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El-Nabulsi动力学模型下Birkhoff系统Noether对称性的摄动与绝热不变量*

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基于El-Nabulsi动力学模型, 研究了小扰动作用下Birkhoff系统Noether对称性的摄动与绝热不变量问题. 首先, 将El-Nabulsi提出的在分数阶微积分框架下基于Riemann-Liouville分数阶积分的非保守系统动力学模型拓展到Birkhoff系统, 建立El-Nabulsi-Birkhoff方程; 其次, 基于在无限小变换下El-Nabulsi-Pfaff作用量的不变性, 给出Noether准对称性的定义和判据, 得到了Noether对称性导致的精确不变量; 再次, 引入力学系统的绝热不变量概念, 研究El-Nabulsi动力学模型下受小扰动作用的Birkhoff系统Noether对称性的摄动与绝热不变量之间的关系, 得到了对称性摄动导致的绝热不变量的条件及其形式. 作为特例, 给出了El-Nabulsi动力学模型下相空间中非保守系统和经典Birkhoff系统的Noether对称性的摄动与绝热不变量. 以著名的Hojman-Urrutia问题为例, 研究其在El-Nabulsi动力学模型下的Noether对称性, 得到了相应的精确不变量和绝热不变量.

关键词: Birkhoff系统, El-Nabulsi动力学模型, Noether对称性摄动, 绝热不变量

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1 引 言

1927年, 美国著名数学家Birkhoff[1]提出了比Hamilton方程更为普遍的一类新型动力学方程和比Hamilton原理更为普遍的一类新型积分变分原理. 美国强子物理学家Santilli[2]将Birkhoff得到的结果推广到包含时间的情形, 并提出了Birkhoff力学一词, 同时指出Birkhoff力学是由Hamilton力学经过变换理论构造出的最一般可能的力学. 实际上, 所有完整约束系统和非完整约束系统都能纳入Birkhoff 系统[3]. Galiullan[4]认为, 对Birkhoff方程的研究是近代分析力学的一个重要发展方向[4]. 梅凤翔[5,6]指出: Birkhoff力学是量子力学出现之后经典力学的新发展. 关于Birkhoff 系统动力学的研究已经取得了一系列重要成果[7−20]. 2005年, El-Nabulsi[21]针对非保守系统的动力学建模提出了一种新的建模方法, 即在分数阶微积分框架下, 基于Riemann-Liouville分数阶积分而建立的非保守系统动力学模型, 称之为El-Nabulsi动力学模型. 该模型所得到的Euler-Lagrange方程形式简单且类似于经典非保守系统的动力学方程, 其新颖之处在于方程中出现相应于耗散力的广义分数阶外力而不出现分数阶导数, 且分数阶时间积分只需要一个参数, 而在其他模型中会出现任意个分数阶参数. 近年来, 基于El-Nabulsi动力学模型的约束系统动力学及其对称性研究取得了重要进展[22−30]. 最近, Zhang和Zhou[31]将El-Nabulsi动力学模型拓展到Birkhoff系统, 并给出了该模型下Birkhoff系统的Noether定理.

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本文进一步研究在El-Nabulsi动力学模型下受小扰动作用的Birkhoff系统Noether对称性的摄动与绝热不变量问题. 将经典的Birkhoff系统Noether对称性推广到El-Nabulsi动力学模型, 给出该模型下Birkhoff系统的精确不变量和绝热不变量的存在条件及其形式. 作为特例, 给出了El-Nabulsi动力学模型下相空间中非保守力学系统和经典Birkhoff系统的Noether对称性的摄动与绝热不变量, 并研究了著名的Hojman-Urrutia问题[32].

2 基于El-Nabulsi动力学模型的Birkhoff方程

为研究非保守系统的动力学建模, El-Nabulsi[21]提出了基于Riemann-Liouville分数阶积分的变分问题. 对于Birkhoff系统, 下面给出基于Riemann-Liouville分数阶积分的Pfaff变分问题[31].

