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含时滞的非保守系统动力学的 Noether 对称性*

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提出并研究含时滞的非保守系统动力学的 Noether 对称性与守恒量. 首先, 建立含时滞的非保守系统的 Hamilton 原理, 得到含时滞的 Lagrange 方程; 其次, 基于含时滞的 Hamilton 作用量在依赖于广义速度的无限小群变换下的不变性, 定义系统的 Noether 对称变换和准对称变换, 建立 Noether 对称性的判据; 最后, 研究对称性与守恒量之间的关系, 建立含时滞的非保守系统的 Noether 理论. 文末举例说明结果的应用.

关键词: 时滞系统, 非保守力学, Noether 对称性, 守恒量

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1 引 言

由于时滞现象普遍存在于自然界和工程实际中, 即使一个很简单的问题, 一旦考虑时滞的影响, 其动力学行为将可能变得非常复杂 [1−3]. 而力学系统的对称性对其动力学行为及其基本性质都具有深刻的影响, 从基本理论到具体应用都显示出对称性的极端重要性 [4−22]. 因此, 研究含时滞的力学系统的动力学对称性对于深入理解力学系统的复杂动力学行为及其内在的物理本质具有重要意义. 考虑时滞的变分问题研究可以追溯到 El′sgol′c 的工作 [23]; 1968 年, Hughes[24] 研究了含时滞的变分和最优控制问题, 建立了含时滞的 Euler-Lagrange 方程; Palm 和 Schmitendorf[25] 研究了含时滞参数的变分问题的共轭点条件, Rosenblueth[26] 以及 Chan 和 Yung[27] 给出了含时滞的变分问题的充分条件, Lee 和 Yung[28] 给出了含时滞的最优控制问题的充分条件. 尽管含时滞的变分和最优控制问题的研究已经取得了许多重要成果, 但是研究状态变量或控制变量具有时滞的变分和控制系统的对称性与守恒量问题还是一个开放的课题 [29]. 2012 年, Frederico 和 Torres[29] 首次初步研究了含时滞的变分和最优控制问题的 Noether 对称性, 给出了含时滞 Lagrange 系统的对称变换导致的守恒量. 但是, 其研究尚限于 Lagrange 系统, 且仅考虑无限小变换为不依赖于广义速度的对称变换.

本文将进一步研究含时滞的非保守力学系统的 Noether 对称性与守恒量. 基于完整非保守系统的 Hamilton 原理, 建立了受非势广义力作用的含时滞的 Lagrange 方程; 在依赖于广义速度的无限小群变换下, 给出了含时滞的 Hamilton 作用量变分的两个公式, 建立了含时滞的 Noether 准对称变换的定义和判据; 研究了含时滞的 Noether 对称性与守恒量之间的内在关系, 建立了含时滞的非保守力学系统的 Noether 定理.

2 系统的运动微分方程

假设力学系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 来确定, 完整非保守系统的 Hamilton 原理为 [30]

$$\int_{t_1}^{t_2}(\delta L+Q_s''\delta q_s)\,\mathrm{d}t=0, \tag{1}$$

其中 L 为 Lagrange 函数, Q_s'' 为广义非势力. 考虑系

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统具有时滞, 即 Lagrange 函数为

$$\begin{aligned}L &= L(t, q_s(t), \dot{q}_s(t), q_s(t-\tau), \dot{q}_s(t-\tau)) \\ &\triangleq L(t, q_s, \dot{q}_s, q_{s\tau}, \dot{q}_{s\tau}),\end{aligned} \tag{2}$$

广义非势力为

$$Q_s'' = Q_s''(t, q_k(t), \dot{q}_k(t), q_k(t-\tau), \dot{q}_k(t-\tau)), \tag{3}$$

且满足边界条件

$$q_s(t) = \delta_s(t), \text{当} t \in [t_1 - \tau, t_1], \tag{4}$$

$$q_s(t) = q_{s2}, \text{当} t = t_2, \tag{5}$$

其中时滞量 $\tau < t_2 - t_1$ 是给定的正实数, $\delta_s(t)$ 是在区间 $[t_1 - \tau, t_1]$ 上的已知分段光滑函数. 则原理 (1) 可写为

$$\begin{aligned}\int_{t_1}^{t_2} \bigg(&\frac{\partial L}{\partial q_s}\delta q_s + \frac{\partial L}{\partial \dot{q}_s}\delta \dot{q}_s + \frac{\partial L}{\partial q_{s\tau}}\delta q_{s\tau} \\ &+ \frac{\partial L}{\partial \dot{q}_{s\tau}}\delta \dot{q}_{s\tau} + Q_s''\delta q_s \bigg) \mathrm{d}t = 0.\end{aligned} \tag{6}$$

对 (6) 式的第三项, 第四项进行变量替换 $t = \theta + \tau$, 并考虑条件 (4), 有

$$\begin{aligned}&\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_{s\tau}}(t)\delta q_{s\tau} + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\delta \dot{q}_{s\tau} \right) \mathrm{d}t \\ =&\int_{t_1-\tau}^{t_2-\tau} \left(\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\delta \dot{q}_s \right) \mathrm{d}\theta \\ =&\int_{t_1}^{t_2-\tau} \left(\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s + \frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\delta \dot{q}_s \right) \mathrm{d}\theta.\end{aligned} \tag{7}$$

