相对运动变质量完整系统的共形不变性与守恒量*

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研究了相对运动变质量完整系统的共形不变性与守恒量, 提出了该系统共形不变性的概念, 推导出了相对运动变质量完整系统的运动微分方程具有共形不变性并且是 Lie 对称性的充要条件, 借助规范函数满足的结构方程导出系统相应的守恒量, 并给出应用算例.

关键词: 变质量, 相对运动, 共形不变性, 守恒量

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1 引 言

寻求对称性原理和守恒量是认识世界和解决问题的有效途径之一, 对称性在动力学系统中也称不变性. 在研究动力学系统的守恒量时, 对称性方法已在分析力学研究中得到广泛应用 [1−5]. 20 世纪末, Galiullin 等在研究 Birkhoff 系统分析动力学时, 提出了 Birkhoff 方程的共形不变性和共形因子的概念, 并讨论了 Lie 对称性与共形不变性之间的关系 [6]. 近年来, 我国许多学者广泛深入地研究了动力学系统的 Noether 对称性, Lie 对称性和形式不变性 [7−14], 文献 [15] 和 [16] 研究了变质量完整动力学系统的共形不变性与守恒量, 文献 [17] 研究了相对运动完整动力学系统的共形不变性与守恒量, 文献 [18] 研究了相对运动变质量完整系统 Appell 方程的 Mei 对称性与 Mei 守恒量. 本文研究相对运动变质量完整系统的共形不变性与守恒量, 推导出相对运动变质量完整系统的运动微分方程具有共形不变性并且是 Lie 对称性的充分必要条件, 借助规范函数满足的结构方程导出系统相应的守恒量, 并给出应用算例.

2 系统运动的微分方程

载体的角速度为 ω, 载体极点 O 的速度为 v_0, 假设变质量完整力学系统由 N 个质点组成, 在瞬时 t, 第 i 个质点的质量为 m_i $(i=1,\cdots,N)$, 在瞬时 $t+\mathrm{d}t$, 由质点并入 (或分离) 的微粒质量为 $\mathrm{d}m_i$. 假设系统的位形由 n 个广义坐标 q_s 来确定, 质点质量是时间 t 和广义坐标 q 的函数

$$m_i=m_i(t,\boldsymbol{q}),\quad (i=1,\cdots,N). \tag{1}$$

系统的微分方程可写为

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_{\mathrm{r}}}{\partial\dot{q}_s}-\frac{\partial T_{\mathrm{r}}}{\partial q_s}\\&=Q_s+P_s-\frac{\partial(V^o+V^\omega)}{\partial q_s}+Q_s^{\dot{\omega}}\\&\quad+\Gamma_s,\quad (s=1,\cdots,n),\end{aligned} \tag{2}$$

方程中 $T_{\mathrm{r}}=T_{\mathrm{r}}(t,q,\dot{q})$ 为系统的相对运动动能, $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 为广义非势力.

$$P_s=(\dot{m}_i\boldsymbol{u}_i+\dot{m}_i\dot{\boldsymbol{r}}_i)\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_s}-\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_i}{\partial q_s} \tag{3}$$

为广义反推力.

$$V^o=M(\boldsymbol{v}_0^*+\boldsymbol{\omega}\times\boldsymbol{v}_0)\cdot\boldsymbol{r}_{\mathrm{c}}' \tag{4}$$

为均匀力场势能, 式中 M 为系统总质量, r_{c}' 为质心

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在动系中的矢径, v_0^* 为 v_0 的相对导数.

$$V^{\omega}=-\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\theta}^{\circ}\cdot\boldsymbol{\omega} \tag{5}$$

为离心势能, 式中 $\boldsymbol{\theta}^{\circ}$ 为系统在 O 点的惯性张量.

$$Q_s^{\dot{\omega}}=-(\dot{\boldsymbol{\omega}}\times m_i\boldsymbol{r}_i')\cdot\frac{\partial\boldsymbol{r}_i'}{\partial q_s} \tag{6}$$

为广义回转惯性力, m_i 为第 i 个质点的质量, $\boldsymbol{r}_i'$ 为它的相对矢径.

$$\Gamma_s=\gamma_{sk}\dot{q}_k,\quad \gamma_{sk}=2\boldsymbol{\omega}\cdot\left(m_i\frac{\partial\boldsymbol{r}_i'}{\partial q_s}\times\frac{\partial\boldsymbol{r}_i'}{\partial q_k}\right), \tag{7}$$

Γ_s 为广义陀螺力.

