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三层密度分层流体毛细重力波二阶Stokes波解*

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以小振幅波理论为基础, 利用摄动方法研究了三层密度分层流体的毛细重力波, 给出了三层成层状态下各层流体速度势的二阶渐近解及毛细重力波波面位移的二阶 Stokes 波解. 结果表明: 一阶解及二阶解除了依赖于各层流体的厚度及密度, 与表面张力也有很重要的关系.

关键词: 三层密度分层流体, 毛细重力波, 二阶 Stokes 波解, 小振幅波理论

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1 引 言

近些年来, 人们比较重视毛细重力波的研究, 这是因为遥感技术已经成为观测海况的有效手段, 显然它优于用船舶或浮标所进行的测量. 譬如: 1978 年, 海洋卫星 SEASAT 的成功测量, 我国 20 世纪 70 年代末将机载微波辐射计应用于海洋环境遥感以及 1889—1894 年间开展的世界海洋环流试验 (WOCE) 都可证明. 由于 Bragg 散射机制, 毛细波对于海面微波发射率有显著影响. 另一方面, 研究破碎波与波的不稳定性时, 波比较陡, 这时表面张力效应亦是不可忽略的.

界面内波是毛细重力波的一种特殊情形, 通常借助于两层流体模型来研究. 例如, Umeyama[1,2] 导出了沿两层密度不同的有限深度流体界面传播的界面内波的二阶和三阶 Stokes 解, 且利用一个造波水槽进行了实验, 并将实验结果与理论结果做了比较; Song[3] 得到了两层密度分层流体系统中界面内波的二阶随机波解; 程友良[4] 导出了两层流体中二维非线性界面波的演化方程; 尤云祥等[5] 讨论了两层流体中水波在垂直薄板上的反射与透射; 魏岗等[6] 研究了分层流体中内孤立波在台阶上的反射与透射等. 这些研究都是针对两层密度成层水域界面内波进行的. 然而, 实际上真实海洋密度成层现象非常复杂, 有时明显呈多层成层状态. 这时, 利用两层界面内波理论常常难以合理地描述海洋内部的波动规律. 因此, 开展多层密度成层流体界面内波研究是非常必要的. 章守宇和杨红[7] 在线性情形下研究了三层密度成层水域界面内波, 陈小刚等[8] 给出了三层流体界面波的二阶 Stokes 波解, Chen 和 Song[9] 导出了 N 层密度成层流体界面内波的二阶随机波解, 庞晶等[10] 给出了有流存在时三层流体界面波的二阶 Stokes 波解, 但是上述研究中忽略了表面张力. 近期, Gui[11] 和 Tinao[12] 对毛细重力波展开了研究. 本文以小振幅波理论为基础, 利用摄动方法研究了三层密度分层状态下的毛细重力波, 求得了三层密度成层状态下各层流体速度势的二阶解及毛细重力波波面位移的二阶 Stokes 波解.

2 基本方程和边界条件

我们考虑水深 H 为一常数, 密度呈三层分层的不可混溶流体. 设流体为无黏性不可压缩, 并忽略地球旋转的影响. 如图 1 所示, 取静止水面向右

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为 x 轴正方向, 垂直向上为 z 轴正向. 自水面而下, 密度成层流体的厚度分别为 h_1, h_2 和 h_3, 密度分别为 $\rho^{(1)}$, $\rho^{(2)}$ 和 $\rho^{(3)}$. 当水面静止时, 不同密度流体之间构成的各界面水深坐标分别为 z_0, z_1, z_2 和 z_3, $z_0=0$ 表示水面, z_3 表示底面.

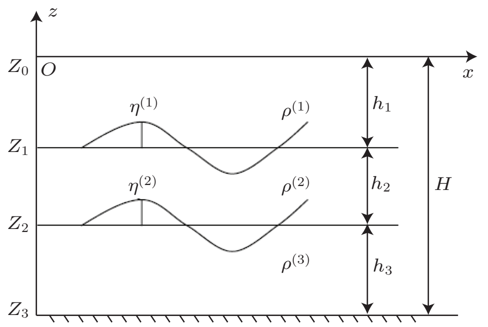

图 1 三层密度成层水域结构及毛细重力波示意图

假定流体是无旋的, 其速度由势函数为 $\Phi^{(i)}(x,z,t)(i=1,2,3)$, $\eta^{(1)},\eta^{(2)}$ 分别为界面 1, 2 处的波面位移, 分别为相对于未受扰动的水面 $z=z_1$ 及 $z=z_2$ 位置的位移. 各层流体速度势满足的 Laplace 方程为

$$\frac{\partial^2\Phi^{(i)}}{\partial x^2}+\frac{\partial^2\Phi^{(i)}}{\partial z^2}=0\quad(z_i+\eta^{(i)}\leqslant z\leqslant z_{i-1}+\eta^{(i-1)},\ i=1,2,3).\tag{1}$$