将 (7) 式代入 (6) 式, 得到

$$\begin{aligned}&\int_{t_1}^{t_2-\tau} \left[\left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau) + Q_s''(t) \right) \delta q_s \right. \\ &\left. + \left(\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \right) \delta \dot{q}_s \right] \mathrm{d}t \\ &+ \int_{t_2-\tau}^{t_2} \left[\left(\frac{\partial L}{\partial q_s}(t) + Q_s''(t) \right) \delta q_s + \frac{\partial L}{\partial \dot{q}_s}(t)\delta \dot{q}_s \right] \mathrm{d}t \\ =&0.\end{aligned} \tag{8}$$

进行分部积分计算, 并利用边界条件 (4) 和 (5), 我们有如下关系:

$$\begin{aligned}&\int_{t_1}^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(t) + \frac{\partial L}{\partial q_{s\tau}}(t+\tau) + Q_s''(t) \right) \delta q_s \mathrm{d}t \\ =&-\left[\delta q_s \int_t^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(\theta) + \frac{\partial L}{\partial q_{s\tau}}(\theta+\tau) \right.\right. \\ &\left.\left. + Q_s''(\theta) \right) \mathrm{d}\theta \right] \Bigg|_{t_1}^{t_2-\tau} \\ &+ \int_{t_1}^{t_2-\tau} \delta \dot{q}_s \left[\int_t^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(\theta) + \frac{\partial L}{\partial q_{s\tau}}(\theta+\tau) \right.\right. \\ &\left.\left. + Q_s''(\theta) \right) \mathrm{d}\theta \right] \mathrm{d}t \\ =&\int_{t_1}^{t_2-\tau} \delta \dot{q}_s \left[\int_t^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(\theta) + \frac{\partial L}{\partial q_{s\tau}}(\theta+\tau) \right.\right. \\ &\left.\left. + Q_s''(\theta) \right) \mathrm{d}\theta \right] \mathrm{d}t,\end{aligned} \tag{9}$$

以及

$$\begin{aligned}&\int_{t_2-\tau}^{t_2} \left(\frac{\partial L}{\partial q_s}(t) + Q_s''(t) \right) \delta q_s \mathrm{d}t \\ =&\left[\delta q_s \int_{t_2-\tau}^{t} \left(\frac{\partial L}{\partial q_s}(\theta) + Q_s''(\theta) \right) \mathrm{d}\theta \right] \Bigg|_{t_2-\tau}^{t_2} \\ &- \int_{t_2-\tau}^{t_2} \delta \dot{q}_s \left[\int_{t_2-\tau}^{t} \left(\frac{\partial L}{\partial q_s}(\theta) + Q_s''(\theta) \right) \mathrm{d}\theta \right] \mathrm{d}t \\ =&- \int_{t_2-\tau}^{t_2} \delta \dot{q}_s \left[\int_{t_2-\tau}^{t} \left(\frac{\partial L}{\partial q_s}(\theta) + Q_s''(\theta) \right) \mathrm{d}\theta \right] \mathrm{d}t.\end{aligned} \tag{10}$$

将 (9) 式和 (10) 式代入 (8) 式, 得到

$$\begin{aligned}&\int_{t_1}^{t_2-\tau} \left[\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \right. \\ &+ \int_t^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(\theta) + \frac{\partial L}{\partial q_{s\tau}}(\theta+\tau) \right. \\ &\left.\left. + Q_s''(\theta) \right) \mathrm{d}\theta \right] \delta \dot{q}_s \mathrm{d}t \\ &+ \int_{t_2-\tau}^{t_2} \left[\frac{\partial L}{\partial \dot{q}_s}(t) - \int_{t_2-\tau}^{t} \left(\frac{\partial L}{\partial q_s}(\theta) \right.\right. \\ &\left.\left. + Q_s''(\theta) \right) \mathrm{d}\theta \right] \delta \dot{q}_s \mathrm{d}t = 0.\end{aligned} \tag{11}$$

由积分区间的任意性, $\delta \dot{q}_s$ 的独立性, 从 (11) 式可得

$$\begin{aligned}&\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) \\ &+ \int_t^{t_2-\tau} \left(\frac{\partial L}{\partial q_s}(\theta) + \frac{\partial L}{\partial q_{s\tau}}(\theta+\tau) + Q_s''(\theta) \right) \mathrm{d}\theta \\ =&0, \quad (t_1 \leqslant t \leqslant t_2 - \tau; s = 1, 2, \cdots, n), \\ &\frac{\partial L}{\partial \dot{q}_s}(t) - \int_{t_2-\tau}^{t} \left(\frac{\partial L}{\partial q_s}(\theta) + Q_s''(\theta) \right) \mathrm{d}\theta = 0, \\ &(t_2 - \tau < t \leqslant t_2; s = 1, 2, \cdots, n).\end{aligned} \tag{12}$$

将 (12) 式对时间 t 求导, 有

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau) - \frac{\partial L}{\partial q_s}(t) - \frac{\partial L}{\partial q_{s\tau}}(t+\tau) \\ =&Q_s''(t), \quad (t_1 \leqslant t \leqslant t_2 - \tau; s = 1, 2, \cdots, n), \\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t) - \frac{\partial L}{\partial q_s}(t) = Q_s''(t), \\ &(t_2 - \tau < t \leqslant t_2; s = 1, 2, \cdots, n).\end{aligned} \tag{13}$$

方程 (13) 可称为含时滞的非保守力学系统的 Lagrange 方程. 如果广义非势力 $Q_s'' = 0$, 则方程 (13)

成为

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)-\frac{\partial L}{\partial q_s}(t)-\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\\&=0,\quad (t_1\leqslant t\leqslant t_2-\tau; s=1,2,\cdots,n),\\&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)-\frac{\partial L}{\partial q_s}(t)=0,\\&\quad (t_2-\tau<t\leqslant t_2; s=1,2,\cdots,n).\end{aligned}\tag{14}$$

方程 (14) 为含时滞的 Lagrange 系统的运动方程 [24]. 如果时滞量 $\tau=0$, 则方程 (13) 给出标准的 Lagrange 方程.