将广义力 Q_s 分为有势的 Q_s' 和非势的 Q_s'', 有

$$Q_s=Q_s'+Q_s'',\quad Q_s'=-\frac{\partial V}{\partial q_s}. \tag{8}$$

令

$$L_{\mathrm{r}}=T_{\mathrm{r}}-V-V^{o}-V^{\omega}, \tag{9}$$

则方程 (2) 可写成如下形式:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\mathrm{r}}}{\partial\dot{q}_s}-\frac{\partial L_{\mathrm{r}}}{\partial q_s}\\&=Q_s''+P_s+Q_s^{\dot{\omega}}+\Gamma_s,\quad(s=1,\cdots,n).\end{aligned} \tag{10}$$

展开方程 (10), 有

$$\begin{aligned}F_s=&A_{sk}\ddot{q}_k+B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\Gamma_s\\=&0,\quad(s,k=1,\cdots,n),\end{aligned} \tag{11}$$

其中

$$\begin{aligned}A_{sk}&=\frac{\partial^2L_{\mathrm{r}}}{\partial\dot{q}_s\partial\dot{q}_k},\\B_s&=\frac{\partial^2L_{\mathrm{r}}}{\partial\dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2L_{\mathrm{r}}}{\partial\dot{q}_s\partial t}-\frac{\partial L_{\mathrm{r}}}{\partial q_s}.\end{aligned} \tag{12}$$

假设系统 (10) 非奇异, 即

$$\det\left(\frac{\partial^2L_{\mathrm{r}}}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq0, \tag{13}$$

由方程 (11) 可以求得广义加速度

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad(s=1,\cdots,n). \tag{14}$$

3 系统的共形不变性与共形因子

引入时间和广义坐标的无限小变换

$$\begin{aligned}&t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\&q_s^*(t^*)=q_s+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad(s=1,\cdots,n).\end{aligned} \tag{15}$$

方程 (15) 中, ε 为无限小参数, ξ_0,ξ_s 为无限小变换生成元.

引入无限小变换生成元向量及其一次和二次展式

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{16}$$

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}, \tag{17}$$

$$X^{(2)}=X^{(1)}+(\ddot{\xi}_s-2\ddot{q}_s\dot{\xi}_0-\dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial\ddot{q}_s}. \tag{18}$$

定义1 二阶微分方程 F_s, 在无限小生成元 $\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 的变换下, 若非退化矩阵 B_s^l 满足

$$X^{(2)}(F_s)=B_s^lF_l,\quad(s,l=1,\cdots,n), \tag{19}$$

则称二阶微分方程为共形不变的, (19) 式是方程 (11) 共形不变的确定方程, B_s^l 称为共形因子.

在相对运动变质量完整系统 (11) 中, 如无限小生成元 ξ_0,ξ_s 满足如下确定方程

$$X^{(2)}(F_s)|_{F_s=0}=0, \tag{20}$$

称这种对称性为系统的 Lie 对称性.

下面求共形因子 B_s^l. 因为

$$\begin{aligned}X^{(2)}(F_s)=&X^{(2)}(A_{sk}\ddot{q}_k+B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\Gamma_s)\\=&A_{sk}(\ddot{\xi}_k-2\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0)+X^{(0)}(A_{sk})\ddot{q}_k\\&+X^{(0)}(B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\Gamma_s)\\&+(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0)\\&\times\frac{\partial(B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\Gamma_s)}{\partial\dot{q}_k},\end{aligned} \tag{21}$$

又

$$\begin{aligned}\dot{\xi}_k=&\frac{\partial\xi_k}{\partial t}+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}\dot{q}_{\mathrm{r}}+\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}\ddot{q}_{\mathrm{r}},\\&(k=0,1,\cdots,n),\end{aligned} \tag{22}$$

