_r}{\partial \dot{q}_l}\Big).\end{aligned} \tag{27}$$

如果两组广义非势力, 广义约束反力, 广义反推力满足条件

$$\begin{aligned}&\frac{\partial}{\partial \dot{q}_l}(\bar{Q}_s + \bar{\Lambda}_s + \bar{P}_s) \\ &= A_s^r \frac{\partial}{\partial \dot{q}_l}(Q_r + \Lambda_r + P_r),\end{aligned} \tag{28}$$

引入两个矩阵 T 和 $\bar{T}$, 元素分别为

$$\begin{aligned}T_{rl} &= \frac{\partial^2 L}{\partial \dot{q}_r \partial q_l} - \frac{\partial^2 L}{\partial q_r \partial \dot{q}_l}, \\ \bar{T}_{rl} &= \frac{\partial^2 \bar{L}}{\partial \dot{q}_r \partial q_l} - \frac{\partial^2 \bar{L}}{\partial q_r \partial \dot{q}_l}.\end{aligned} \tag{29}$$

由 (27) 式 —(29) 式, 得到 $\dot{A} = -\bar{T}U + ATU$, 其中 A 和 U 分别为 $n \times n$ 阶矩阵, 元素分别为 $A = (A_s^r)$, $U = (U^{sk})$, 对于正整数 m 有

$$\begin{aligned}\dot{A}A^{m-1} &= \big(-\bar{T}U + ATU\big)A^{m-1} \\ &= -\bar{T}U\big(\bar{W}U\big)^{m-1} \\ &\quad + \bar{W}UTU\big(\bar{W}U\big)^{m-1}.\end{aligned} \tag{30}$$

因为 $\overline{W}$ 和 U 为对称矩阵, 且 T 和 $\bar{T}$ 为反对称矩阵, 则据矩阵迹的性质得 $\mathrm{tr}[\bar{T}U(\bar{W}U)^{m-1}] = 0$, $\mathrm{tr}[\bar{W}UTU(\bar{W}U)^{m-1}] = 0$, 所以 $\mathrm{tr}(\dot{A}A^{m-1}) = 0$, 因此有 $\mathrm{tr}[\frac{\mathrm{d}}{\mathrm{d}}t(A^m)] = 0$, 即 $\frac{\mathrm{d}}{\mathrm{d}}t[\mathrm{tr}(A^m)] = 0$, 故得

$$\mathrm{tr}(A^m) = \mathrm{const}, \tag{31}$$

为守恒量.

于是有如下命题:

命题 对于变质量非完整系统 (3), 如果广义非势力, 广义反推力, 广义约束反力 Q_r, Λ_r, P_r 和 $\bar{Q}_s$, $\bar{\Lambda}_s, \bar{P}_s$ 满足方程 (28), 系统的 Lagrange 对称性可导致守恒量 (31).

由上述命题可得到下列推论:

推论 1 对于变质量完整系统, 如果广义非势力 $\bar{Q}_s$ 和 Q_r, 广义反推力 $\bar{P}_s$ 和 P_r 满足

$$\frac{\partial}{\partial \dot{q}_l}(\bar{Q}_s+\bar{P}_s)=A_s^r\frac{\partial}{\partial \dot{q}_l}(Q_r+P_r), \tag{32}$$

系统的 Lagrange 对称性可导致守恒量 (31).

推论 2 对于变质量的 Lagrange 系统, 广义反推力 $\bar{P}_s$ 和 P_r 满足

$$\frac{\partial}{\partial \dot{q}_l}\bar{P}_s=A_s^r\frac{\partial}{\partial \dot{q}_l}P_r, \tag{33}$$

系统的 Lagrange 对称性可导致守恒量 (31).

4 算 例

设系统的 Lagrange 函数为

$$L=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2),\quad (m(t)=m_0(1-\alpha t),\alpha=\text{const}). \tag{34}$$

系统所受非完整约束为

$$f=\dot{q}_1+bt\dot{q}_2-bq_2+t=0,\quad (b=\text{const}), \tag{35}$$

微粒分离的相对速度为 $\boldsymbol{u}_i=-\boldsymbol{r}_i$, 广义非势力为

$$Q_1=-\alpha m_0\dot{q}_1,\quad Q_2=-\alpha m_0\dot{q}_2. \tag{36}$$

试研究系统的 Lagrange 对称性及其导致的守恒量.