求积分泛函

$$S(\gamma)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}[R_\mu(\tau,a^\nu)\dot a^\mu-B(\tau,a^\nu)]\times(t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{1}$$

在给定边界条件

$$a^\mu|_{\tau=t_1}=a_1^\mu,a^\mu|_{\tau=t_2}=a_2^\mu\quad(\mu=1,2,\cdots,2n) \tag{2}$$

下的极值问题, 其中, $a^\mu(\mu=1,2,\cdots,2n)$为Birkhoff变量, $\dot a^\mu=\mathrm{d}a^\mu/\mathrm{d}\tau$, $B=B(\tau,a^\nu)$ 为Birkhoff函数, $R_\mu=R_\mu(\tau,a^\nu)$为Birkhoff函数组, Γ是Euler-Gamma函数, $0<\alpha\leqslant1$, τ为固有时间, t为观察者时间, $\tau\neq t$, 光滑函数B和R_μ是其变量的C^2类函数, γ为一曲线.

上述变分问题可称为基于Riemann-Liouville分数阶积分的El-Nabulsi-Pfaff变分问题, 泛函(1)式称为El-Nabulsi-Pfaff作用量. 当$\alpha=1$时, 上述问题成为经典的Pfaff变分问题.

如果El-Nabulsi变分问题的极值是$a^\mu(\tau)$, 则其满足如下方程[31]:

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}=\frac{1-\alpha}{t-\tau}R_\mu\quad(\mu=1,2,\cdots,2n). \tag{3}$$

方程(3)称为基于Riemann-Liouville分数阶积分的El-Nabulsi-Pfaff变分问题的Birkhoff方程或El-Nabulsi-Birkhoff方程. 如果取$\alpha=1$, 方程(3)为经典的Birkhoff方程, 即

$$\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}=0\quad(\mu=1,2,\cdots,2n). \tag{4}$$

3 Noether对称性与精确不变量

引进时间τ和Birkhoff变量a^μ的无限小变换,

$$\bar\tau=\tau+\Delta\tau,$$
$$\bar a^\mu(\bar\tau)=a^\mu(\tau)+\Delta a^\mu\quad(\mu=1,2,\cdots,2n), \tag{5}$$

其展开式为

$$\bar\tau=\tau+\varepsilon\xi_0^0(\tau,a^\nu),$$
$$\bar a^\mu(\bar\tau)=a^\mu(\tau)+\varepsilon\xi_\mu^0(\tau,a^\nu)\quad(\mu=1,2,\cdots,2n). \tag{6}$$

这里ε是无限小参数, ξ_0^0,ξ_μ^0是无限小变换的生成函数或生成元. 在无限小变换(5)式作用下, 曲线γ将变换为邻近曲线$\bar\gamma$, 相应地积分泛函$S(\gamma)$变换为$S(\bar\gamma)$, 设ΔS为差$S(\bar\gamma)-S(\gamma)$相对ε的主线性部分, 则有[31]

$$\begin{aligned}\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\Bigg\{\Bigg[\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\Delta\tau\\&+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu+R_\mu\Delta\dot a^\mu\\&+(R_\mu\dot a^\mu-B)\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\Bigg](t-\tau)^{\alpha-1}\\&-(R_\mu\dot a^\mu-B)(\alpha-1)\Delta\tau(t-\tau)^{\alpha-2}\Bigg\}\\&\times\mathrm{d}\tau.\end{aligned} \tag{7}$$

注意到用Δ表示的非等时变分与用δ表示的等时变分之间有如下关系:

$$\delta a^\mu=\Delta a^\mu-\dot a^\mu\Delta\tau,$$
$$\Delta\dot a^\mu=\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta a^\mu)-\dot a^\mu\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta\tau), \tag{8}$$

(7)式可进一步表示为

$$\begin{aligned}\Delta S=&\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\varepsilon\Bigg\{\frac{\mathrm{d}}{\mathrm{d}\tau}[(R_\mu\xi_\mu^0-B\xi_0^0)\\&\times(t-\tau)^{\alpha-1}]+\Bigg[\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu\\&-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}-\frac{1-\alpha}{t-\tau}R_\mu\Bigg]\\&\times(t-\tau)^{\alpha-1}(\xi_\mu^0-\dot a^\mu\xi_0^0)\Bigg\}\mathrm{d}\tau.\end{aligned} \tag{9}$$

(7) 和 (9) 式是El-Nabulsi-Pfaff作用量变分的两个基本公式.