上表面及下表面分别是刚性边界, 即

$$\frac{\partial\Phi^{(1)}}{\partial z}=0\quad(z=z_0=0),\tag{2}$$

$$\frac{\partial\Phi^{(3)}}{\partial z}=0\quad(z=z_3).\tag{3}$$

各层流体界面上的运动学边界条件为

$$\frac{\partial\Phi^{(i)}}{\partial z}=\frac{\partial\eta^{(i)}}{\partial t}+\frac{\partial\eta^{(i)}}{\partial x}\frac{\partial\Phi^{(i)}}{\partial x}\quad(z=z_i+\eta^{(i)},\quad i=1,2),\tag{4}$$

$$\frac{\partial\Phi^{(i)}}{\partial z}=\frac{\partial\eta^{(i-1)}}{\partial t}+\frac{\partial\eta^{(i-1)}}{\partial x}\frac{\partial\Phi^{(i)}}{\partial x}\quad(z=z_{i-1}+\eta^{(i-1)},\ i=2,3);\tag{5}$$

动力学边界条件为

$$\rho^{(i)}\left\{g\eta^{(i)}+\frac{\partial\Phi^{(i)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(i)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(i)}}{\partial z}\right)^2\right]\right\}=\rho^{(i+1)}\left\{g\eta^{(i)}+\frac{\partial\Phi^{(i+1)}}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi^{(i+1)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(i+1)}}{\partial z}\right)^2\right]\right\}-\Gamma^{(i)}\frac{\partial^2\eta^{(i)}}{\partial x^2}\times\left[1+\left(\frac{\partial\eta^{(i)}}{\partial x}\right)^2\right]^{-3/2}\quad(z=z_i+\eta^{(i)},\quad i=1,2),\tag{6}$$

这里, $\eta^{(0)}(x,t)=0$, $\eta^{(3)}(x,t)=0$.

3 毛细重力波二阶 Stokes 波解

类似于 Umeyama[1,2] 和 Song[3] 对两层流体系统中毛细重力波的研究, 利用摄动方法来求解上述基本方程和边界条件 (1)—(6) 式. 我们首先将上述方程及边界条件全部无因次化. 引入下列无因次量:

$$\begin{aligned}&x'=kx,\quad z'=kz,\quad t'=\omega t,\\&\Phi^{(i)'}(x',z',t')=\frac{2}{H}\sqrt{\frac{k}{g}}\Phi^{(i)}(x,z,t)\\&h_i'=kh_i\quad i=1,2,3,\\&\eta^{(i)'}(x',t')=\frac{2}{H}\eta^{(i)}(x,t)\quad i=1,2,\\&\omega'=\frac{\omega}{\sqrt{gk}},\quad\Gamma^{(i)'}=\frac{k^2\Gamma^{(i)}}{\rho^{(i)}g}\quad i=1,2,\end{aligned}\tag{7}$$

这里, k 为波数, ω 为角频率, H 为特征波幅. 将 (7) 式代入方程 (1) 及条件 (2)—(6) 式, 可得由无因次量表示的方程及边界条件 (这里, 为了简单起见, 略去表示无因次量的一撇) 为

$$\frac{\partial^2\Phi^{(i)}}{\partial x^2}+\frac{\partial^2\Phi^{(i)}}{\partial z^2}=0\quad(z_i+\varepsilon\eta^{(i)}\leqslant z\leqslant z_{i-1}+\varepsilon\eta^{(i-1)},\ i=1,2,3),\tag{8}$$

$$\frac{\partial\Phi^{(1)}}{\partial z}=0\quad(z=z_0=0),\tag{9}$$

$$\frac{\partial\Phi^{(i)}}{\partial z}=\omega\frac{\partial\eta^{(i)}}{\partial t}+\varepsilon\frac{\partial\eta^{(i)}}{\partial x}\frac{\partial\Phi^{(i)}}{\partial x}\quad(z=z_i+\varepsilon\eta^{(i)},\ i=1,2),\tag{10}$$