$$\begin{aligned}\ddot{\xi}_k=&\frac{\partial^2\xi_k}{\partial t^2}+2\frac{\partial^2\xi_k}{\partial q_{\mathrm{r}}\partial t}\dot{q}_{\mathrm{r}}+2\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial t}\ddot{q}_{\mathrm{r}}\\&+\frac{\partial^2\xi_k}{\partial q_{\mathrm{r}}\partial q_j}\dot{q}_{\mathrm{r}}\dot{q}_j+2\frac{\partial^2\xi_k}{\partial q_{\mathrm{r}}\partial\dot{q}_j}\dot{q}_{\mathrm{r}}\ddot{q}_j\\&+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}\ddot{q}_{\mathrm{r}}+\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}\ddot{q}_{\mathrm{r}}\ddot{q}_j\\&+\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}\left(\frac{\partial\alpha_{\mathrm{r}}}{\partial t}+\frac{\partial\alpha_{\mathrm{r}}}{\partial q_j}\dot{q}_j+\frac{\partial\alpha_{\mathrm{r}}}{\partial\dot{q}_j}\ddot{q}_j\right),\\&(k=0,1,\cdots,n;r,j=1,\cdots,n),\end{aligned} \tag{23}$$

将 (22), (23) 式代入 (21) 式, 并考虑到 $F_s=0$ 时, $\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 得到

$$\begin{aligned}&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}\\=&A_{sk}\left\{\left[2\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\right]\right.\end{aligned}$$

$$
\begin{aligned}
&\times(\ddot{q}_{\mathrm{r}}-\alpha_{\mathrm{r}})+\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\ddot{q}_{\mathrm{r}}\ddot{q}_j-\alpha_{\mathrm{r}}\alpha_j)\\
&-2(\ddot{q}_k-\alpha_k)\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\dot{q}_{\mathrm{r}}\right)-2\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}(\ddot{q}_k\ddot{q}_{\mathrm{r}}-\alpha_k\alpha_{\mathrm{r}})\\
&-\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\right)\\
&\times(\ddot{q}_{\mathrm{r}}-\alpha_{\mathrm{r}})-\dot{q}_k\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\ddot{q}_{\mathrm{r}}\ddot{q}_j-\alpha_{\mathrm{r}}\alpha_j)\Bigg\}\\
&+X^{(0)}(A_{sk})(\ddot{q}_k-\alpha_k)+\left(\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}-\dot{q}_k\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}\right)\\
&\times(\ddot{q}_{\mathrm{r}}-\alpha_{\mathrm{r}})\frac{\partial(B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\varGamma_s)}{\partial\dot{q}_k},\\
&(s,k,r,j=1,\cdots,n).
\end{aligned}
\tag{24}
$$

由于

$$
\begin{aligned}
\ddot{q}_k-\alpha_k=&\ddot{q}_k+A^{kl}(B_l-Q_l''-P_l-Q_l^{\dot{\omega}}-\varGamma_l)\\
=&A^{kl}(A_{lm}\ddot{q}_m+B_l-Q_l''-P_l-Q_l^{\dot{\omega}}-\varGamma)\\
=&A^{kl}F_{l,}\\
\ddot{q}_k\ddot{q}_j=&(A^{kl}F_l+\alpha_k)(A^{jm}F_m+\alpha_j)\\
=&A^{kl}F_lA^{jm}F_m+\alpha_kA^{jm}F_m+\alpha_jA^{kl}F_l\\
&+\alpha_k\alpha_j,\quad(s,k,r,j,m,l=1,\cdots,n).
\end{aligned}
\tag{25}
$$

所以

$$
\begin{aligned}
&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}\\
=&\Bigg\{A_{sk}\Bigg[2\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\\
&-\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\right)\Bigg]A^{rl}\\
&-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\dot{q}_{\mathrm{r}}\right)+\left(\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}-\dot{q}_k\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}\right)\\
&\times A^{rl}\frac{\partial[B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\varGamma_s]}{\partial\dot{q}_k}\\
&+X^{(0)}(A_{sk})A^{kl}+A_{sk}\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl})\\
&-2A_{sk}\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}(\alpha_kA^{rl}+\alpha_{\mathrm{r}}A^{kl})\\
&-A_{sk}\dot{q}_k\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl})\Bigg\}F_l.
\end{aligned}
\tag{26}
$$

令

$$
\begin{aligned}
M_s^l=&A_{sk}\Bigg[2\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\\
&-\dot{q}_k\Bigg(2\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\\
&+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\Bigg)\Bigg]A^{rl}-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\dot{q}_{\mathrm{r}}\right)\\
&+\left(\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}-\dot{q}_k\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}\right)\\
&\times A^{rl}\frac{\partial(B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\varGamma_s)}{\partial\dot{q}_k}\\
&+X^{(0)}(A_{sk})A^{kl}+A_{sk}\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl})\\
&-2A_{sk}\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}(\alpha_kA^{rl}+\alpha_{\mathrm{r}}A^{kl})-A_{sk}\dot{q}_k\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}\\
&\times(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl}),\\
&(s,k,r,j,l=1,\cdots,n),
\end{aligned}
\tag{27}
$$