定义1 如果El-Nabulsi-Pfaff作用量 (1) 式是无限小变换 (5) 式的准不变量, 即对每一个无限小变换,

$$\Delta S=-\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)\mathrm{d}\tau \qquad (10)$$

始终成立, 其中 $\Delta G=\varepsilon G^0$, $G^0=G^0(\tau,a^\nu)$ 称为规范函数, 则称无限小变换 (5) 式为El-Nabulsi动力学模型下Birkhoff系统的Noether 准对称变换或者El-Nabulsi-Noether准对称变换.

判据1 对于无限小变换 (5) 式, 如果满足

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\Delta\tau+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\Delta a^\nu\\&+R_\mu\Delta\dot a^\mu+(R_\mu\dot a^\mu-B)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau+\frac{1-\alpha}{t-\tau}\Delta\tau\right)\\=&-\frac{\mathrm{d}}{\mathrm{d}\tau}(\Delta G)(t-\tau)^{1-\alpha},\end{aligned}\qquad (11)$$

则无限小变换 (5) 式是Birkhoff系统 (3) 的El-Nabulsi-Noether准对称变换.

由 (5) 和 (6) 式, (11) 式可进一步表示为

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\xi_0^0+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^0\\&+R_\mu(\dot\xi_\mu^0-\dot a^\mu\dot\xi_0^0)+(R_\mu\dot a^\mu-B)\left(\dot\xi_0^0+\frac{1-\alpha}{t-\tau}\xi_0^0\right)\\=&-\dot G^0(t-\tau)^{1-\alpha}.\end{aligned}\qquad (12)$$

(12) 式称为El-Nabulsi动力学模型下Birkhoff系统的Noether等式或El-Nabulsi-Noether等式. 当 $\alpha=1$ 时, (12) 式为经典Birkhoff系统的Noether等式, 即

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\xi_0^0+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^0\\&+R_\mu(\dot\xi_\mu^0-\dot a^\mu\dot\xi_0^0)+(R_\mu\dot a^\mu-B)\dot\xi_0^0\\=&-\dot G^0.\end{aligned}\qquad (13)$$

因此, 就有定理1.

定理1 对于El-Nabulsi动力学模型下的Birkhoff系统 (3), 如果无限小变换的生成元 ξ_0^0, ξ_μ^0 满足El-Nabulsi-Noether等式 (12), 则该系统未受扰动时的Noether 对称性直接导致如下守恒量:

$$\begin{aligned}I_0&=(R_\mu\xi_\mu^0-B\xi_0^0)(t-\tau)^{\alpha-1}+G^0\\&=\mathrm{const}.\end{aligned}\qquad (14)$$

守恒量 (14) 式是精确不变量. 定理1揭示了El-Nabulsi动力学模型下未受扰动的Birkhoff系统 (3) 的Noether对称性与精确不变量之间的关系.

4 Noether对称性的摄动与绝热不变量

首先, 我们引入Birkhoff系统的绝热不变量概念 [33,34].

定义2 若 $I_z(\tau,a^\nu,\rho)$ 是Birkhoff系统的一个含有小参数 ρ 的最高次幂为 z 的物理量, 其对时间 τ 的一阶导数正比于 ρ^{z+1}, 则称 I_z 为Birkhoff系统的 z 阶绝热不变量.