$$\frac{\partial\Phi^{(i)}}{\partial z}=\omega\frac{\partial\eta^{(i-1)}}{\partial t}+\varepsilon\frac{\partial\eta^{(i-1)}}{\partial x}\frac{\partial\Phi^{(i)}}{\partial x}\quad(z=z_{i-1}+\varepsilon\eta^{(i-1)},\ i=2,3),\tag{11}$$

$$\frac{\partial\Phi^{(3)}}{\partial z}=0\quad(z=z_3),\tag{12}$$

$$\rho^{(i)}\left\{\eta^{(i)}+\omega\frac{\partial\Phi^{(i)}}{\partial t}+\frac{1}{2}\varepsilon\left[\left(\frac{\partial\Phi^{(i)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(i)}}{\partial z}\right)^2\right]\right\}$$

$$
\begin{aligned}
&=\rho^{(i+1)}\left\{\eta^{(i)}+\omega\frac{\partial\Phi^{(i+1)}}{\partial t}\right.\\
&\quad\left.+\frac{1}{2}\varepsilon\left[\left(\frac{\partial\Phi^{(i+1)}}{\partial x}\right)^2+\left(\frac{\partial\Phi^{(i+1)}}{\partial z}\right)^2\right]\right\}\\
&\quad-\Gamma^{(i)}\frac{k^2}{g}\frac{\partial^2\eta^{(i)}}{\partial x^2}\left[1+\left(\varepsilon\frac{\partial\eta^{(i)}}{\partial x}\right)^2\right]^{-3/2}\\
&\quad(z=z_i+\varepsilon\eta^{(i)},\ i=1,2),
\end{aligned}
\tag{13}
$$

其中$\varepsilon=kH/2$为小参数(这是由于本文讨论的是小振幅波, $H/L\ll1$, 而$k=2\pi/L$, L为特征波长).

将$\Phi^{(i)}(i=1,2,3)$, $\eta^{(i)}(i=1,2)$按小参数ε展开为

$$
\begin{aligned}
\Phi^{(i)}={}&\Phi_{\mathrm{I}}^{(i)}+\varepsilon\Phi_{\mathrm{II}}^{(i)}\\
&+\varepsilon^2\Phi_{\mathrm{III}}^{(i)}+O(\varepsilon^3)\quad i=1,2,3,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\eta^{(i)}={}&\eta_{\mathrm{I}}^{(i)}+\varepsilon\eta_{\mathrm{II}}^{(i)}\\
&+\varepsilon^2\eta_{\mathrm{III}}^{(i)}+O(\varepsilon^3)\quad i=1,2,
\end{aligned}
\tag{15}
$$

这里, O是阶符号, 下标I, II和III分别表示一阶, 二阶和三阶近似. 在推导基本方程组时, 为方便处理, 将各界面$z=z_i+\varepsilon\eta^{(i)}(i=1,2)$处的边界条件分别替代为$z=z_i(i=1,2)$处的边界条件. 为此, 将$\Phi_{\mathrm{I}}^{(i)}(i=1,2,3)$分别关于$z$在$z=z_i(i=1,2)$展开为Taylor级数, 连同(14)和(15)式一起代入(8)—(13)式, 比较ε同次幂的系数, 即可得到$\Phi_{\mathrm{I}}^{(i)}(i=1,2,3)$, $\eta_{\mathrm{I}}^{(i)}(i=1,2)$满足的控制方程及边界条件.

一阶方程及边界条件为

$$
\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial z^2}=0
$$

$$
(z_i+\varepsilon\eta^{(i)}\leqslant z\leqslant z_{i-1}+\varepsilon\eta^{(i-1)},\ i=1,2,3),\tag{16}
$$

$$
\frac{\partial\Phi_{\mathrm{I}}^{(1)}}{\partial z}=0\quad(z=z_0=0),\tag{17}
$$

$$
\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial z}=\omega\frac{\partial\eta_{\mathrm{I}}^{(i)}}{\partial t}\quad(z=z_i,\ i=1,2),\tag{18}
$$

$$
\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial z}=\omega\frac{\partial\eta_{\mathrm{I}}^{(i-1)}}{\partial t}\quad(z=z_{i-1},\ i=2,3),\tag{19}
$$