求 (35) 式关于 t 的导数, 得

$$\ddot{q}_1+bt\ddot{q}_2+1=0. \tag{37}$$

根据 (4) 式和 (5) 式, 可给出

$$P_1=P_2=0, \tag{38}$$

$$\Lambda_1=\Lambda\frac{\partial f}{\partial \dot{q}_1},\quad \Lambda_2=\Lambda\frac{\partial f}{\partial \dot{q}_2}. \tag{39}$$

由 (6) 式和 (34) 式, 并注意到 (38) 式和 (39) 式, 有

$$\begin{aligned} L_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_1}L-\frac{\partial}{\partial q_1}L-Q_1-P_1-\Lambda_1 \\ &= -\alpha m_0\dot{q}_1+m(t)\ddot{q}_1-Q_1-P_1-\Lambda_1 \\ &= m(t)\ddot{q}_1-\Lambda\frac{\partial f}{\partial \dot{q}_1}, \end{aligned} \tag{40}$$

$$\begin{aligned} L_2 &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_2}L-\frac{\partial}{\partial q_2}L-Q_2-\Lambda_2-P_2 \\ &= m(t)\ddot{q}_2-\Lambda\frac{\partial f}{\partial \dot{q}_2}. \end{aligned} \tag{41}$$

根据 (40) 式, (41) 式和 (3) 式, 可以得到

$$\ddot{q}_1=\Lambda\frac{\dfrac{\partial f}{\partial \dot{q}_1}}{m(t)},\quad \ddot{q}_2=\Lambda\frac{\dfrac{\partial f}{\partial \dot{q}_2}}{m(t)}. \tag{42}$$

把 (42) 式代入 (37) 式, 有

$$\Lambda=-\frac{m(t)}{1+b^2t^2}. \tag{43}$$

由 (43) 式和 (39) 式可给出

$$\begin{aligned} \Lambda_1 &= -\frac{1}{1+b^2t^2}m(t), \\ \Lambda_2 &= -\frac{bt}{1+b^2t^2}m(t). \end{aligned} \tag{44}$$

根据 (36) 式, (38) 式和 (44) 式, 得

$$\begin{aligned} Q_1+\Lambda_1+P_1 &= -\alpha m_0\dot{q}_1-\frac{1}{1+b^2t^2}m(t), \\ Q_2+\Lambda_2+P_2 &= -\alpha m_0\dot{q}_2-\frac{bt}{1+b^2t^2}m(t). \end{aligned} \tag{45}$$

将 (45) 式代入 (40) 式和 (41) 式, 得

$$\begin{aligned} L_1 &= m(t)(\ddot{q}_1+\frac{1}{1+b^2t^2}), \\ L_2 &= m(t)(\ddot{q}_2+\frac{bt}{1+b^2t^2}). \end{aligned} \tag{46}$$

假设有另一组 Lagrange 函数为

$$\begin{aligned} \bar{L} &= m(t)\bigg\{\frac{1}{6}\Big(\dot{q}_1+\frac{1}{b}\arctan bt\Big)^3 \\ &\quad +\frac{1}{2}\Big[\dot{q}_2+\frac{1}{2b}\ln(1+b^2t^2)\Big]^2\bigg\}, \end{aligned} \tag{47}$$

其广义非势力, 广义约束反力和广义反推力为

$$\begin{aligned} &\bar{Q}_1+\bar{\Lambda}_1+\bar{P}_1 \\ &= -\alpha m_0\Big[\frac{1}{2}\Big(\dot{q}_1+\frac{1}{b}\arctan bt\Big)^2\Big], \\ &\bar{Q}_2+\bar{\Lambda}_2+\bar{P}_2 \\ &= -\alpha m_0\Big[\dot{q}_2+\frac{1}{2b}\ln(1+b^2t^2)\Big]. \end{aligned} \tag{48}$$

则 $\bar{L}_1$ 和 $\bar{L}_2$ 分别为

$$\begin{aligned} \bar{L}_1 &= \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}_1}-\frac{\partial \bar{L}}{\partial q_1}-\bar{Q}_1-\bar{\Lambda}_1-\bar{P}_1 \\ &= -\alpha m_0\Big[\frac{1}{2}(\dot{q}_1+\frac{1}{b}\arctan bt)^2\Big] \\ &\quad +m(t)\Big[\Big(\dot{q}_1+\frac{1}{b}\arctan bt\Big) \\ &\quad \times\Big(\ddot{q}_1+\frac{1}{1+b^2t^2}\Big)\Big]-\bar{Q}_1-\bar{\Lambda}_1-\bar{P}_1, \end{aligned} \tag{49}$$

$$\begin{aligned} \bar{L}_2 &= \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{q}_2}-\frac{\partial \bar{L}}{\partial q_2}-\bar{Q}_2-\bar{\Lambda}_2-\bar{P}_2 \\ &= -\alpha m_0\Big[\dot{q}_2+\frac{1}{2b}\ln(1+b^2t^2)\Big]+m(t) \end{aligned}$$

$$\times\Big(\ddot{q}_2+\frac{bt}{1+b^2t^2}\Big)-\bar{Q}_2-\bar{\varLambda}_2-\bar{P}_2. \quad (50)$$