假设基于El-Nabulsi动力学模型的Birkhoff系统 (3) 受到小扰动 ρQ_μ 的作用, 则该系统的运动正轨满足如下运动微分方程:

$$\begin{aligned}&\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial\tau}\\=&\frac{1-\alpha}{t-\tau}R_\mu+\rho Q_\mu\quad(\mu=1,2,\cdots,2n).\end{aligned}\qquad (15)$$

在小扰动 ρQ_μ 的作用下, 此系统原有的对称性与不变量将会发生相应的改变. 假设受扰动后的无限小生成元 ξ_0,ξ_μ 是在系统未受扰动时的对称变换生成元基础上发生的小摄动, 即

$$\begin{aligned}\xi_0&=\xi_0^0+\rho\xi_0^1+\rho^2\xi_0^2+\cdots,\\\xi_\mu&=\xi_\mu^0+\rho\xi_\mu^1+\rho^2\xi_\mu^2+\cdots,\end{aligned}\qquad (16)$$

规范函数 G 也相应地发生了小摄动, 即

$$G=G^0+\rho G^1+\rho^2G^2+\cdots.\qquad (17)$$

于是, 就有定理2.

定理2 设El-Nabulsi动力学模型下的Birkhoff系统 (3) 受到小扰动 ρQ_μ 的作用, 如果存在规范函数 $G^j(\tau,a^\nu)$ 使无限小变换的生成元 $\xi_0^j(\tau,a^\nu)$ 和 $\xi_\mu^j(\tau,a^\nu)$ 满足

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\xi_0^j+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^j\\&+R_\mu(\dot\xi_\mu^j-\dot a^\mu\dot\xi_0^j)+(R_\mu\dot a^\mu-B)\\&\times\left(\dot\xi_0^j+\frac{1-\alpha}{t-\tau}\xi_0^j\right)-Q_\mu(\xi_\mu^{j-1}-\dot a^\mu\xi_0^{j-1})\\=&-\dot G^j(t-\tau)^{1-\alpha}\quad(j=0,1,2,\cdots,2n),\end{aligned}\qquad (18)$$

当 $j=0$ 时, 约定 $\xi_0^{-1}=\xi_\mu^{-1}=0$, 则

$$I_z=\sum_{j=0}^{z}\rho^j[(R_\mu\xi_\mu^j-B\xi_0^j)(t-\tau)^{\alpha-1}+G^j]\qquad (19)$$

是该系统的一个 z 阶绝热不变量.

证明 对 (19) 式等号两端求微分, 有

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}\tau} =& \sum_{j=0}^{z}\rho^j\Bigg[\bigg(\frac{\partial R_\mu}{\partial\tau}\xi_\mu^j + \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\xi_\mu^j + R_\mu\dot{\xi}_\mu^j \\ &- \frac{\partial B}{\partial\tau}\xi_0^j - \frac{\partial B}{\partial a^\mu}\dot{a}^\mu\xi_0^j - B\dot{\xi}_0^j\bigg)(t-\tau)^{\alpha-1} \\ &+ \frac{1-\alpha}{t-\tau}(R_\mu\xi_\mu^j - B\xi_0^j)(t-\tau)^{\alpha-1} + \dot{G}^j\Bigg] \\ =& \sum_{j=0}^{z}\rho^j\Bigg\{\bigg[\frac{\partial R_\mu}{\partial\tau}\xi_\mu^j + \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\xi_\mu^j \\ &+ R_\mu\dot{\xi}_\mu^j - \frac{\partial B}{\partial\tau}\xi_0^j - \frac{\partial B}{\partial a^\mu}\dot{a}^\mu\xi_0^j - B\dot{\xi}_0^j \\ &+ \frac{1-\alpha}{t-\tau}(R_\mu\xi_\mu^j - B\xi_0^j) - \bigg(\frac{\partial R_\mu}{\partial\tau}\dot{a}^\mu \\ &- \frac{\partial B}{\partial\tau}\bigg)\xi_0^j - \bigg(\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\mu - \frac{\partial B}{\partial a^\nu}\bigg)\xi_\nu^j \\ &- R_\mu(\dot{\xi}_\mu^j - \dot{a}^\mu\dot{\xi}_0^j) - (R_\mu\dot{u}^\mu - B) \\ &\times\bigg(\dot{\xi}_0^j + \frac{1-\alpha}{t-\tau}\xi_0^j\bigg) \\ &+ Q_\mu(\xi_\mu^{j-1} - \dot{a}^\mu\xi_0^{j-1})\bigg]\Bigg\}(t-\tau)^{\alpha-1} \\ =& \sum_{j=0}^{z}\rho^j\Bigg\{\bigg[-\bigg(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\bigg)\dot{a}^\nu \\ &+ \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial\tau} + \frac{1-\alpha}{t-\tau}R_\mu\bigg] \\ &\times(\xi_\mu^j - \dot{a}^\mu\xi_0^j) \\ &+ Q_\mu(\xi_\mu^{j-1} - \dot{a}^\mu\xi_0^{j-1})\Bigg\}(t-\alpha)^{\alpha-1} \\ =& \sum_{j=0}^{z}\rho^j\Big[-\rho Q_\mu(\xi_\mu^j - \dot{a}^\mu\xi_0^j) \\ &+ Q_\mu(\xi_\mu^{j-1} - \dot{a}^\mu\xi_0^{j-1})\Big](t-\tau)^{\alpha-1} \\ =& -\rho^{z+1}Q_\mu(\xi_\mu^z - \dot{a}^\mu\xi_0^z)(t-\tau)^{\alpha-1}. \end{aligned} \tag{20}$$