得

$$
\begin{aligned}
&X^{(2)}(F_s)-X^{(2)}(F_s)|_{F_s=0}\\
=&M_s^lF_l,\quad(s,l=1,\cdots,n).
\end{aligned}
\tag{28}
$$

如系统具有共形不变性且是 Lie 对称性, 由 (19), (20) 和 (28) 式可得

$$
B_s^lF_l-M_s^lF_l=X^{(2)}(F_s)|_{F_s=0}=0,\tag{29}
$$

即

$$
B_s^l=M_s^l,\quad(s,l=1,\cdots,n).\tag{30}
$$

命题1 对于相对运动变质量完整系统 (11), 既是共形不变性又是 Lie 对称性的充分必要条件是生成元 ξ_0,ξ_s 满足

$$
\begin{aligned}
B_s^l=&M_s^l\\
=&A_{sk}\Bigg[2\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_k}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j+\frac{\partial\xi_k}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\\
&-\dot{q}_k\Bigg(2\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial t}+2\frac{\partial^2\xi_0}{\partial q_j\partial\dot{q}_{\mathrm{r}}}\dot{q}_j\\
&+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_{\mathrm{r}}}\Bigg)\Bigg]A^{rl}-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\dot{q}_{\mathrm{r}}\right)\\
&+\left(\frac{\partial\xi_k}{\partial\dot{q}_{\mathrm{r}}}-\dot{q}_k\frac{\partial\xi_0}{\partial q_{\mathrm{r}}}\right)A^{rl}\frac{\partial[B_s-Q_s''-P_s-Q_s^{\dot{\omega}}-\varGamma_s]}{\partial\dot{q}_k}\\
&+X^{(0)}(A_{sk})A^{kl}+A_{sk}\frac{\partial^2\xi_k}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl})\\
&-2A_{sk}\frac{\partial\xi_0}{\partial\dot{q}_{\mathrm{r}}}(\alpha_kA^{rl}+\alpha_{\mathrm{r}}A^{kl})\\
&-A_{sk}\dot{q}_k\frac{\partial^2\xi_0}{\partial\dot{q}_{\mathrm{r}}\partial\dot{q}_j}(\alpha_{\mathrm{r}}A^{jl}+\alpha_jA^{rl}).
\end{aligned}
\tag{31}
$$

4 共形不变性与守恒量

根据相对运动变质量完整系统的共形不变性, 由 Noether 对称性可求出 Noether 守恒量, 有如下结论:

命题2 对于相对运动变质量完整系统 (11), 若共形不变性的无限小生成元 $\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ 和规范函数 G 满足 Noether 等式

$$\begin{aligned}&L_\mathrm{r}\dot{\xi}_0+X^{(1)}(L_\mathrm{r})+(Q_s''+P_s+Q_s^{\dot{\omega}}+\varGamma_s)\\&\times(\xi_s-\dot{q}_s\xi_0)+\dot{G}=0.\end{aligned}\tag{32}$$

则共形不变性导致 Noether 守恒量

$$I=L_\mathrm{r}\xi_0+\frac{\partial L_\mathrm{r}}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G=\mathrm{const}.\tag{33}$$

证明

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\dot{L}_\mathrm{r}\xi_0+L_\mathrm{r}\dot{\xi}_0+\frac{\partial L_\mathrm{r}}{\partial\dot{q}_s}(\dot{\xi}_s-\ddot{q}_s\xi_0-\dot{q}_s\dot{\xi}_0)\\&+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\mathrm{r}}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)-L_\mathrm{r}\dot{\xi}_0\\&-X^{(1)}(L_\mathrm{r})-(Q_s''+P_s+Q_s^{\dot{\omega}}+\varGamma_s)(\xi_s-\dot{q}_s\xi_0)\\=&(\xi_s-\dot{q}_s\xi_0)\left[\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_\mathrm{r}}{\partial\dot{q}_s}-\frac{\partial L_\mathrm{r}}{\partial q_s}\right.\\&\left.-(Q_s''+P_s+Q_s^{\dot{\omega}}+\varGamma_s)\right]=0.\end{aligned}$$

5 算 例

设一相对运动变质量力学系统为

$$\begin{aligned}L_\mathrm{r}=&\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2)+\frac{1}{2}m\omega^2(q_1^2+q_2^2)\\&-\frac{1}{2}km(q_1^2+q_2^2),\\P_1=&0,\ P_2=\dot{m}\dot{q}_2,\\V^0=&\varGamma_s=Q_s^{\dot{\omega}}=Q_s''=0,\quad(s=1,2),\end{aligned}\tag{34}$$

其中 $m=m_0e^{-t}$, k,ω,m_0 为常数, 试研究系统的共形不变性与守恒量.