3 含时滞的 Hamilton 作用量变分

含时滞的 Hamilton 作用量为

$$\begin{aligned}S(\gamma)&=\int_{t_1}^{t_2}L(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))\,\mathrm{d}t\\&\triangleq\int_{t_1}^{t_2}L(t,q_s,\dot{q}_s,q_{s\tau},\dot{q}_{s\tau})\,\mathrm{d}t.\end{aligned}\tag{15}$$

引入 r 参数有限变换群的无限小变换

$$\begin{aligned}&\bar{t}=t+\Delta t,\\&\bar{q}_s(\bar{t})=q_s(t)+\Delta q_s,\quad (s=1,2,\cdots,n),\end{aligned}\tag{16}$$

其展开式为

$$\begin{aligned}&\bar{t}=t+\varepsilon_\sigma\xi_0^\sigma(t,q_k,\dot{q}_k),\\&\bar{q}_s(\bar{t})=q_s(t)+\varepsilon_\sigma\xi_s^\sigma(t,q_k,\dot{q}_k),\ (s=1,2,\cdots,n),\end{aligned}\tag{17}$$

其中 $\varepsilon_\sigma(\sigma=1,2,\cdots,r)$ 为无限小参数, ξ_0^σ, ξ_s^σ 为无限小变换的生成函数或生成元. 在变换 (16) 作用下, Hamilton 作用量 (15) 变为

$$\begin{aligned}S(\bar{\gamma})=\int_{\bar{t}_1}^{\bar{t}_2}&L(\bar{t},\bar{q}_s(\bar{t}),\dot{\bar{q}}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),\\&\dot{\bar{q}}_s(\bar{t}-\tau))\,\mathrm{d}\bar{t},\end{aligned}\tag{18}$$

其中 $\bar{\gamma}$ 为 γ 的邻近曲线, 则变换前后的差 $S(\bar{\gamma})-S(\gamma)$ 相对 ε 的主线性部分为

$$\begin{aligned}\Delta S=\int_{t_1}^{t_2}&\left[\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s\right.\\&\left.+\frac{\partial L}{\partial q_{s\tau}}\Delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\Delta\dot{q}_{s\tau}+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right]\mathrm{d}t.\end{aligned}\tag{19}$$

注意到关系 [16]

$$\begin{aligned}&\Delta\dot{q}_s=\frac{\mathrm{d}}{\mathrm{d}t}\Delta q_s-\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\Delta t,\\&\delta q_s=\Delta q_s-\dot{q}_s\Delta t,\end{aligned}\tag{20}$$

(19) 式可写为

$$\begin{aligned}\Delta S=\int_{t_1}^{t_2}&\left[\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t)+\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_s\right.\\&\left.+\frac{\partial L}{\partial q_{s\tau}}\delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_{s\tau}\right]\mathrm{d}t.\end{aligned}\tag{21}$$

对 (21) 式的第四项, 第五项进行变量替换 $t=\theta+\tau$, 并考虑到条件 (4), 有

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_{s\tau}}\delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_{s\tau}\right)\mathrm{d}t\\&=\int_{t_1}^{t_2-\tau}\left[\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\delta q_s(\theta)\right.\\&\quad\left.+\frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\frac{\mathrm{d}}{\mathrm{d}\theta}\delta q_s(\theta)\right]\mathrm{d}\theta.\end{aligned}\tag{22}$$

将 (22) 式代入 (21) 式, 得到

$$\begin{aligned}\Delta S=&\int_{t_1}^{t_2-\tau}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[L\Delta t+\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right.\right.\right.\\&\left.\left.+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\delta q_s\right]\\&+\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right.\\&\left.\left.-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\right]\delta q_s\right\}\mathrm{d}t\\&+\int_{t_2-\tau}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(L\Delta t+\frac{\partial L}{\partial \dot{q}_s}(t)\delta q_s\right)\right.\\&\left.+\left(\frac{\partial L}{\partial q_s}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)\right)\delta q_s\right]\mathrm{d}t\\=&\int_{t_1}^{t_2-\tau}\varepsilon_\sigma\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right.\right.\right.\\&\left.\left.+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)\right]\\&+\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}(t)\right.\right.\\&\left.\left.\left.+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\right](\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)\right\}\mathrm{d}t\\&+\int_{t_2-\tau}^{t_2}\varepsilon_\sigma\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma+\frac{\partial L}{\partial \dot{q}_s}(t)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)\right]\right.\\&\left.+\left(\frac{\partial L}{\partial q_s}(t)-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(t)\right)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)\right\}\mathrm{d}t.\end{aligned}\tag{23}$$

对 (19) 式的第四项, 第五项进行变量替换 $t=\theta+\tau$, 并考虑条件 (4), 有

$$\begin{aligned}&\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_{s\tau}}(t)\Delta q_{s\tau}+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t)\Delta\dot{q}_{s\tau}\right)\mathrm{d}t\\&=\int_{t_1}^{t_2-\tau}\left(\frac{\partial L}{\partial q_{s\tau}}(\theta+\tau)\Delta q_s+\frac{\partial L}{\partial \dot{q}_{s\tau}}(\theta+\tau)\Delta\dot{q}_s\right)\mathrm{d}\theta.\end{aligned}\tag{24}$$

将 (24) 式代入 (19) 式, 得到

$$\begin{aligned}\Delta S=&\int_{t_1}^{t_2-\tau}\left\{\frac{\partial L}{\partial t}(t)\Delta t+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\Delta q_s\right.\\&\left.+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right\}\mathrm{d}t\\&+\int_{t_2-\tau}^{t_2}\left\{\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s\right.\\&\left.+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\right\}\mathrm{d}t.\end{aligned}\tag{25}$$

(25) 和 (23) 式是含时滞的 Hamilton 作用量变分的两个基本公式.