$$
\frac{\partial\Phi_{\mathrm{I}}^{(3)}}{\partial z}=0\quad(z=z_3),\tag{20}
$$

$$
\begin{aligned}
&\rho^{(i)}\left\{\eta_{\mathrm{I}}^{(i)}+\omega\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial t}\right\}=\rho^{(i+1)}\left\{\eta_{\mathrm{I}}^{(i)}+\omega\frac{\partial\Phi_{\mathrm{I}}^{(i+1)}}{\partial t}\right\}\\
&-\Gamma^{(i)}\frac{k^2}{g}\frac{\partial^2\eta_{\mathrm{I}}^{(i)}}{\partial x^2}\quad(z=z_i,\ i=1,2);
\end{aligned}
\tag{21}
$$

二阶方程及边界条件为

$$
\frac{\partial^2\Phi_{\mathrm{II}}^{(i)}}{\partial x^2}+\frac{\partial^2\Phi_{\mathrm{II}}^{(i)}}{\partial z^2}=0
$$

$$
\begin{aligned}
&(z_i+\varepsilon\eta^{(i)}\leqslant z\leqslant z_{i-1}+\varepsilon\eta^{(i-1)},\\
&\ i=1,2,3),
\end{aligned}
\tag{22}
$$

$$
\frac{\partial\Phi_{\mathrm{II}}^{(1)}}{\partial z}=0\quad(z=z_0=0),\tag{23}
$$

$$
\begin{aligned}
&\frac{\partial\Phi_{\mathrm{II}}^{(i)}}{\partial z}+\eta_{\mathrm{I}}^{(i)}\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial z^2}=\omega\frac{\partial\eta_{\mathrm{II}}^{(i)}}{\partial t}\\
&+\frac{\partial\eta_{\mathrm{I}}^{(i)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial x}\quad(z=z_i,\ i=1,2),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&\frac{\partial\Phi_{\mathrm{II}}^{(i)}}{\partial z}+\eta_{\mathrm{I}}^{(i-1)}\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial z^2}=\omega\frac{\partial\eta_{\mathrm{II}}^{(i-1)}}{\partial t}\\
&+\frac{\partial\eta_{\mathrm{I}}^{(i-1)}}{\partial x}\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial x}\quad(z=z_i,\ i=2,3),
\end{aligned}
\tag{25}
$$

$$
\frac{\partial\Phi_{\mathrm{II}}^{(3)}}{\partial z}=0\quad(z=z_3),\tag{26}
$$

$$
\begin{aligned}
&\rho^{(i)}\left\{\eta_{\mathrm{II}}^{(i)}+\omega\frac{\partial\Phi_{\mathrm{II}}^{(i)}}{\partial t}+\omega\eta_{\mathrm{I}}^{(i)}\frac{\partial^2\Phi_{\mathrm{I}}^{(i)}}{\partial t\partial z}\right.\\
&\left.+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(i)}}{\partial z}\right)^2\right]\right\}\\
=\ &\rho^{(i+1)}\left\{\eta_{\mathrm{II}}^{(i)}+\omega\frac{\partial\Phi_{\mathrm{II}}^{(i+1)}}{\partial t}+\omega\eta_{\mathrm{I}}^{(i)}\frac{\partial^2\Phi_{\mathrm{I}}^{(i+1)}}{\partial t\partial z}\right.\\
&\left.+\frac{1}{2}\left[\left(\frac{\partial\Phi_{\mathrm{I}}^{(i+1)}}{\partial x}\right)^2+\left(\frac{\partial\Phi_{\mathrm{I}}^{(i+1)}}{\partial z}\right)^2\right]\right\}\\
&-\Gamma^{(i)}\frac{k^2}{g}\frac{\partial^2\eta_{\mathrm{II}}^{(i)}}{\partial x^2}\quad(z=z_i,i=1,2).
\end{aligned}
\tag{27}
$$