将 (48) 式代入 (49) 式和 (50) 式, 得

$$\begin{aligned}\bar{L}_1 &= m(t)\Big[\Big(\dot{q}_1+\frac{1}{b}\arctan bt\Big)\Big(\ddot{q}_1+\frac{1}{1+b^2t^2}\Big)\Big]\\ &= \Big(\dot{q}_1+\frac{1}{b}\arctan bt\Big)L_1,\\ \bar{L}_2 &= m(t)(\ddot{q}_2+\frac{bt}{1+b^2t^2})=L_2. \end{aligned} \quad (51)$$

并且可以得到

$$A_1^1=\dot{q}_1+\frac{1}{b}\arctan bt,\quad A_2^2=1. \quad (52)$$

易知 (45) 式和 (48) 式满足条件 (28), 故由命题得守恒量

$$I_L=tr(A)=1+\dot{q}_1+\frac{1}{b}\arctan bt=\text{const}. \quad (53)$$

5 结 论

本文研究了变质量非完整系统的 Lagrange 对称性, 得到了变质量非完整系统 Lagrange 对称性导致的守恒量及其存在条件. 本文结果更具一般意义, 对变质量非完整系统, 常质量非完整系统, 常质量完整系统都适用. 当质点的质量为常数时, 本文结果将回到文献 [11] 的结果, 当系统的约束为完整约束且质点质量为常数时, 本文结果自然地回到常质量完整约束下的结果.

[1] Guo Y X, Shang M, Luo S K 2003 *Applied Mathematics & Mechanics* **24** 62 (in Chinese) [郭永新, 尚玫, 罗绍凯 2003 应用数学和力学 **24** 62]

[2] Mei F X, Xu X J, Zhang Y F 2004 *Acta Mech. Sin.* **20** 668

[3] Mei F X 2004 *Symmetries and Conserved Quantities of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese) [梅凤翔 2004 约束力学系统的对称性与守恒量 (北京: 北京理工大学出版社)]

[4] Fang J H, Chen P S, Zhang J 2005 *Applied Mathematics & Mechanics* **26** 187 (in Chinese) [方建会, 陈培胜, 张军 2005 应用数学和力学 **26** 187]

[5] Jia L Q, Zhang Y Y, Zheng S W 2007 *J. Yunnan Univ.* **29** 589 (in Chinese) [贾利群, 张耀宇, 郑世旺 2007 云南大学学报 (自然科学版) **29** 589]

[6] Cai J L 2008 *Chin. Phys. Lett.* **25** 1523

[7] Fu J L, Wang X J, Xie F P 2008 *Chin. Phys. Lett.* **25** 2413

[8] Currie D G, Saletan E J 1966 *J. Math. Phys.* **7** 967

[9] Hojman S, Harleston H 1981 *J. Math. Phys.* **22** 1414

[10] Zhao Y Y, Mei F X 1999 *Symmetries and invaiants of mechanical systems* (Beijing: Science Press) (in Chinese) [赵跃宇, 梅凤翔 1999 力学系统的对称性与不变量 (北京: 科学出版社)]

[11] Mei F X, Wu H B 2008 *Phys. Lett.* A **372** 2174

[12] Mei F X, Wu H B 2009 *Acta Phys. Sin.* **58** 5916 (in Chinese) [梅凤翔, 吴惠彬 2009 物理学报 **58** 5919]

[13] Wu H B, Mei F X 2009 *Chin. Phys.* B **18** 3145

[14] Zhang Y, Ge W K 2009 *Acta Phys. Sin.* **58** 7447 (in Chinese) [张毅, 葛伟宽 2009 物理学报 **58** 7447]

[15] Wu H B, Mei F X 2010 *Chin. Phys.* B **19** 030303

[16] Xia L L, Cai J L 2010 *Chin. Phys. Lett.* **27** 080201

Symmetry and conserved quantity of Lagrangians for nonholonomic variable mass system*

Zhang Bin Fang Jian-Hui† Zhang Ke-Jun

(*College of Physics Science and Technology, China University of Petroleum (East China), Qingdao 266555, China*)

(Received 18 March 2011; revised manuscript received 25 April 2011)

Abstract

In this paper are study the symmetry of Lagrangians and the conserved quantities for a nonholonomic variable mass system. Firstly, the criterion of the symmetry of Lagrangians for a nonholonomic variable mass system is given. Secondly, the conditions under which there exist a conserved quantity and the form of the conserved quantity, are obtained. Finally, an example is given to illustrate the application of the results.

Keywords: variable mass, nonholonomic system, symmetry of Lagrangians, conserved quantity

PACS: 11.30.−j, 11.10.Ef, 02.40.−k

* Project supported by the Fundamental Research Funds for the Central Universities (Grant No. 09CX04018A), the Natural Science Foundation of Shandong Province, China (Grant No. ZR2011AM012), and the Postgraduate's Innovation Foundation of China University of Petroleum (East China) (Grant No. CXYB11-12).

† E-mail: fangjh@upc.edu.cn