4 含时滞的力学系统的 Noether 对称性

我们首先来建立含时滞的力学系统的 Noether 对称变换的定义和判据.

定义1 如果含时滞的 Hamilton 作用量 (15) 在无限小变换 (16) 作用下, 满足条件

$$\Delta S=0,\tag{26}$$

则称无限小变换为含时滞的力学系统的 Noether 对称变换.

由定义 1 和 (25) 式, 容易得到如下判据:

判据1 如果无限小变换 (16), 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 满足条件

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\Delta t+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\Delta q_s\\&+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=0.\end{aligned}\tag{27}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 满足条件

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)\\&=0.\end{aligned}\tag{28}$$

则变换 (16) 是所论含时滞的力学系统的 Noether 对称变换.

由 (17) 式, 利用 (20) 式, 并考虑参数 ε_σ 的独立性, (27) 和 (28) 式也可表为: 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 有

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s^\sigma\\&+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)+L\dot{\xi}_0^\sigma\\&=0,\quad(\sigma=1,2,\cdots,r),\end{aligned}\tag{29}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\frac{\partial L}{\partial q_s}(t)\xi_s^\sigma+\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)\\&+L\dot{\xi}_0^\sigma=0,\quad(\sigma=1,2,\cdots,r),\end{aligned}\tag{30}$$

当 $r=1$ 时, (29) 和 (30) 式称为含时滞的力学系统的 Noether 等式.

其次, 我们来建立含时滞的力学系统的 Noether 准对称变换的定义和判据.

设 L_1 是某个另外的含时滞的 Lagrange 函数, 如果变换 (16) 精确到一阶小量满足如下关系:

$$\begin{aligned}&\int_{t_1}^{t_2}L(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))\mathrm{d}t\\&=\int_{\bar{t}_1}^{\bar{t}_2}L_1(\bar{t},\bar{q}_s(\bar{t}),\dot{\bar{q}}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),\dot{\bar{q}}_s(\bar{t}-\tau))\mathrm{d}\bar{t}.\end{aligned}\tag{31}$$

那么称这种不变性为含时滞的 Hamilton 作用量 (15) 在无限小变换 (16) 作用下的准不变性. 由此确定的 L_1 与 L 具有相同的运动微分方程, 因而变换 (16) 可称为含时滞的力学系统的 Noether 准对称变换. 于是有

定义2 如果含时滞的 Hamilton 作用量 (15) 在无限小变换 (16) 作用下, 满足条件

$$\Delta S=-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)\mathrm{d}t,\tag{32}$$

其中 $G=G(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))$, 则称无限小变换为含时滞的力学系统的 Noether 准对称变换.

由定义 2 和 (25) 式, 容易得到如下判据:

判据2 如果无限小变换 (16), 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 满足条件

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\Delta t+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\Delta q_s\\&+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\Delta\dot{q}_s\\&+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G).\end{aligned}\tag{33}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 满足条件

$$\begin{aligned}&\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s\\&+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G).\end{aligned}\tag{34}$$

则变换 (16) 是所论含时滞的力学系统的 Noether 准对称变换.

(33) 和 (34) 式也可表为: 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 有

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s^\sigma$$

$$+\left(\frac{\partial L}{\partial \dot{q}_s}t(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)+L\dot{\xi}_0^\sigma$$
$$=-\dot{G}^\sigma,\quad (\sigma=1,2,\cdots,r). \tag{35}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\frac{\partial L}{\partial q_s}(t)\xi_s^\sigma+\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)$$
$$+L\dot{\xi}_0^\sigma=-\dot{G}^\sigma,\quad (\sigma=1,2,\cdots,r), \tag{36}$$

其中 $\Delta G=\varepsilon_\sigma G^\sigma$. 当 $r=1$ 时, (35) 和 (36) 式也称为含时滞的力学系统的 Noether 等式.

最后, 我们来建立含时滞的力学系统的 Noether 广义准对称变换的定义和判据.

假设所论含时滞的力学系统受到广义非势力 (3) 的作用, 如果精确到一阶小量满足如下条件:

$$\int_{t_1}^{t_2}L(t,q_s(t),\dot{q}_s(t),q_s(t-\tau),\dot{q}_s(t-\tau))\,\mathrm{d}t$$
$$=\int_{\bar{t}_1}^{\bar{t}_2}L_1(\bar{t},\bar{q}_s(\bar{t}),\dot{\bar{q}}_s(\bar{t}),\bar{q}_s(\bar{t}-\tau),\dot{\bar{q}}_s(\bar{t}-\tau))\,\mathrm{d}\bar{t}$$
$$+\int_{t_1}^{t_2}Q_s''\delta q_s\,\mathrm{d}t, \tag{37}$$

则相应不变性称为含时滞的 Hamilton 作用量 (15) 在无限小变换 (16) 下的广义准不变性, 而变换 (16) 称为含时滞的力学系统的 Noether 广义准对称变换. 于是有

定义3 如果含时滞的 Hamilton 作用量 (15) 在无限小变换 (16) 作用下, 满足条件

$$\Delta S=-\int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G)+Q_s''\delta q_s\right)\mathrm{d}t. \tag{38}$$

则称无限小变换为含时滞的力学系统的 Noether 广义准对称变换.