3.1 一阶方程的解

$$
\Phi_{\mathrm{I}}^{(1)}=-\omega\frac{b^{(1)}\cosh z}{\sinh h_1}\sin(x-t),\tag{28}
$$

$$
\begin{aligned}
\Phi_{\mathrm{I}}^{(2)}={}&\omega\frac{b^{(1)}\cosh(z-z_2)-b^{(2)}\cosh(z-z_1)}{\sinh h_2}\\
&\times\sin(x-t),
\end{aligned}
\tag{29}
$$

$$
\Phi_{\mathrm{I}}^{(3)}=\omega\frac{b^{(2)}\cosh(z-z_3)}{\sinh h_3}\sin(x-t),\tag{30}
$$

$$
\eta_{\mathrm{I}}^{(1)}=b^{(1)}\cos(x-t),\tag{31}
$$

$$
\eta_{\mathrm{I}}^{(2)}=b^{(2)}\cos(x-t).\tag{32}
$$

将(28)—(32)式代入到(27)式中得到关于$b^{(1)},b^{(2)}$的二元一次方程组:

$$
\begin{aligned}
&\left[(\rho^{(1)}-\rho^{(2)})+\omega^2(\rho^{(1)}\coth h_1\right.\\
&\left.+\rho^{(2)}\coth h_2)-\Gamma^{(1)}\frac{k^2}{g}\right]b^{(1)}\\
&-\frac{\rho^{(2)}\omega^2}{\sinh h_2}b^{(2)}=0,
\end{aligned}
\tag{33}
$$

$$-\frac{\rho^{(2)}\omega^2}{\sinh h_2}b^{(1)}+\Big[(\rho^{(2)}-\rho^{(3)})+\omega^2(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)-\varGamma^{(2)}\frac{k^2}{g}\Big]b^{(2)}=0. \tag{34}$$

上述方程组有解的条件是

$$\begin{vmatrix}\Big[(\rho^{(1)}-\rho^{(2)})+\omega^2(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)-\varGamma^{(1)}\frac{k^2}{g}\Big] & -\frac{\rho^{(2)}\omega^2}{\sinh h_2}\\ -\frac{\rho^{(2)}\omega^2}{\sinh h_2} & \Big[(\rho^{(2)}-\rho^{(3)})+\omega^2(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)-\varGamma^{(2)}\frac{k^2}{g}\Big]\end{vmatrix}=0. \tag{35}$$

由 (35) 式可以得到频散关系为

$$\begin{aligned}&\omega^4\Big[(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)-\frac{{\rho^{(2)}}^2}{(\sinh h_2)^2}\Big]\\&+\omega^2[(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}\coth h_2+\rho^{(3)}\coth h_3)+g(\rho^{(2)}-\rho^{(3)})(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)]\\&+(\rho^{(1)}-\rho^{(2)})(\rho^{(2)}-\rho^{(3)})-(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)\varGamma^{(2)}\frac{k^2}{g}-(\rho^{(2)}\coth h_2\\&+\rho^{(3)}\coth h_3)\varGamma^{(1)}\frac{k^2}{g}-(\rho^{(1)}-\rho^{(2)})\varGamma^{(2)}\frac{k^2}{g}-(\rho^{(2)}-\rho^{(3)})\varGamma^{(1)}\frac{k^2}{g}+\varGamma^{(1)}\varGamma^{(2)}\frac{k^4}{g^2}=0,\end{aligned} \tag{36}$$

这里

$$b^{(2)}=\alpha b^{(1)}, \tag{37}$$

其中

$$\alpha=\frac{\Big[(\rho^{(1)}-\rho^{(2)})+\omega^2(\rho^{(1)}\coth h_1+\rho^{(2)}\coth h_2)-\varGamma^{(1)}\frac{k^2}{g}\Big]\sinh h_2}{\rho^{(2)}\omega^2}. \tag{38}$$

3.2 二阶方程的解

$$\varPhi_{\rm II}^{(1)}=-\omega\Big[d^{(1)}+\frac{1}{2}{b^{(1)}}^2\coth h_1\Big]\times\frac{\cosh 2z}{\sinh 2h_1}\sin 2(x-t), \tag{39}$$

$$\begin{aligned}\varPhi_{\rm II}^{(2)}=\omega\Big[&\Big(d^{(1)}-\frac{1}{2}b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\Big)\times\frac{\cosh 2(z-z_2)}{\sinh 2h_2}\\&-\Big(d^{(2)}-\frac{1}{2}b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\Big)\times\frac{\cosh 2(z-z_1)}{\sinh 2h_2}\Big]\sin 2(x-t),\end{aligned} \tag{40}$$

$$\varPhi_{\rm II}^{(3)}=\omega\Big[d^{(2)}-\frac{1}{2}{b^{(2)}}^2\coth h_3\Big]\times\frac{\cosh 2(z-z_3)}{\sinh 2h_3}\sin 2(x-t), \tag{41}$$