因此, I_z 是El-Nabulsi动力学模型下Birkhoff系统 (3) 的一个 z 阶绝热不变量. 证毕.

由 (20) 式可知, 若 $Q_\mu = 0$, 则 $\dfrac{\mathrm{d}I_z}{\mathrm{d}\tau} = 0$. 因此, El-Nabulsi动力学模型下Birkhoff系统 (3) 的精确不变量是一个特殊的绝热不变量.

下面我们讨论两种重要的特殊情况.

首先, 基于El-Nabulsi动力学模型, 研究相空间中的非保守系统. 取

$$a^\mu = \begin{cases} q_\mu & (\mu = 1,2,\cdots,n), \\ p_{\mu-n} & (\mu = n+1,\cdots,2n), \end{cases}$$

$$R_\mu = \begin{cases} p_\mu & (\mu = 1,2,\cdots,n), \\ 0 & (\mu = n+1,\cdots,2n), \end{cases}$$

$$B = H, \tag{21}$$

则Birkhoff方程 (3) 变换为

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} - \frac{1-\alpha}{t-\tau}p_k \quad (k = 1,2,\cdots,n), \end{aligned} \tag{22}$$

其中, q_k 为广义坐标, p_k 为广义动量, H 为Hamilton函数.

此系统受到小扰动 ρQ_k 后的运动微分方程 (15) 变换为

$$\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k}, \\ \dot{p}_k &= -\frac{\partial H}{\partial q_k} - \frac{1-\alpha}{t-\tau}p_k \\ &\quad + \rho Q_k \quad (k = 1,2,\cdots,n), \end{aligned} \tag{23}$$

于是, 定理1和定理2成为定理3和定理4.

定理3 对于El-Nabulsi动力学模型下相空间中非保守系统 (22), 如果无限小变换的生成元 ξ_0^0, ξ_k^0 和 η_k^0 满足El-Nabulsi-Noether等式,

$$\begin{aligned} &-\frac{\partial H}{\partial\tau}\xi_0^0 - \frac{\partial H}{\partial q_k}\xi_k^0 + \bigg(\dot{q}_k - \frac{\partial H}{\partial p_k}\bigg)\eta_k^0 \\ &+ p_k(\dot{\xi}_k^0 - \dot{q}_k\dot{\xi}_0^0) + (p_k\dot{q}_k - H)\bigg(\dot{\xi}_0^0 + \frac{1-\alpha}{t-\tau}\xi_0^0\bigg) \\ =& -\dot{G}^0(t-\tau)^{1-\alpha}, \end{aligned} \tag{24}$$