由定义 3 和 (25) 式, 容易得到如下判据.

判据3 如果无限小变换 (16), 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 满足条件

$$\frac{\partial L}{\partial t}(t)\Delta t+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\Delta q_s$$
$$+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)\Delta\dot{q}_s.$$
$$+Q_s''(t)(\Delta q_s-\dot{q}_s\Delta t)+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G). \tag{39}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 满足条件

$$\frac{\partial L}{\partial t}(t)\Delta t+\frac{\partial L}{\partial q_s}(t)\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}(t)\Delta\dot{q}_s$$
$$+Q_s''(t)(\Delta q_s-\dot{q}_s\Delta t)+L\frac{\mathrm{d}}{\mathrm{d}t}(\Delta t)=-\frac{\mathrm{d}}{\mathrm{d}t}(\Delta G). \tag{40}$$

则变换 (16) 是所论含时滞的力学系统的 Noether 广义准对称变换.

(39) 和 (40) 式也可表为: 当 $t_1\leqslant t\leqslant t_2-\tau$时, 有

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\left(\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right)\xi_s^\sigma$$
$$+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)$$
$$+Q_s''(t)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)+L\dot{\xi}_0^\sigma$$
$$=-\dot{G}^\sigma,\quad (\sigma=1,2,\cdots,r), \tag{41}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\frac{\partial L}{\partial t}(t)\xi_0^\sigma+\frac{\partial L}{\partial q_s}(t)\xi_s^\sigma+\frac{\partial L}{\partial \dot{q}_s}(t)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)$$
$$+Q_s''(t)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)+L\dot{\xi}_0^\sigma$$
$$=-\dot{G}^\sigma,\quad (\sigma=1,2,\cdots,r). \tag{42}$$

当 $r=1$ 时, (41) 和 (42) 式称为含时滞的力学系统的 Noether 等式.

利用判据 1— 判据 3 或 Noether 等式 (29) 和 (30), (35) 和 (36), (41) 和 (42) 可以分别判断含时滞的力学系统的 Noether 对称性, Noether 准对称性和 Noether 广义准对称性.

5 含时滞的力学系统的 Noether 定理

我们首先给出含时滞的力学系统的守恒量的定义.

定义4 函数 $I(t,t+\tau,q_s(t),q_s(t-\tau),q_s(t+\tau),\dot{q}_s(t),\dot{q}_s(t-\tau),\dot{q}_s(t+\tau))$ 称为含时滞的力学系统 (13) 的守恒量, 当且仅当沿着运动方程 (13) 的解曲线恒成立

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t,t+\tau,q_s(t),q_s(t-\tau),q_s(t+\tau),\dot{q}_s(t),$$
$$\dot{q}_s(t-\tau),\dot{q}_s(t+\tau))=0. \tag{43}$$

对于含时滞的 Lagrange 系统 (14), 如果能找到系统的 Noether 对称变换或 Noether 准对称变换, 便可求得相应的守恒量. 有如下定理.

定理1 对于含时滞的 Lagrange 系统 (14), 如果无限小变换 (16) 是系统的 Noether 对称变换, 则系统存在 r 个线性独立的守恒量, 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 形如

$$I^\sigma=L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot{q}_s}(t)+\frac{\partial L}{\partial \dot{q}_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot{q}_s\xi_0^\sigma)$$
$$=c^\sigma,\quad (\sigma=1,2,\cdots,r), \tag{44}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 形如

$$\begin{aligned}I^\sigma &= L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\\&=c^\sigma,\quad (\sigma=1,2,\cdots,r).\end{aligned}\tag{45}$$

证明　由于无限小变换 (16) 是系统的 Noether 对称变换, 由定义 1, 得

$$\Delta S=0.$$

根据 (23) 式, 考虑到积分区间的任意性和无限小参数 ε_σ 的独立性, 得到: 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\right]\\&+\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)-\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial L}{\partial \dot q_s}(t)\right.\right.\\&\left.\left.+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)\right](\xi_s^\sigma-\dot q_s\xi_0^\sigma)=0,\\&(\sigma=1,2,\cdots,r),\end{aligned}\tag{46}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\right]\\&+\left(\frac{\partial L}{\partial q_s}(t)-\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial \dot q_s}(t)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)=0,\\&(\sigma=1,2,\cdots,r).\end{aligned}\tag{47}$$

对于含时滞的 Lagrange 系统, 有方程 (14), 因此方程 (46) 和 (47) 给出: 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma\right]\\&=0,\quad (\sigma=1,2,\cdots,r),\end{aligned}\tag{48}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\right]\\&=0,\quad (\sigma=1,2,\cdots,r).\end{aligned}\tag{49}$$

积分之, 便得到守恒量 (44) 和 (45). 证毕.

定理2　对于含时滞的 Lagrange 系统 (14), 如果无限小变换 (16) 是系统的 Noether 准对称变换, 则系统存在 r 个线性独立的守恒量, 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 形如

$$\begin{aligned}I^\sigma =&L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\\&+G^\sigma=c^\sigma,\quad (\sigma=1,2,\cdots,r),\end{aligned}\tag{50}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 形如

$$\begin{aligned}I^\sigma &= L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)+G^\sigma\\&=c^\sigma,\quad (\sigma=1,2,\cdots,r).\end{aligned}\tag{51}$$

证明　由于无限小变换 (16) 是系统的 Noether 准对称变换, 由定义 2, 得

$$\Delta S=-\int_{t_1}^{t_2}\frac{\mathrm d}{\mathrm dt}(\Delta G)\,\mathrm dt.$$

将 (23) 式代入上式, 由积分区间的任意性和 ε_σ 的独立性, 并利用含时滞的 Lagrange 方程 (14), 易知定理成立. 证毕.