$$\eta_{\rm II}^{(1)}=d^{(1)}\cos 2(x-t), \tag{42}$$

$$\eta_{\rm II}^{(2)}=d^{(2)}\cos 2(x-t), \tag{43}$$

其中

$$d^{(1)}=\frac{l_1a_{22}-l_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}, \tag{44}$$

$$d^{(2)}=\frac{l_2a_{11}-l_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}, \tag{45}$$

$$a_{11}=(\rho^{(1)}-\rho^{(2)})+2\omega^2(\rho^{(1)}\coth 2h_1+\rho^{(2)}\coth 2h_2)-4\varGamma^{(1)}\frac{k^2}{g}, \tag{46}$$

$$a_{12}=a_{21}=-\frac{2\rho^{(2)}\omega^2}{\sinh 2h_2}, \tag{47}$$

$$a_{22}=(\rho^{(2)}-\rho^{(3)})+2\omega^2(\rho^{(2)}\coth 2h_2+\rho^{(3)}\coth 2h_3)-4\varGamma^{(2)}\frac{k^2}{g}, \tag{48}$$

$$l_1=-\rho^{(1)}\omega^2{b^{(1)}}^2\Big[\frac{1}{4}(\coth h_1)^2+\coth h_1\cdot\coth 2h_1-\frac{3}{4}\Big]$$

$$
\begin{aligned}
&+\rho^{(2)}\omega^2\left\{b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\tanh 2h_2}\right.\\
&-b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\sinh 2h_2}\\
&+\frac{1}{4}\left(\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2}\right)^2\\
&\left.-\frac{3}{4}b^{(1)^2}\right\},
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
l_2=&-\rho^{(2)}\omega^2\left\{b^{(1)}\frac{b^{(1)}\cosh h_2-b^{(2)}}{\sinh h_2\sinh 2h_2}\right.\\
&-b^{(2)}\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2\tanh 2h_2}\\
&+\frac{1}{4}\left(\frac{b^{(1)}-b^{(2)}\cosh h_2}{\sinh h_2}\right)^2\\
&\left.-\frac{3}{4}b^{(2)^2}\right\}+\rho^{(3)}\omega^2 b^{(2)^2}\left[\frac{1}{4}(\coth h_3)^2\right.\\
&\left.+\coth h_3\coth 2h_3-\frac{3}{4}\right].
\end{aligned}
\tag{50}
$$

4 讨 论

1) 当 $\Gamma=0$, 即不考虑表面张力时, 此波为重力波. 利用 (7) 式中有因次量和无因次量之间的关系, 易知所得一阶、二阶解的表达式与陈小刚等[9] 给出的三层流体界面波的二阶 Stokes 波解的相一致.

2) 一阶解是正弦波解, 与传统线性理论的结果相一致; 二阶解描述了毛细重力波的二阶非线性修正及两毛细重力波之间的非线性相互作用.

3) 若已知波动周期 T 及初始振幅 $b^{(1)}$, 由关系式 (33) 和 (34) 可以求出波数 k 及 $b^{(2)}$, 再将 k, $b^{(1)}$, $b^{(2)}$ 代入到 (44)—(50) 式中可以求出 $d^{(1)}$, $d^{(2)}$, 从而得到各层流体速度势及波面位移的一阶解及二阶解; 由于 $b^{(1)}$, $b^{(2)}$, $d^{(1)}$, $d^{(2)}$ 式中均含有表面张力 Γ, 由此可见表面张力在水波的研究当中是不可忽略的.

5 结 论

以小振幅波理论为基础, 利用摄动方法研究了三层密度成层状态下的毛细重力波, 求得了三层密度成层状态下各层流体速度势的二阶解及毛细重力波波面位移的二阶 Stokes 波解. 结果表明: 一阶解及二阶解除了依赖于各层流体的厚度及密度, 和表面张力也有非常重要的关系.

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Second-order Stokes wave solutions for gravity capillary water waves in three-layer dendity-stratified fluid*

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Abstract

In this paper, gravity-capillary water waves in a three-layer stratified fluid are investigated by using a perturbation method, and the second-order asymptotic solutions of the velocity potentials and the second-order Stokes solutions of the associated elevations of the gravity-capillary water waves are presented based on the small amplitude wave theory. As expected, both the first-order and second-order solutions derived depend on not only the depth and density of the three-layer fluid but also the surface tension.

Keywords: three-layer density-stratified fluid, gravity-capillary water waves, second-order Stokes solutions, small amplitude wave theory

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