则该系统未受扰动时的Noether对称性直接导致如下精确不变量:

$$\begin{aligned} I_0 &= (p_k\xi_k^0 - H\xi_0^0)(t-\tau)^{\alpha-1} + G^0 \\ &= \mathrm{const}. \end{aligned} \tag{25}$$

这里 $\Delta\tau = \varepsilon\xi_0^0$, $\Delta q_k = \varepsilon\xi_k^0$, $\Delta p_k = \varepsilon\eta_k^0$.

定理4 设El-Nabulsi动力学模型下相空间中非保守系统 (22) 受到小扰动 ρQ_k 的作用, 如果存在规范函数 $G^j(\tau, q_k, p_k)$ 使无限小变换的生成元 ξ_0^j, ξ_k^j 和 η_k^j 满足

$$\begin{aligned} &-\frac{\partial H}{\partial\tau}\xi_0^j - \frac{\partial H}{\partial q_k}\xi_k^j + \bigg(\dot{q}_k - \frac{\partial H}{\partial p_k}\bigg)\eta_k^j \\ &+ p_k(\dot{\xi}_k^j - \dot{q}_k\dot{\xi}_0^j) + (p_k\dot{q}_k - H) \\ &\times\bigg(\dot{\xi}_0^j + \frac{1-\alpha}{t-\tau}\xi_0^j\bigg) + Q_k(\xi_k^{j-1} - \dot{q}_k\xi_0^{j-1}) \\ =& -\dot{G}^j(t-\tau)^{1-\alpha} \quad (j = 0,1,2,\cdots,n), \end{aligned} \tag{26}$$

当 $j=0$ 时, 约定 $\xi_0^{-1}=\xi_k^{-1}=0$, 则

$$I_z=\sum_{j=0}^{z} v^j[(p_k\xi_k^j-H\xi_0^j)(t-\tau)^{\alpha-1}+G^j] \quad (27)$$

是该系统的一个 z 阶绝热不变量.

如果取 $\alpha=1$, 定理3和定理4退化为与经典Hamilton系统相应的结论[33].

其次, 研究经典的Birkhoff系统. 取 $\alpha=1$, 则定理1和定理2成为定理5和定理6.

定理5 对于经典Birkhoff系统 (4), 如果无限小变换的生成元 ξ_0^0, ξ_μ^0 满足Noether等式(13), 则该系统未受扰动时的Noether 对称性直接导致如下精确不变量:

$$\begin{aligned}I_0&=R_\mu\xi_\mu^0-B\xi_0^0+G^0\\&=\text{const.}\end{aligned} \quad (28)$$

定理6 设经典Birkhoff系统 (4) 受到小扰动 ρQ_μ 的作用, 如果存在规范函数 $G^j(\tau,a^\nu)$ 使无限小变换的生成元 $\xi_0^j(\tau,a^\nu)$ 和 $\xi_\mu^j(\tau,a^\nu)$ 满足

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial\tau}\dot a^\mu-\frac{\partial B}{\partial\tau}\right)\xi_0^j+\left(\frac{\partial R_\mu}{\partial a^\nu}\dot a^\mu-\frac{\partial B}{\partial a^\nu}\right)\xi_\nu^j\\&+R_\mu(\dot\xi_\mu^j-\dot a^\mu\dot\xi_0^j)+(R_\mu\dot a^\mu-B)\dot\xi_0^j\\&-Q_\mu(\xi_\mu^{j-1}-\dot a^\mu\xi_0^{j-1})\\=&-\dot G^j \quad (j=0,1,2,\cdots,2n),\end{aligned} \quad (29)$$

当 $j=0$ 时, 约定 $\xi_0^{-1}=\xi_\mu^{-1}=0$, 则

$$I_z=\sum_{j=0}^{z}\rho^j(R_\mu\xi_\mu^j-B\xi_0^j+G^j) \quad (30)$$

是该系统的一个 z 阶绝热不变量.