由方程 (10) 可得

$$m\ddot{q}_1-m\dot{q}_1+(k-\omega^2)mq_1=0,\tag{35}$$

$$m\ddot{q}_2+(k-\omega^2)mq_2=0.\tag{36}$$

系统的微分方程为

$$\begin{aligned}&\ddot{q}_1=\dot{q}_1-(k-\omega^2)q_1=\alpha_1,\\&\ddot{q}_2=-(k-\omega^2)q_2=\alpha_2.\end{aligned}\tag{37}$$

或

$$\begin{aligned}&F_1=\ddot{q}_1-\dot{q}_1+(k-\omega^2)q_1=0,\\&F_2=\ddot{q}_2+(k-\omega^2)q_2=0.\end{aligned}\tag{38}$$

取无限小变换生成元为

$$\xi_0=0,\quad\xi_1=q_1,\quad\xi_2=q_2.\tag{39}$$

则

$$\begin{aligned}X^{(2)}(\boldsymbol{F})=&\left(\xi_s\frac{\partial}{\partial q_s}+\dot{\xi}_s\frac{\partial}{\partial\dot{q}_s}+\ddot{\xi}_s\frac{\partial}{\partial\ddot{q}_s}\right)\\&\times\begin{pmatrix}\ddot{q}_1-\dot{q}_1+(k-\omega^2)q_1\\\ddot{q}_2+(k-\omega^2)q_2\end{pmatrix}\\=&\begin{pmatrix}\ddot{q}_1-\dot{q}_1+(k-\omega^2)q_1\\\ddot{q}_2+(k-\omega^2)q_2\end{pmatrix}\\=&\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}.\end{aligned}$$

因此, 共形因子为

$$B_s^l=\begin{pmatrix}1&0\\0&1\end{pmatrix},\tag{40}$$

也可从 (31) 式求出共形因子

$$B_s^l=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

显然, 该结果与 (40) 式的一样, 共形不变性的确定方程可表示为

$$X^{(2)}(\boldsymbol{F})=\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}.\tag{41}$$

这时, 相对运动变质量完整系统 (11) 不仅是共形不变性的, 同时又是 Lie 对称性的.

将 (34) 和 (39) 式代入 (32) 式可得

$$G=\dot{q}_2^2+(k-\omega^2)q_2^2-mq_1\dot{q}_1-mq_2\dot{q}_2,\tag{42}$$

将 (42) 式代入 (33) 式, 得到 Noether 守恒量

$$I=\dot{q}_2^2+(k-\omega^2)q_2^2=\mathrm{const}.\tag{43}$$

显然, (43) 式可改写为

$$\frac{1}{2}m_0\dot{q}_2^2+\frac{1}{2}m_0kq_2^2-\frac{1}{2}m_0q_2^2\omega^2=\mathrm{const}.\tag{44}$$

(44) 式左端第一项不是动能, 但和动能有关; (44) 式左端第二项不是弹性势能, 但和弹性势能有关; (44) 式左端第三项不是离心力势能, 但和离心力势能有关. 因此, (44) 式或 (43) 式的守恒量可认为是关于广义坐标 q_2 的广义能量守恒.

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Conformal invariance and conserved quantity for a variable mass holonomic system in relative motion*

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Abstract

Conformal invariance and conserved quantity for a variable mass holonomic system in relative motion have been studied. The definition and the determining equations of conformal invariance for a variable mass holonomic system in relative motion are given. The necessary and sufficient conditions that the system's conformal invariance be of Lie symmetry are deduced. With the aid of a structure equation which the gauge function should satisfy, the system's corresponding conserved quantity is obtained. Finally, an illustrative example is given to verify the results.

Keywords: variable mass, relative motion, conformal invariance, conserved quantity

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