定理 1 和定理 2 称为含时滞的 Lagrange 系统的 Noether 定理. 由 Noether 定理可知, 对于含时滞的 Lagrange 系统, 如果能找到系统的一个 Noether 对称变换或准对称变换, 便有可能得到系统的一个守恒量.

下面, 我们进一步建立含时滞的完整非保守力学系统的 Noether 定理.

定理3　对于含时滞的非保守力学系统 (13), 如果无限小变换 (16) 是 Noether 广义准对称变换, 则系统存在 r 个线性独立的守恒量 (50), (51).

证明　由于无限小变换 (16) 是系统的 Noether 广义准对称变换, 由定义 3, 并利用条件 (4), 得到

$$\begin{aligned}\Delta S=&-\int_{t_1}^{t_2}\left[\frac{\mathrm d}{\mathrm dt}(\Delta G)+Q_s''(t)\delta q_s\right]\mathrm dt\\=&-\int_{t_1}^{t_2-\tau}\left[\frac{\mathrm d}{\mathrm dt}(\Delta G)+Q_s''(t)\delta q_s\right]\mathrm dt\\&-\int_{t_2-\tau}^{t_2}\left[\frac{\mathrm d}{\mathrm dt}(\Delta G)+Q_s''(t)\delta q_s\right]\mathrm dt\\=&-\int_{t_1}^{t_2-\tau}\varepsilon_\sigma[\dot G^\sigma+Q_s''(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)]\,\mathrm dt\\&-\int_{t_2-\tau}^{t_2}\varepsilon_\sigma[\dot G^\sigma+Q_s''(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)]\,\mathrm dt.\end{aligned}\tag{52}$$

将 (23) 式代入上式, 并考虑积分区间的任意性和 ε_σ 的独立性, 得到: 当 $t_1\leqslant t\leqslant t_2-\tau$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)\right.\\&\left.\times(\xi_s^\sigma-\dot q_s\xi_0^\sigma)+G^\sigma\right]+\left[\frac{\partial L}{\partial q_s}(t)+\frac{\partial L}{\partial q_{s\tau}}(t+\tau)\right.\\&\left.-\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)+Q_s''(t)\right]\\&\times(\xi_s^\sigma-\dot q_s\xi_0^\sigma)=0,\\&(\sigma=1,2,\cdots,r).\end{aligned}\tag{53}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\left[L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)+G^\sigma\right]\\&+\left(\frac{\partial L}{\partial q_s}(t)-\frac{\mathrm d}{\mathrm dt}\frac{\partial L}{\partial \dot q_s}(t)+Q_s''(t)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)\end{aligned}$$

$$=0,\quad (\sigma=1,2,\cdots,r). \tag{54}$$

利用含时滞的非保守力学系统的 Lagrange 方程 (13), 得到: 当 $t_1 \leqslant t \leqslant t_2-\tau$ 时, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}(t)+\frac{\partial L}{\partial \dot q_{s\tau}}(t+\tau)\right)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)+G^\sigma\right]=0,\quad (\sigma=1,2,\cdots,r). \tag{55}$$

当 $t_2-\tau<t\leqslant t_2$ 时, 有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L\xi_0^\sigma+\frac{\partial L}{\partial \dot q_s}(t)(\xi_s^\sigma-\dot q_s\xi_0^\sigma)+G^\sigma\right]=0,\quad (\sigma=1,2,\cdots,r). \tag{56}$$

积分之, 便得到守恒量. 证毕.

6 算 例

例1 设力学系统的 Lagrange 函数为

$$L=\frac{1}{2}[\dot q(t)+\dot q(t-\tau)]^2, \tag{57}$$

非势广义力为

$$Q''=-\frac{1}{t}q(t)\dot q(t), \tag{58}$$

其中 $t\in[t_1,t_2]$, 时滞常量 $\tau<t_2-t_1$. 且满足边界条件: 当 $t\in[t_1-\tau,t_1]$ 时, $q(t)=\delta(t)$, 这里 $\delta(t)$ 是区间 $[t_1-\tau,t_1]$ 上的已知分段光滑函数; 当 $t=t_2$ 时, $q(t)=q(t_2)$, 这里 $q(t_2)$ 是某一确定值.

系统的运动微分方程 (13) 给出

$$\begin{aligned}&2\ddot q(t)+\ddot q(t-\tau)+\ddot q(t+\tau)\\&=-\frac{1}{t}q(t)\dot q(t),\quad (t_1\leqslant t\leqslant t_2-\tau),\\&\ddot q(t)+\ddot q(t-\tau)\\&=-\frac{1}{t}q(t)\dot q(t),\quad (t_2-\tau<t\leqslant t_2).\end{aligned} \tag{59}$$

Noether 等 (41) 和 (42) 式给出

$$\begin{aligned}&[2\dot q(t)+\dot q(t-\tau)+\dot q(t+\tau)](\dot\xi_1-\dot q(t)\dot\xi_0)\\&-\frac{1}{t}q(t)\dot q(t)(\xi_1-\dot q(t)\xi_0)+L\dot\xi_0=-\dot G,\\&(t_1\leqslant t\leqslant t_2-\tau),\\&[\dot q(t)+\dot q(t-\tau)](\dot\xi_1-\dot q(t)\dot\xi_0)-\frac{1}{t}q(t)\dot q(t)\\&\times(\xi_1-\dot q(t)\xi_0)+L\dot\xi_0\\&=-\dot G,\quad (t_2-\tau<t\leqslant t_2).\end{aligned} \tag{60}$$