5 算 例

研究Hojman-Urrutia问题[32]. 系统的Birkhoff函数 B 和Birkhoff函数组 R_μ 为[3]

$$\begin{aligned}B&=\frac{1}{2}[(a^3)^2+2a^2a^3-(a^4)^2],\\R_1&=a^2+a^3,\\R_2&=0,\\R_3&=a^4,\\R_4&=0.\end{aligned} \quad (31)$$

研究在El-Nabulsi动力学模型下上述系统Noether对称性的摄动与绝热不变量.

根 据 Hojman 和 Urrutia[32] 以 及 Santilli[2] 的研究结果, 系统(31)具有Birkhoff表示. 但是这个系统本质上不是自伴随的, 因此没有Hamilton表示. 由El-Nabulsi-Noether等式(12)给出

$$\begin{aligned}&(\dot a^1-a^3)\xi_2^0+(\dot a^1-a^2-a^3)\xi_3^0+(\dot a^3+a^4)\xi_4^0\\&+(a^2+a^3)(\dot\xi_1^0-\dot a^1\dot\xi_0^0)+a^4(\dot\xi_3^0-\dot a^3\dot\xi_0^0)\\&+(a^2\dot a^1+a^3\dot a^1+a^4\dot a^3-B)\left(\dot\xi_0^0+\frac{1-\alpha}{t-\tau}\xi_0^0\right)\\=&-\dot G^0(t-\tau)^{1-\alpha}.\end{aligned} \quad (32)$$

利用方程(3), 方程(32)有下列解:

$$\begin{aligned}(\xi_0^0)^1&=0,\\(\xi_1^0)^1&=0,\\(\xi_2^0)^1&=1,\\(\xi_3^0)^1&=0,\\(\xi_4^0)^1&=0,\\(G^0)^1&=0;\end{aligned} \quad (33)$$

$$\begin{aligned}(\xi_0^0)^2&=0,\\(\xi_1^0)^2&=1,\\(\xi_2^0)^2&=0,\\(\xi_3^0)^2&=0,\\(\xi_4^0)^2&=0,\\(G^0)^2&=0.\end{aligned} \quad (34)$$

由定理1可知, 系统(31)有如下精确不变量:

$$(I_0)^1=0. \quad (35)$$

$$\begin{aligned}(I_0)^2&=(a^2+a^3)(t-\tau)^{\alpha-1}\\&=\text{const.}\end{aligned} \quad (36)$$

下面讨论系统(31)的绝热不变量. 假设相应于Noether对称性(33)式, 系统(31)受到的小扰动为

$$\begin{aligned}\rho Q_1&=-\rho a^2,\\\rho Q_2&=-\rho a^3,\\\rho Q_3&=-\rho(a^3a^2+a^2),\\\rho Q_4&=-\rho a^4,\end{aligned} \quad (37)$$

条件 (18) 式给出

$$\begin{aligned}&(\dot a^1-a^3)\xi_2^1+(\dot a^1-a^2-a^3)\xi_3^1+(\dot a^3+a^4)\xi_4^1\\&+(a^2+a^3)(\dot\xi_1^1-\dot a^1\dot\xi_0^1)+a^4(\dot\xi_3^1-\dot a^3\dot\xi_0^1)\\&+(a^2\dot a^1+a^3\dot a^1+a^4\dot a^3-B)\left(\dot\xi_0^1+\frac{1-\alpha}{t-\tau}\xi_0^1\right)\\&-Q_1(\xi_1^0-\dot a^1\xi_0^0)-Q_2(\xi_2^0-\dot a^2\xi_0^0)\\&-Q_3(\xi_3^0-\dot a^3\xi_0^0)-Q_4(\xi_4^0-\dot a^4\xi_0^0)\end{aligned}$$

$$= -\dot{G}^1(t-\tau)^{1-\alpha}, \tag{38}$$

方程(38)有解:

$$\begin{aligned}
(\xi_0^1)^1 &= 0,\\
(\xi_1^1)^1 &= -\tau,\\
(\xi_2^1)^1 &= 1,\\
(\xi_3^1)^1 &= -1,\\
(\xi_4^1)^1 &= 0,\\
(G^1)^1 &= 0.
\end{aligned} \tag{39}$$

由定理2可知, 系统(31)有如下一阶绝热不变量:

$$\begin{aligned}
(I_1)^1 = & -\rho(a^2\tau + a^3\tau + a^4)(t-\tau)^{\alpha-1}\\
& = \text{const.}
\end{aligned} \tag{40}$$

假设相应于Noether对称性(34)式, 系统(31)受到的小扰动为

$$\begin{aligned}
\rho Q_1 &= -\rho a^2,\\
\rho Q_2 &= 0,\\
\rho Q_3 &= -\rho(a^3 a^2 + a^2),\\
\rho Q_4 &= -\rho a^4,
\end{aligned} \tag{41}$$

则方程(38)有解:

$$\begin{aligned}
(\xi_0^1)^2 &= 0,\\
(\xi_1^1)^2 &= 1,\\
(\xi_2^1)^2 &= -1,\\
(\xi_3^1)^2 &= 1,\\
(\xi_4^1)^2 &= 0,\\
(G^1)^2 &= 0.
\end{aligned} \tag{42}$$

由定理2可知, 系统(31)有如下一阶绝热不变量:

$$\begin{aligned}
(I_1)^2 = & (a^2 + a^3)(t-\tau)^{\alpha-1}\\
& + \rho(a^2 + a^3 + a^4)(t-\tau)^{\alpha-1}\\
= & \text{const.}
\end{aligned} \tag{43}$$

进一步可求得系统(31)的更高阶绝热不变量.

6 结 论

本文基于El-Nabulsi动力学模型提出并研究了受扰Birkhoff系统Noether对称性的摄动与绝热不变量问题. 将非保守系统的El-Nabulsi动力学模型拓展到Birkhoff系统, 给出了该模型下Birkhoff系统的精确不变量, 研究了在小扰动作用下系统对称性的摄动, 得到了一阶至高阶绝热不变量的存在条件及其形式. 本文方法和结果具有普遍性, 文中给出了与相空间中非保守系统和经典Birkhoff系统相应的结论, 并且还可以将此方法拓展到广义Birkhoff系统、非完整约束系统等.

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Perturbation to Noether symmetries and adiabatic invariants for Birkhoffian systems based on El-Nabulsi dynamical models*

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Abstract

In this paper, we study the problem of perturbation to Noether symmetries and adiabatic invariants for a Birkhoffian system under small disturbance based on the El-Nabulsi dynamical model. First, the dynamical model presented by El-Nabulsi, which is based on the Riemann-Liouville fractional integral under the framework of the fractional calculus, is extended to the Birkhoffian system, and El-Nabulsi-Birkhoff equations for the Birkhoffian system are established. Then, by using the invariance of the El-Nabulsi-Pfaff action under the infinitesimal transformations, the definition and criterion of the Noether quasi-symmetric transformation are given, and the exact invariant caused directly by the Noether symmetry is obtained. Furthermore, by introducing the concept of high-order adiabatic invariant of a mechanical system, the relationship between the perturbation to the Noether symmetry and the adiabatic invariant after the action of small disturbance is studied, the condition that the perturbation of symmetry leads to the adiabatic invariant and its formulation are presented. As a special case, the perturbation to Noether symmetries and corresponding adiabatic invariants mechanics of non-conservative systems in phase space under El-Nabulsi models and classical Birkhoffian systems are discussed. At the end of the paper, taking the well-known Hojman-Urrutia problem for example, its Noether symmetries under the El-Nabulsi dynamical model is investigated and corresponding exact invariants and adiabatic invariants are presented.

Keywords: Birkhoffian system, El-Nabulsi dynamical model, perturbation of Noether symmetry, adiabatic invariant

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