方程 (60) 有解

$$\xi_0=0,\quad \xi_1=t. \tag{61}$$

将生成元 (61) 代入 (60) 式有

$$\begin{aligned}&G=\frac{1}{2}q^2(t)-2q(t)-q(t-\tau)\\&\quad -q(t+\tau),\quad (t_1\leqslant t\leqslant t_2-\tau),\\&G=\frac{1}{2}q^2(t)-q(t)-q(t-\tau),\\&\quad (t_2-\tau<t\leqslant t_2).\end{aligned} \tag{62}$$

因此, 生成元 (61) 相应于系统的 Noether 广义准对称性. 由生成元 (61) 和规范函数 (62), 根据定理 2, 系统有如下守恒量:

$$\begin{aligned}&[2\dot q(t)+\dot q(t-\tau)+\dot q(t+\tau)]t+\frac{1}{2}q^2(t)-2q(t)\\&-q(t-\tau)-q(t+\tau)=\text{const.},\quad t\in[t_1,t_2-\tau],\\&[\dot q(t)+\dot q(t-\tau)]t+\frac{1}{2}q^2(t)-q(t)-q(t-\tau)\\&=\text{const.},\quad t\in(t_2-\tau,t_2].\end{aligned} \tag{63}$$

这是所论含时滞的非保守力学系统的 Noether 广义准对称性 (61) 直接导致的守恒量.

例2 研究单自由度阻尼振子系统, 其 Lagrange 函数为

$$L=\frac{1}{2}m\dot q^2(t), \tag{64}$$

广义非势力为

$$Q''=-c\dot q(t-\tau), \tag{65}$$

其中质量 m, 阻尼系数 c 以及时滞量 τ 均为常数.

系统的运动微分方程给出

$$m\ddot q(t)=-c\dot q(t-\tau), \tag{66}$$

Noether 等式 (41) 和 (42) 给出

$$m\dot q(t)(\dot\xi_1-\dot q(t)\dot\xi_0)-c\dot q(t-\tau)(\xi_1-\dot q(t)\xi_0)+\frac{1}{2}m\dot q^2(t)\dot\xi_0=-\dot G. \tag{67}$$

方程 (67) 有解

$$\xi_0^1=1,\quad \xi_1^1=\dot q(t),\quad G^1=-\frac{1}{2}m\dot q^2(t), \tag{68}$$

$$\xi_0^2=0,\quad \xi_1^2=1,\quad G^2=cq(t-\tau). \tag{69}$$

生成元 (68) 和 (69) 都相应于系统的 Noether 广义准对称变换. 将生成元 (68) 和 (69) 分别代入 (50) 和 (51) 式, 得到

$$I^1=0, \tag{70}$$

$$I^2=m\dot q(t)+cq(t-\tau)=\text{const.} \tag{71}$$

因此, 生成元 (68) 相应的守恒量是平庸的.

例3 已知含时滞的两自由度 Lagrange 系统的 Lagrange 函数为

$$
\begin{aligned}
L=&\frac{m}{2}\left\{[\dot{q}_1(t)+\dot{q}_1(t-\tau)]^2+[\dot{q}_2(t)+\dot{q}_2(t-\tau)]^2\right\}\\
&-\frac{k}{2}\left\{[q_1(t)+q_1(t-\tau)]^2\right.\\
&\left.+[q_2(t)+q_2(t-\tau)]^2\right\},
\end{aligned}\tag{72}
$$

系统的运动微分方程为

$$
\begin{aligned}
&2m\ddot{q}_1(t)+m\ddot{q}_1(t-\tau)+m\ddot{q}_1(t+\tau)+2kq_1(t)\\
&+kq_1(t-\tau)+kq_1(t+\tau)=0,\\
&2m\ddot{q}_2(t)+m\ddot{q}_2(t-\tau)+m\ddot{q}_2(t+\tau)+2kq_2(t)\\
&+kq_2(t-\tau)+kq_2(t+\tau)=0,\\
&t\in[t_1,t_2-\tau],
\end{aligned}\tag{73}
$$

$$
\begin{aligned}
&m\ddot{q}_1(t)+m\ddot{q}_1(t-\tau)+kq_1(t)+kq_1(t-\tau)=0,\\
&m\ddot{q}_2(t)+m\ddot{q}_2(t-\tau)+kq_2(t)+kq_2(t-\tau)\\
=&0,\quad t\in(t_2-\tau,t_2].
\end{aligned}\tag{74}
$$

当 $t\in[t_1,t_2-\tau]$ 时, 由 Noether 等式 (35), 有

$$
\begin{aligned}
&-k[2q_1(t)+q_1(t-\tau)+q_1(t+\tau)]\xi_1\\
&-k[2q_2(t)+q_2(t-\tau)+q_2(t+\tau)]\xi_2\\
&+m[2\dot{q}_1(t)+\dot{q}_1(t-\tau)+\dot{q}_1(t+\tau)](\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)\\
&+m[2\dot{q}_2(t)+\dot{q}_2(t-\tau)+\dot{q}_2(t+\tau)](\dot{\xi}_2-\dot{q}_2\dot{\xi}_0)\\
&+L\dot{\xi}_0=-\dot{G},
\end{aligned}\tag{75}
$$

方程 (75) 有解

$$
\begin{aligned}
\xi_0=&0,\quad \xi_1=2\dot{q}_1(t)+\dot{q}_1(t-\tau)+\dot{q}_1(t+\tau),\\
\xi_2=&2\dot{q}_2(t)+\dot{q}_2(t-\tau)+\dot{q}_2(t+\tau),\\
G=&-\frac{1}{2}m\left\{[2\dot{q}_1(t)+\dot{q}_1(t-\tau)+\dot{q}_1(t+\tau)]^2\right.\\
&\left.+[2\dot{q}_2(t)+\dot{q}_2(t-\tau)+\dot{q}_2(t+\tau)]^2\right\}\\
&+\frac{1}{2}k\left\{[2q_1(t)+q_1(t-\tau)+q_1(t+\tau)]^2\right.\\
&\left.+[2q_2(t)+q_2(t-\tau)+q_2(t+\tau)]^2\right\}.
\end{aligned}\tag{76}
$$

当 $t\in(t_2-\tau,t_2]$ 时, 由 Noether 等式 (36), 有

$$
\begin{aligned}
&-k[q_1(t)+q_1(t-\tau)]\xi_1-k[q_2(t)+q_2(t-\tau)]\xi_2\\
&+m[\dot{q}_1(t)+\dot{q}_1(t-\tau)](\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)\\
&+m[\dot{q}_2(t)+\dot{q}_2(t-\tau)](\dot{\xi}_2-\dot{q}_2\dot{\xi}_0)+L\dot{\xi}_0=-\dot{G},
\end{aligned}\tag{77}
$$

方程 (77) 有解

$$
\begin{aligned}
\xi_0=&0,\xi_1=\dot{q}_1(t)+\dot{q}_1(t-\tau),\\
\xi_2=&\dot{q}_2(t)+\dot{q}_2(t-\tau),\\
G=&-\frac{1}{2}m\left\{[\dot{q}_1(t)+\dot{q}_1(t-\tau)]^2+[\dot{q}_2(t)\right.\\
&\left.+\dot{q}_2(t-\tau)]^2\right\}\\
&+\frac{1}{2}k\left\{[q_1(t)+q_1(t-\tau)]^2\right.\\
&\left.+[q_2(t)+q_2(t-\tau)]^2\right\},
\end{aligned}\tag{78}
$$

生成元 (76) 和 (78) 相应于系统的 Noether 准对称性. 由定理 2, 我们得到: 当 $t\in[t_1,t_2-\tau]$ 时, 有

$$
\begin{aligned}
&\frac{1}{2}m\left\{[2\dot{q}_1(t)+\dot{q}_1(t-\tau)+\dot{q}_1(t+\tau)]^2\right.\\
&\left.+[2\dot{q}_2(t)+\dot{q}_2(t-\tau)+\dot{q}_2(t+\tau)]^2\right\}\\
&+\frac{1}{2}k\left\{[2q_1(t)+q_1(t-\tau)+q_1(t+\tau)]^2\right.\\
&\left.+[2q_2(t)+q_2(t-\tau)+q_2(t+\tau)]^2\right\}=\text{const}.
\end{aligned}\tag{79}
$$

当 $t\in(t_2-\tau,t_2]$ 时, 有

$$
\begin{aligned}
&\frac{1}{2}m\left\{[\dot{q}_1(t)+\dot{q}_1(t-\tau)]^2+[\dot{q}_2(t)+\dot{q}_2(t-\tau)]^2\right\}\\
&+\frac{1}{2}k\left\{[q_1(t)+q_1(t-\tau)]^2+[q_2(t)+q_2(t-\tau)]^2\right\}\\
=&\text{const}.
\end{aligned}\tag{80}
$$

(79) 和 (80) 式是所论含时滞的 Lagrange 系统相应于 Noether 准对称性 (76) 和 (78) 的 Noether 守恒量.

7 结 论

我们在文中研究了含时滞的非保守系统动力学的 Noether 对称性与守恒量. 从非保守系统的 Hamilton 原理出发, 建立了含时滞的非保守系统的变分原理 (11) 和 Euler-Lagrange 方程 (13), 含时滞的 Lagrange 系统的方程是方程 (13) 的特例; 建立了含时滞的 Hamilton 作用量 (15) 在依赖于广义速度的无限小群变换 (17) 下的变分的两个基本公式 (23) 和 (25), 该公式为建立含时滞 Noether 对称性的判据奠定了基础; 建立了含时滞的力学系统的 Noether 对称变换和 Noether 准对称变换的定义和判据; 给出了含时滞的力学系统的守恒量的定义, 建立了含时滞的非保守力学系统的 Noether 定理. 文章从三个方面拓展了 Frederico 和 Torres[29] 的结果: 从 Lagrange 系统拓展到一般非保守系统; 从相应于广义坐标和时间的点变换群拓展到一般的相应于依赖广义速度, 广义坐标和时间的无限小变换群; 从 Noether 对称变换拓展到 Noether 准对称变换. 本文方法具有普遍性, 我们可以进一步研究含时滞的最优控制系统, 含时滞的非完整约束系统, 以及含时滞的 Birkhoff 系统等.

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Noether symmetries of dynamics for non-conservative systems with time delay*

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Abstract

The Noether symmetries and the conserved quantities of dynamics for non-conservative systems with time delay are proposed and studied. Firstly, the Hamilton principle for non-conservative systems with time delay is established, and the Lagrange equations with time delay are obtained. Secondly, based upon the invariance of the Hamilton action with time delay under a group of infinitesimal transformations which depends on the generalized velocities, the generalized coordinates and the time, the Noether symmetric transformations and the Noether quasi-symmetric transformations of the system are defined and the criteria of the Noether symmetries are established. Finally, the relationship between the symmetries and the conserved quantities are studied, and the Noether theory of non-conservative systems with time delay is established At the end of the paper, some examples are given to illustrate the application of the results.

Keywords: system with time delay, non-conservative mechanics, Noether symmetry, conserved quantity

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